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SPECTRAL PARAMETER ESTIMATION FOR LINEAR SYSTEM IDENTIFICATION

by

A.C. LUCIA

1970

Joint Nuclear Research Center
Ispra Establishment - Italy

Reactor Physics Department
Research Reactors

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ABSTRACT

The aim of this report is to examine the properties and statistical errors of the estimators of some spectral parameters of stationary, ergodic, linear processes (power and cross-power spectral density, Fourier transform, transfer functions).

In addition, procedures and computing programmes are supplied for the estimation of these parameters by means of the statistical dynamics analyzer S.D.A. - Mod. 040. The case of deterministic signals is also taken into consideration.

KEYWORDS

MATHEMATICS
PARAMETERS
FOURIER TRANSFORMATIONS
COMPUTERS
STATISTICS
SPECTRA

CONTENTS

SPECTRAL PARAMETER ESTIMATION FOR LINEAR SYSTEM IDENTIFICATION

INTRODUCTION *)

The techniques for determining the power and cross-power spectral density, the Fourier transforms and the transfer functions assume considerable importance in the identification of a system or process [1] [2] [3]. From these functions it is in fact often possible to arrive at some basic parameters of the system or process under examination. For this reason it is interesting to know the characteristics of the estimators we use and the accuracy obtainable with them.

This report studies the statistical properties of some continuous estimators in general use which allow direct determination of Fourier transforms, power and cross-power spectral densities and transfer functions. The determination of these functions is treated categorically for the cases of random signals and of aperiodic and periodic signals.

The estimators we deal with in this work also constitute the algorithms by which are obtained the determinations made on the statistical dynamics analyzer S.D.A. (a general purpose analyzer, designed and built at the Euratom Joint Research Center of Ispra) [4]. In the sections 1.3; 2.3 and 3.2 and in the Appendix, in which processing procedures and computing programmes are given, reference is made exclusively to the S.D.A.

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1.) SPECTRAL ANALYSIS OF RANDOM SIGNALS

The random signals we will consider are assumed to be stationary and ergodic, even when this is not explicity stated.

We know that in the case of random signals, no Fourier transform (of the classic type) exists, and that for this reason the power spectra are defined in terms of Fourier transforms of the correlation functions [5] [16] :

$$\phi_{xx}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}x(t)\,x(t+\tau)dt\right]e^{-j\omega\tau}d\tau \qquad (1)$$

$$\phi_{yy}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}y(t)\,y(t+\tau)dt\right]e^{-j\omega\tau}d\tau \qquad (2)$$

$$\phi_{xy}(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}x(t)\,y(t+\tau)dt\right]e^{-j\omega\tau}d\tau \qquad (3)$$

Nevertheless the validity of the direct determination (by the Fourier transform of the signals under examination) of the power and cross-power spectral density, is demonstrated [6] [7].

The estimators by which the direct determinations can be carried out have the following expressions:

$$\hat{\phi}_{xx}(\omega) = \frac{1}{2\pi}\,\frac{1}{T}\left|\int_{0}^{T}x(t)\,e^{-j\omega t}dt\right|^{2} \qquad (4)$$

$$\hat{\phi}_{yy}(\omega) = \frac{1}{2\pi}\,\frac{1}{T}\left|\int_{0}^{T}y(t)\,e^{-j\omega t}dt\right|^{2} \qquad (5)$$

$$\hat{\psi}_{xy}(j\omega) = \frac{1}{2\pi}\,\frac{1}{T}\int_{o}^{T} x(t)\, e^{j\omega t} dt \,.\, \int_{o}^{T} y(t)\, e^{-j\omega t} dt \tag{6}$$

which are the estimators we are considering here; they are likewise used in the S.D.A. for the spectral analysis of random signals [4]).

1.1.) Power spectral density estimation

Let expression (4):

$$\hat{\psi}_{xx}(\omega) = \frac{1}{2\pi}\,\frac{1}{T}\left|\int_{o}^{T} x(t)\, e^{-j\omega t} dt\right|^{2} \tag{4}$$

be the estimator of the power spectral density of a stationary, ergodic random signal $x(t)$.

We will see what the properties of this estimator are, particularly as far as the bias and the variance of the estimation are concerned.

The mathematical expectation of the estimator is:

$$E\left[\hat{\psi}_{xx}(\omega)\right] = \frac{1}{2\pi}\,\frac{1}{T}\,E\left[\left|\int_{o}^{T} x(t)\, e^{-j\omega t} dt\right|^{2}\right] \tag{7}$$

Considering the square of the modulus of the integral at the right-hand member of (7) as a double integral and remembering that:

$$E\left[x(t)\; x(\theta)\right] = R_{xx}(t-\theta) \tag{8}$$

where R_{xx} is the autocorrelation function of $x(t)$, we can write [8]):

$$E\left[\hat{\psi}_{xx}(\omega)\right] = \frac{1}{2\pi}\,\frac{1}{T}\int_{o}^{T}\int_{o}^{T} R_{xx}(t-\theta)\, e^{-j\omega(t-\theta)} dt\, d\theta \tag{9}$$

which gives finally [9]) :

$$E\left[\hat{\psi}_{xx}(\omega)\right] = \frac{1}{2\pi}\int_{-T}^{T}\left(1-\frac{|\tau|}{T}\right)R_{xx}(\tau)\,e^{-j\omega\tau}d\tau \qquad (10)$$

For an analysis time T tending to the infinite, wherever the auto-correlation function can be integrated absolutely, we obtain from (10):

$$\lim_{T\to\infty} E\left[\hat{\psi}_{xx}(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}R_{xx}(\tau)\,e^{-j\omega\tau}d\tau = \psi_{xx}(\omega) \qquad (11)$$

i.e. the estimator (4) is unbiased in the limit, but for a finite analysis time T , it is biased: that is, it contains a systematic error. Let us now see what the variance of the estimate is:

$$\mathrm{var}\left[\hat{\psi}_{xx}(\omega)\right] = E\left[\hat{\psi}^2_{xx}(\omega)\right] - \left\{E\left[\hat{\psi}_{xx}(\omega)\right]\right\}^2 \qquad (12)$$

where the mathematical expectation of the square of the estimate is [8]):

$$E\left[\hat{\psi}^2_{xx}(\omega)\right] = E\left\{\left[\frac{1}{2\pi}\,\frac{1}{T}\left|\int_0^T x(t)\,e^{-j\omega t}dt\right|^2\right]^2\right\} =$$

$$= \frac{1}{4\pi^2}\,\frac{1}{T^2}\int_0^T\int_0^T\int_0^T\int_0^T E\left[x(t)\,x(\theta)\,x(\eta)\,x(\xi)\,e^{-j\omega(t+\theta-\eta-\xi)}\right]dt\,d\theta\,d\eta\,d\xi \qquad (13)$$

the expression at the third member of (13) having been deduced by considering the square of the integral as a double integral, and by using the property of interchangeability of the operation of finding the mathematical expectation and the operation of integration.

Remembering that for any four normal variables x_1, x_2, x_3, x_4 we have [10] [11] :

$$E(x_1 x_2 x_3 x_4) = E(x_1 x_2)\, E(x_3 x_4) + E(x_1 x_3) E(x_2 x_4) + E(x_1 x_4) E(x_2 x_3) - 2\, \bar{x}_1 \bar{x}_2 \bar{x}_3 \bar{x}_4 =$$

$$= R_{12} R_{34} + R_{13} R_{24} + R_{14} R_{23} - 2\, \bar{x}_1 \bar{x}_2 \bar{x}_3 \bar{x}_4 \qquad (14)$$

where R_{ij} is the correlation function of the ith and jth random variables, we obtain from expression (13):

$$E(\hat{\psi}^2_{xx}(\omega)) = \frac{1}{4\pi^2}\, \frac{1}{T^2} \int_o^T \int_o^T \int_o^T \int_o^T \Big[R_{xx}(t-\theta)\, R_{xx}(\eta-\xi) + R_{xx}(t-\eta)\, R_{xx}(\theta-\xi) +$$

$$+ R_{xx}(t-\xi)\, R_{xx}(\theta-\eta) - 2\, x \Big]\, e^{-j\omega(t+\theta-\eta-\xi)} dt\, d\theta\, d\eta\, d\xi \qquad (15)$$

Taking into consideration expression (9) , expansion of the right-hand member of (15) gives:

$$E(\hat{\psi}^2_{xx}(\omega)) = 2\, (E\, (\hat{\psi}_{xx}(\omega)))^2 + \frac{1}{T^2}\, \frac{1}{4\pi^2} \left| \int_o^T \int_o^T R_{xx}(t-\theta)\, e^{j\omega(t+\theta)} dt\, d\theta \right|^2 +$$

$$- 2\, \bar{x}^4 \frac{1}{T^2}\, \frac{1}{4\pi^2} \int_o^T \int_o^T \int_o^T \int_o^T e^{-j\omega(t+\theta-\eta-\xi)} dt\, d\theta\, d\eta\, d\xi \qquad (16)$$

As T tends to the infinite, equation (16) becomes:

$$\lim_{T\to\infty} E(\hat{\psi}^2_{xx}(\omega)) = \lim_{T\to\infty} 2(E(\hat{\psi}_{xx}(\omega)))^2 \qquad (17)$$

so that by virtue of (12) and (17) one can write:

$$\lim_{T\to\infty} \mathrm{var}(\hat{\psi}_{xx}(\omega)) = \lim_{T\to\infty} (\, E(\hat{\psi}_{xx}(\omega))\,)^2 \qquad (18)$$

Hence from (11) :

$$\lim_{T\to\infty} \text{var}\,[\hat{\psi}_{xx}(\omega)] = \psi^2_{xx}(\omega) \tag{19}$$

which means that the random variable constituted by the estimator $\hat{\psi}_{xx}(\omega)$ does not converge in the mean upon the value $\psi_{xx}(\omega)$ of the power spectral density, and the accuracy of the measurement does not improve with an increase of integration time T. The statistical error thus remains, even with an infinite analysis-time.

Let us however repeat this measurement on successive times, and use then the estimator:

$${}_k\hat{\psi}_{xx}(\omega) = \frac{1}{k}\,\frac{1}{2\pi}\,\frac{1}{T}\,\sum_{i=1}^{k}\left|\int_{t_i}^{t_i+T} x(t)\, e^{-j\omega t} dt\right|^2 = \frac{1}{k}\sum_{i=1}^{k}\hat{\psi}_{xx,i}(\omega) \tag{20}$$

which is unbiased in the limit, as estimator (4) was already; in fact:

$$E\,[{}_k\hat{\psi}_{xx}(\omega)] = \frac{1}{k}\sum_{i=1}^{k} E\,[\hat{\psi}_{xx,i}(\omega)] \tag{21}$$

and remembering (11):

$$\lim_{T\to\infty} E\,[{}_k\hat{\psi}_{xx}(\omega)] = \frac{1}{k}\sum_{i=1}^{k}\psi_{xx,i}(\omega) = \psi_{xx}(\omega) \tag{22}$$

As far as the variance of the new estimator is concerned, we must bear in mind that if the initial instants t_i of each integration are spaced sufficiently so that the k determinations can be considered independent, one can write [12] [13] :

$$\text{var}\,[{}_k\hat{\psi}_{xx}(\omega)] = \frac{1}{k}\,\text{var}\,[\hat{\psi}_{xx,i}(\omega)] \tag{23}$$

by which, for (19), we have:

$$\lim_{T\to\infty} \operatorname{var}\left({}_k\hat{\psi}_{xx}(\omega)\right) = \frac{1}{k}\psi^2_{xx}(\omega) \tag{24}$$

which clearly means that:

$$\lim_{\substack{T\to\infty \\ k\to\infty}} \operatorname{var}\left({}_k\hat{\psi}_{xx}(\omega)\right) = \lim_{k\to\infty} \frac{1}{k}\psi^2_{xx}(\omega) = 0 \tag{25}$$

As for an infinite analysis time T, the estimator (20) is unbiased (so that mean square arror and variance of the estimate have the same value), expression (25) means also that estimator (20) is consistent for infinite values of both T and k.
It will be observed that the variance of the estimator tends to zero, when k tends to be infinite, even if T remains finite, while the mean square error tends to a nonzero value owing to the bias of the estimate.

Equation (25) has been deduced by considering k independent measurements. Let us take, on the other hand, the general case of correlated measurements:

$$\operatorname{var}\left[{}_k\hat{\psi}_{xx}(\omega)\right] = E\left[{}_k\hat{\psi}^2_{xx}(\omega)\right] - \left\{E\left[{}_k\hat{\psi}_{xx}(\omega)\right]\right\}^2$$

$$= E\left[\left(\frac{1}{k}\sum_{i=1}^{k}\hat{\psi}_{xx,i}(\omega)\right)^2\right] - \left\{E\left[{}_k\hat{\psi}_{xx}(\omega)\right]\right\}^2 \tag{26}$$

that is, developing the square of the summation:

$$\operatorname{var}\left({}_k\hat{\psi}_{xx}(\omega)\right) = E\left[\frac{1}{k^2}\sum_{i=1}^{k}\hat{\psi}^2_{xx,i}(\omega) + \frac{1}{k^2}\sum_{\substack{i,j=1 \\ i\neq j}}^{k}\hat{\psi}_{xx,i}(\omega)\hat{\psi}_{xx,j}(\omega)\right] - \left\{E\left[{}_k\hat{\psi}_{xx}(\omega)\right]\right\}^2 \tag{27}$$

If we consider the determinations $\hat{\psi}_{xx,i}(\omega)$ as random variables and remember that, for two random variables z and w wa can write [9]):

$$E(z^2) = \sigma_z^2 + \bar{z}^2$$

$$E(z^r w^s) = \alpha_{rs} \tag{28}$$

$$\alpha_{11} = R_{zw}(\tau) = \bar{z}.\bar{w} + \rho_{zw}(\tau)\sqrt{C_{zz}(0)C_{ww}(0)} = \bar{z}.\bar{w} + \rho_{zw}(\tau)\,\sigma_z\,\sigma_w$$

where α_{rs} is the moment of order $r+s$, equation (27) can be expressed as:

$$\mathrm{var}\left({}_k\hat{\psi}_{xx}(\omega)\right) = \frac{1}{k}\left\{\mathrm{var}\left[\hat{\psi}_{xx,i}(\omega)\right] + \left(E\left[{}_k\hat{\psi}_{xx}(\omega)\right]\right)^2\right\} + \frac{k-1}{k}\left(E\left[{}_k\hat{\psi}_{xx}(\omega)\right]\right)^2 +$$

$$+ \frac{1}{k^2}\,\mathrm{var}\left[\hat{\psi}_{xx,i}(\omega)\right]\sum_{\substack{i,j=1\\ i\neq j}}^{k}\rho_{ij}(\tau) - \left(E\left[{}_k\hat{\psi}_{xx}(\omega)\right]\right)^2 \tag{29}$$

from which we obtain the final result:

$$\mathrm{var}\left[{}_k\hat{\psi}_{xx}(\omega)\right] = \frac{1}{k}\,\mathrm{var}\left[\hat{\psi}_{xx,i}(\omega)\right]\left[1 + \frac{1}{k}\sum_{\substack{i,j=1\\ i\neq j}}^{k}\rho_{ij}(\tau)\right] \tag{30}$$

where $\rho_{ij}(\tau)$ expresses the degree of correlation existing between the ith and the jth measurement, and can assume values between zero and one.

If the function $\rho_{ij}(\tau)$ is supposed to be constant for all values of i and j, expression (30) of the variance of the estimate reduces to the approximate and simplified form:

$$\mathrm{var}\left[{}_k\hat{\psi}_{xx}(\omega)\right] \cong \mathrm{var}\left[\hat{\psi}_{xx,i}(\omega)\right]\frac{(k-1)\rho_k(\tau)+1}{k} \tag{31}$$

If $\rho_k(\tau)$ is zero, we come back to the theoretical case considered in equation (23). It is however sufficient that $\rho(\tau)$ is less than unity for the relation (25) to be valid, and the estimator to be consistent.

It would be interesting here to examine more closely the spectral estimator, and the physical significance of the error introduced by the finite analysis time T.
To do this, it is not necessary to take the repetitions into account; for simplicity of notation, we will therefore refer to estimator (4) and resume equation (10):

$$E(\hat{\psi}_{xx}(\omega)) = \frac{1}{2\pi}\int_{-T}^{T}\left(1-\frac{|\tau|}{T}\right)R_{xx}(\tau)\,.\,e^{-j\omega\tau}d\tau \tag{10}$$

whose second member can be interpreted as the Fourier transform of the product of the autocorrelation function $R_{xx}(\tau)$ of the signal $x(t)$ to be analyzed, and the function:

$$h(\tau) = 1-\frac{|\tau|}{T} \tag{32}$$

defined for $|\tau| \leqslant T$ and zero elsewhere; that is, the estimated power spectral density is the Fourier transform of the autocorrelation function weighed by a data window $h(\tau)$.

In the frequency domain, remembering the convolution theorem, the Fourier transform of the product is given by the convolution of the factors:

$$E(\hat{\psi}_{xx}(\omega')) = \int_{-\infty}^{\infty}\psi_{xx}(\omega)\,.\,H(\omega'-\omega)d\omega \tag{33}$$

where:

$$H(\omega) = T \left(\frac{\text{sen } \omega \frac{T}{2}}{\omega \frac{T}{2}} \right)^2 \qquad (34)$$

Expression (33) means that the power spectral density estimator has an expected value which corresponds to the theoretical value $\psi_{xx}(\omega)$ seen through a spectral window $H(\omega)$ [14]).
As T tends to infinity, $H(\omega)$ tends to a delta function centered at $\omega = \omega'$; hence the spectrum estimator gives a correct, unbiased estimate.

In practice the spectral window is composed essentially of a slit with a width of the order of $\frac{1}{T}$ (in Hz); hence, for sufficiently large values of T, it is reasonable to assume $\psi_{xx}(\omega)$ quite constant in the frequency range $\frac{1}{T}$, so that:

$$E(\hat{\psi}_{xx}(\omega')) \cong \psi_{xx}(\omega') \int_{-\infty}^{\infty} T \left(\frac{\text{sen } \frac{\omega T}{2}}{\frac{\omega T}{2}} \right)^2 d\omega = \psi_{xx}(\omega') \qquad (35)$$

Once again, it becomes obvious that the error due to the finite analysis time is less serious than, and has nothing to do with, the statistical error, so that having a record of lenght T_t of the signal to analyze, it is better to divide the time T_t into k intervals T_i , suitably spaced out, and to perform k measurements, rather than to perform one continuous analysis for the whole period T_t.

1.2.) Cross-power spectral density estimation

Let us consider the estimator (6):

$$\hat{\psi}_{xy}(j\omega) = \frac{1}{2\pi} \frac{1}{T} \int_0^T x(t)\, e^{j\omega t} dt \cdot \int_0^T y(t)\, e^{-j\omega t} dt \qquad (6)$$

The same considerations that were taken for the analogous estimator (4) of the power spectral density $\psi_{xx}(\omega)$ are valid for this estimator. In fact:

$$E(\hat{\psi}_{xy}(j\omega)) = \frac{1}{2\pi}\,\frac{1}{T}\int_{0}^{T}\int_{0}^{T} E\left[x(t)y(\theta)\, e^{j\omega(t-\theta)}\right] dt\, d\theta \tag{36}$$

from which [9]:

$$E(\hat{\psi}_{xy}(j\omega)) = \frac{1}{2\pi}\int_{-T}^{T}\left(1-\frac{|\tau|}{T}\right) R_{xy}(\tau)\, e^{-j\omega\tau} d\tau \tag{37}$$

which, in the case of the absolute integrality of $R_{xy}(\tau)$, allows us to write:

$$\lim_{T\to\infty} E(\hat{\psi}_{xy}(j\omega)) = \psi_{xy}(j\omega) \tag{38}$$

which means the estimator is unbiased in the limit.

Let us consider now the variance of the estimate:

$$\operatorname{var}(\hat{\psi}_{xy}(j\omega)) = E\left[\hat{\psi}^{2}_{xy}(j\omega)\right] - \left\{E\left[\hat{\psi}_{xy}(j\omega)\right]\right\}^{2} \tag{39}$$

Substituting (6) and (36) in (39) and developping the resulting expression as we have already made in the case of power spectral density estimation (see section 1.1.), we have:

$$\text{var}(\hat{\psi}_{xy}(j\omega)) = \left\{E\,(\hat{\psi}_{xy}(j\omega))\right\}^2 + \frac{1}{T^2}\,\frac{1}{4\pi^2}\left|\int_0^T\int_0^T R_{xy}(t-\theta)e^{j\omega(t+\theta)}dt\;d\theta\right|^2 +$$

$$-2\,\bar{x}^2\,\bar{y}^2\;\frac{1}{T^2}\,\frac{1}{4\pi^2}\int_0^T\int_0^T\int_0^T\int_0^T e^{-j\omega(\xi+\theta-\eta-t)}dt\;d\theta\;d\eta\;d\xi \tag{40}$$

which, for T tending to infinity, gives:

$$\lim_{T\to\infty} \text{var}\,(\hat{\psi}_{xy}(j\omega)) = \lim_{T\to\infty}\left\{E\left[\hat{\psi}_{xy}(j\omega)\right]\right\}^2 = \psi^2_{xy}(j\omega) \tag{41}$$

To reduce the variance of the determination, it is better to use an estimator of the type:

$$_k\hat{\psi}_{xy}(j\omega) = \frac{1}{k}\;\frac{1}{2\pi}\,\frac{1}{T}\sum_{i=1}^{k}\int_{t_i}^{t_i+T} x(t)\;e^{j\omega t}dt\;.\int_{t_i}^{t_i+T} y(t)\;e^{-j\omega t}dt =$$

$$= \frac{1}{k}\sum_{i=1}^{k}\hat{\psi}_{xy,i}(j\omega) \tag{42}$$

for which one has:

$$E(_k\hat{\psi}_{xy}(j\omega)) = \frac{1}{k}\sum_{i=1}^{k} E(\hat{\psi}_{xy,i}(j\omega)) \tag{43}$$

from which, taking equation (38) into account:

$$\lim_{T\to\infty} E(_k\hat{\psi}_{xy}(j\omega)) = \psi_{xy}(j\omega) \tag{44}$$

that is, the estimator is unbiased in the limit.

Proceeding in a manner similar to that employed for $\psi_{xx}(\omega)$ we find:

$$\mathrm{var}({}_k\hat{\psi}_{xy}(j\omega)) \cong \mathrm{var}(\hat{\psi}_{xy,i}(j\omega)) \frac{(k-1)\rho_k(\tau)+1}{k} \tag{45}$$

because of which, if $\rho_k(\tau)$ is less than unity:

$$\lim_{k\to\infty} \mathrm{var}\,({}_k\hat{\psi}_{xy}(j\omega)) = 0 \tag{46}$$

Expression (46) is valid whatever the value of the integration time T, while the mean square error (m.s.e.) tends to zero only if both T and k tends to infinity:

$$\lim_{\substack{T\to\infty \\ k\to\infty}} \mathrm{m.s.e.}\,({}_k\hat{\psi}_{xy}(j\omega)) = 0 \tag{47}$$

The considerations set out for the power spectral density concerning the physical significance of the error due to the finite analysis time are also valid for the cross-power spectral density.

1.3.) Power and cross-power spectral density estimation with the statistical dynamics analyzer S.D.A.

The estimate of the power and cross-power spectral density of random signals is carried out in the S.D.A. statistical dynamics analyzer [4] [15], on the basis of estimators (20) and (42).

If x(t) and y(t) are the random signals being examined, the initial expressions are therefore:

$$\hat{\psi}_{xx}(\omega) = \frac{1}{2\pi}\,\frac{1}{k}\,\frac{1}{T}\sum_{i=1}^{k}\left|\int_{t_i}^{t_i+T} x(t)\,e^{-j\omega t}dt\right|^2 \tag{48}$$

$$\hat{\psi}_{yy}(\omega) = \frac{1}{2\pi} \frac{1}{k} \frac{1}{T} \sum_{i=1}^{k} \left| \int_{t_i}^{t_i+T} y(t)\, e^{-j\omega t} dt \right|^2 \tag{49}$$

$$\hat{\psi}_{xy}(j\omega) = \frac{1}{2\pi} \frac{1}{k} \frac{1}{T} \sum_{i=1}^{k} \int_{t_i}^{t_i+T} x(t) e^{j\omega t} dt \cdot \int_{t_i}^{t_i+T} y(t) e^{-j\omega t} dt \tag{50}$$

where, for simplicity of notation, we have omitted the subscript k at the left-hand side of $\hat{\psi}$.

The S.D.A. system calculates and supplies the following data [4] [15] to the computer which constitutes its final element:

$$\alpha_i = 10^{-3} \int_{t_i}^{t_i+T} f_s(t) f_x(t) \tau_{ke}\, dt \tag{51}$$

$$\beta_i = 10^{-3} \int_{t_i}^{t_i+T} f_c(t) f_x(t)\, \tau_{ke}\, dt \tag{52}$$

$$i = 1,2,\ldots\ldots k$$

$$\gamma_i = 10^{-3} \int_{t_i}^{t_i+T} f_s(t) f_y(t)\, \tau_{ke}\, dt \tag{53}$$

$$\delta_i = 10^{-3} \int_{t_i}^{t_i+T} f_c(t) f_y(t)\, \tau_{ke}\, dt \tag{54}$$

where $f_s(t)$ and $f_c(t)$ are the frequencies, variable with time, of the frequency modulated pulses which constitute the sine and cosine reference signals:

$$\begin{aligned} f_s(t) &= \frac{\omega}{4} K_\tau \text{ sen } \omega t \\ f_c(t) &= \frac{\omega}{4} K_\tau \cos \omega t \end{aligned} \tag{55}$$

in which K_τ is the constant number of pulses per cycle of the two reference signals (the kernels $e^{-j\omega t}$).
The frequencies $f_x(t)$ and $f_y(t)$ are proportional respectively to the analog input signals $\dot{x}(t)$ and $y(t)$ according to the relations:

$$
\begin{aligned}
x(t) &= h \cdot f_x(t) \\
y(t) &= h \cdot f_y(t)
\end{aligned}
\tag{56}
$$

h being the proportionality coefficient in the voltage to frequency conversion.
Finally, τ_{ke} is the length of the sine and cosine pulses. Thus substitution of (55) and (56) in (51), (52), (53) and (54) gives:

$$\alpha_i = 10^{-3} \frac{\omega K_\tau \tau_{ke}}{4 h} \int_{t_i}^{t_i+T} x(t) \text{ sen } \omega t \, dt \tag{57}$$

$$\beta_i = 10^{-3} \frac{\omega K_\tau \tau_{ke}}{4 h} \int_{t_i}^{t_i+T} x(t) \cos \omega t \, dt \tag{58}$$

$$\gamma_i = 10^{-3} \frac{\omega K_\tau \tau_{ke}}{4 h} \int_{t_i}^{t_i+T} y(t) \text{ sen } \omega t \, dt \tag{59}$$

$$\delta_i = 10^{-3} \frac{\omega K_\tau \tau_{ke}}{4 h} \int_{t_i}^{t_i+T} y(t) \cos \omega t \, dt \tag{60}$$

Expressions (57) .....(60) allow to write:

$$\alpha_i^2 + \beta_i^2 = 10^{-6} \frac{\omega^2 K_\tau^2 \tau_{ke}^2}{16 h^2} \left| \int_{t_i}^{t_i+T} x(t) e^{-j\omega t} dt \right|^2 \tag{61}$$

$$\gamma_i^2+\delta_i^2 = 10^{-6} \frac{\omega^2 K_\tau^2 \tau_{ke}^2}{16 h^2} \left| \int_{t_i}^{t_i+T} y(t)\, e^{-j\omega t} dt \right|^2 \tag{62}$$

$$\alpha_i\gamma_i+\beta_i\delta_i = 10^{-6} \frac{\omega^2 K_\tau^2 \tau_{ke}^2}{16 h^2} \mathrm{Re} \left[\int_{t_i}^{t_i+T} x(t) e^{j\omega t} dt \cdot \int_{t_i}^{t_i+T} y(t) e^{-j\omega t} dt \right] \tag{63}$$

$$\alpha_i\delta_i-\beta_i\gamma_i = 10^{-6} \frac{\omega^2 K_\tau^2 \tau_{ke}^2}{16 h^2} \mathrm{Im} \left[\int_{t_i}^{t_i+T} x(t) e^{j\omega t} dt \cdot \int_{t_i}^{t_i+T} y(t) e^{-j\omega t} dt \right] \tag{64}$$

If we now remember the relations (48), (49) and (50), we can see that it is possible to obtain the power spectral density from the quantities α_i , β_i , γ_i and δ_i :

$$\hat{\psi}_{xx}(\omega) = \frac{1}{2\pi} \frac{1}{k} \frac{1}{T} \frac{4 h^2}{10^{-6} \pi^2 f^2 \tau_{ke}^2 K_\tau^2} \sum_{i=1}^{k} (\alpha_i^2+\beta_i^2) = \frac{1}{2\pi} \frac{f}{kn} \frac{1}{h_r^2} \sum_{i=1}^{k} (\alpha_i^2+\beta_i^2) \tag{65}$$

$$\hat{\psi}_{yy}(\omega) = \frac{1}{2\pi} \frac{1}{k} \frac{1}{T} \frac{4 h^2}{10^{-6} \pi^2 f^2 \tau_{ke}^2 K_\tau^2} \sum_{i=1}^{k} (\gamma_i^2+\delta_i^2) = \frac{1}{2\pi} \frac{f}{kn} \frac{1}{h_r^2} \sum_{i=1}^{k} (\gamma_i^2+\delta_i^2) \tag{66}$$

$$\mathrm{Re}(\hat{\psi}_{xy}(j\omega)) = \frac{1}{2\pi} \frac{1}{k} \frac{1}{T} \frac{4 h^2}{10^{-6} \pi^2 f^2 \tau_{ke}^2 K_\tau^2} \sum_{i=1}^{k} (\alpha_i\beta_i+\gamma_i\delta_i) = \frac{1}{2\pi} \frac{f}{kn} \frac{1}{h_r^2} \sum_{i=1}^{k} (\alpha_i\beta_i+\gamma_i\delta_i) \tag{67}$$

$$\mathrm{Im}(\hat{\psi}_{xy}(j\omega)) = \frac{1}{2\pi} \frac{1}{k} \frac{1}{T} \frac{4 h^2}{10^{-6} \pi^2 f^2 \tau_{ke}^2 K_\tau^2} \sum_{i=1}^{k} (\alpha_i\delta_i-\beta_i\gamma_i) = \frac{1}{2\pi} \frac{f}{nk} \frac{1}{h_r^2} \sum_{i=1}^{k} (\alpha_i\delta_i-\beta_i\gamma_i) \tag{68}$$

where power and cross-power spectral densities are expressed in volts squared per unit of angular frequency and, in agreement with their theoretical definition, defined fot both positive and negative frequencies, [16]).

If we want to express the spectral densities in squared volts per hertz, we have to multiply expressions (65).....(68) by a factor 2π.

In the right-hand members of (65), (66), (67) and (68) we have introduced the normalization coefficient h_r for random signal analysis:

$$h_r = 10^{-3} \frac{\pi f \tau_{ke} K_\tau}{2 h} \tag{69}$$

and we have expressed the integration time T as:

$$T = \frac{n}{f} \tag{70}$$

since in the S.D.A. apparatus this time is defined, in the spectral analysis of random signals, as a multiple of the period $1/f$ of the frequency being examined [15]).
The normalisation coefficient h_r depends upon the frequency decade under examination, but does not vary with the frequency, which means that τ_{ke} is inversely proportional to the frequency itself [15]). Its value is however supplied directly from the analyzer to the final computer, together with the quantities α_i , β_i , γ_i , δ_i , the analysis frequency f , the n number of integration cycles, and the k number of repetitions [15]) [17]).

If the signals $x(t)$ and $y(t)$ represent respectively the excitation signal and the response of a linear system, it may be interesting to obtain the transfer function $G(j\omega)$ of the system itself.

This can be obtained on the basis of the definition [16]:

$$G(j\omega) = \frac{\psi_{xy}(j\omega)}{\psi_{xx}(\omega)} \tag{71}$$

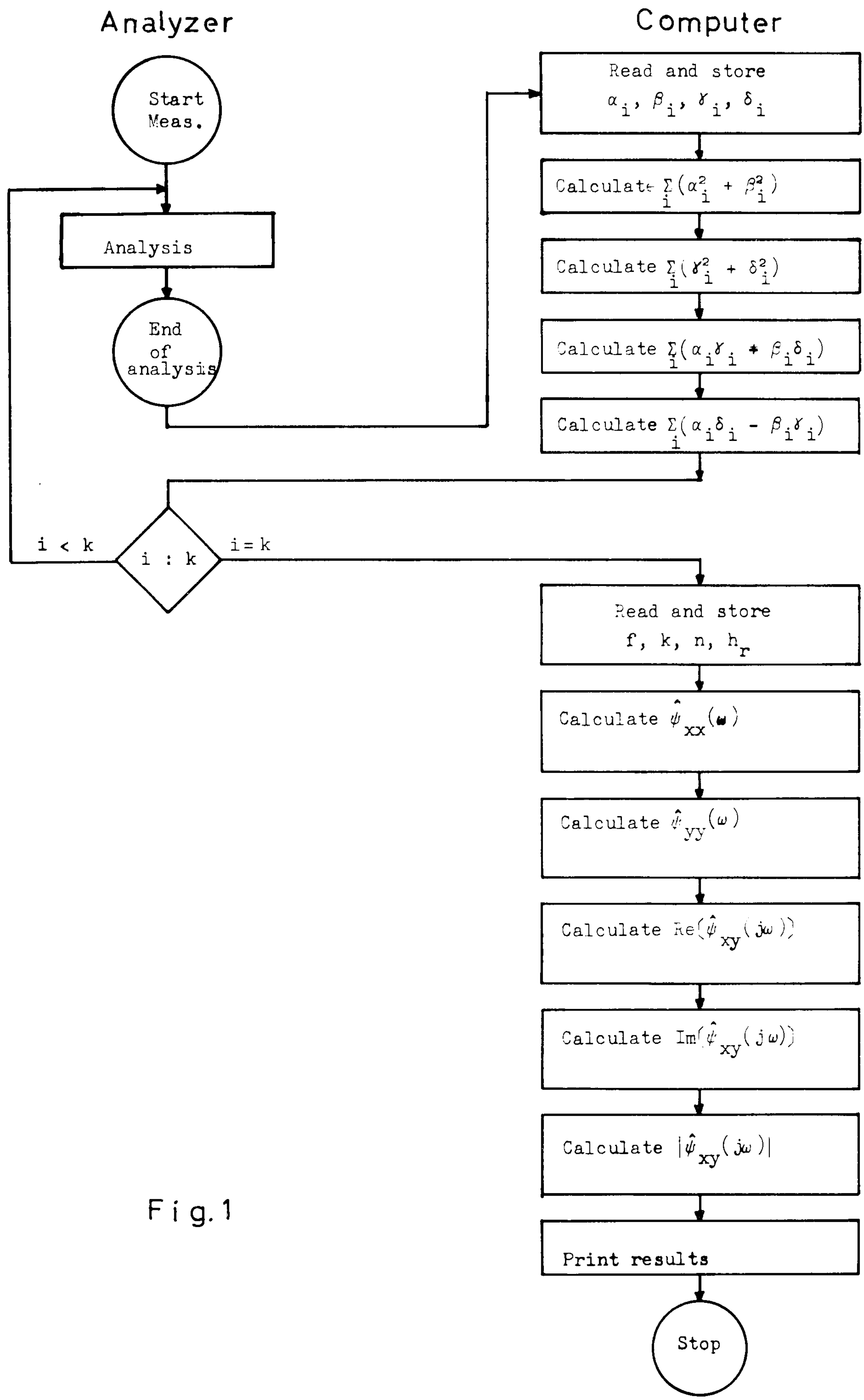
and by rearranging (65), (67) and (68); we have:

$$\mathrm{Re}[\hat{G}(j\omega)] = \mathrm{Re}\left[\frac{\hat{\psi}_{xy}(j\omega)}{\hat{\psi}_{xx}(\omega)}\right] = \frac{\sum_{i=1}^{k}(\alpha_i\gamma_i + \beta_i\delta_i)}{\sum_{i=1}^{k}(\alpha_i^2 + \beta_i^2)} \tag{72}$$

$$\mathrm{Im}[\hat{G}(j\omega)] = \mathrm{Im}\left[\frac{\hat{\psi}_{xy}(j\omega)}{\hat{\psi}_{xx}(\omega)}\right] = \frac{\sum_{i=1}^{k}(\alpha_i\delta_i - \beta_i\gamma_i)}{\sum_{i=1}^{k}(\alpha_i^2 + \beta_i^2)} \tag{73}$$

In fig. 1 is shown a brief sequential diagram of operation, in which the operations performed by the computer, and those performed by that part of the S.D.A. (indicated by the name "analyzer") which processes the signals before the computer does, are listed separately for clarity.

In Appendix A2, two computer programmes for power spectral analysis of random signals are given in full, for cases where an Olivetti P102 is employed as the final computer.



Analyzer
Computer
Start Meas.
Analysis
End of analysis
Read and store $\alpha_i, \beta_i, \gamma_i, \delta_i$
Calculate $\sum_i(\alpha_i^2 + \beta_i^2)$
Calculate $\sum_i(\gamma_i^2 + \delta_i^2)$
Calculate $\sum_i(\alpha_i\gamma_i + \beta_i\delta_i)$
Calculate $\sum_i(\alpha_i\delta_i - \beta_i\gamma_i)$
i < k
i : k
i = k
Read and store f, k, n, h_r
Calculate $\hat{\psi}_{xx}(\omega)$
Calculate $\hat{\psi}_{yy}(\omega)$
Calculate $Re(\hat{\psi}_{xy}(j\omega))$
Calculate $Im(\hat{\psi}_{xy}(j\omega))$
Calculate $|\hat{\psi}_{xy}(j\omega)|$
Print results
Stop

Fig.1

2.) SPECTRAL ANALYSIS OF APERIODIC SIGNALS

Let $x(t)$ and $y(t)$ be two aperiodic signals; their Fourier transforms are defined [16] as:

$$X(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \qquad (74)$$

$$Y(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt \qquad (75)$$

while the power and cross-power spectral densities can be obtained from the following relations:

$$\psi_{xx}(\omega) = 2\pi \left| X(j\omega) \right|^2 \qquad (76)$$

$$\psi_{yy}(\omega) = 2\pi \left| Y(j\omega) \right|^2 \qquad (77)$$

$$\psi_{xy}(j\omega) = 2\pi\, \overline{X(j\omega)} \,.\, Y(j\omega) \qquad (78)$$

The Fourier transforms (74) and (75) are complex continuous spectra and can be divided into amplitude density spectra ($|X(j\omega)|$, $|Y(j\omega)|$) and phase density spectra; it is clear that the amplitude density spectrum of a transient signal does not express the actual amplitudes of the sinusoids composing the signal under examination (as they are infinitesimal), but gives relative magnitudes only [16] [18].

Therefore, the energy spectra (76), (77) and (78) show relative values; in fact the total energy (expressed in volt squared-second) over an infinite range of frequencies is finite [16], so that the energy of each periodic component is an infinitesimal quantity.

2.1.) Fourier transform estimation

For the spectral analysis of aperiodic signals, one can employ formulae of the type:

$$\hat{X}(j\omega) \quad = \quad \frac{1}{2\pi}\int_{t_o}^{t_o+T} x(t)\, e^{-j\omega t}\, dt \tag{79}$$

where the error due to the finite analysis time becomes irrelevant if the time T is chosen in such a way as to cover the whole period during which the signal under examination has an amplitude too great to be ignored; such a time T is independent of the analysis frequencies, whereas the instant t_o at which the analysis starts should be chosen appartunely during the transient in examination.

The real and imaginary parts of the Fourier transform of $x(t)$ and $y(t)$ can then be estimated from the relations:

$$Re(\hat{X}(j\omega)) \quad = \quad \frac{1}{2\pi}\int_{o}^{T} x(t)\cos\omega\, t\, dt \tag{80}$$

$$Im(\hat{X}(j\omega)) \quad = \quad -\frac{1}{2\pi}\int_{o}^{T} x(t)\ \mathrm{sen}\ \omega\, t\, dt \tag{81}$$

and from similar expressions for the signal $y(t)$, having assumed the instant t_o as the origin of the axis of the times.

In expressions (80) and (81) measurement repetitions are not indicated. In effect, even in the case of deterministic signals, such as the aperiodic signals, the averaging out of several measurements is useful, insofar as it serves to reduce the influence of spurious noise which sumperimposes itself upon the signal to be analyzed.

2.2.) Fourier transform estimation with the statistical dynamics analyzer S.D.A.

The evaluation of the Fourier transforms of the aperiodic signals $x(t)$ and $y(t)$ is performed, in the S.D.A. analyzer, on the basis of relations:

$$\mathrm{Re}(\hat{X}(j\omega)) = \frac{1}{k}\frac{1}{2\pi}\sum_{i=1}^{k}\left[\int_{o}^{T} x(t)\cos\omega\, t\, dt\right]_i \tag{82}$$

$$\mathrm{Im}(\hat{X}(j\omega)) = -\frac{1}{k}\frac{1}{2\pi}\sum_{i=1}^{k}\left[\int_{o}^{T} x(t)\,\mathrm{sen}\,\omega\, t\, dt\right]_i \tag{83}$$

$$\mathrm{Re}(\hat{Y}(j\omega)) = \frac{1}{k}\frac{1}{2\pi}\sum_{i=1}^{k}\left[\int_{o}^{T} y(t)\cos\omega\, t\, dt\right]_i \tag{84}$$

$$\mathrm{Im}(\hat{Y}(j\omega)) = -\frac{1}{k}\frac{1}{2\pi}\sum_{i=1}^{k}\left[\int_{o}^{T} y(t)\,\mathrm{sen}\,\omega\, t\, dt\right]_i \tag{85}$$

The data which the S.D.A. pre-processing system supplies to the general purpose digital processor are α_i , β_i , γ_i and δ_i which (see formulae (57)...(60)) can be expressed as:

$$\alpha_i = -10^{-3}\,\frac{\omega\, K_{\tau}\, \tau_{ke}}{4\,h}\, 2\pi\, \mathrm{Im}(\hat{X}(j\omega))_i \tag{86}$$

$$\beta_i = 10^{-3} \frac{\omega K_\tau \tau_{ke}}{4h} 2\pi \, Re(\hat{X}(j\omega))_i \qquad (87)$$

$$\gamma_i = - 10^{-3} \frac{\omega K_\tau \tau_{ke}}{4h} 2\pi \, Im(\hat{Y}(j\omega))_i \qquad (88)$$

$$\delta_i = 10^{-3} \frac{\omega K_\tau \tau_{ke}}{4h} 2\pi \, Re(\hat{Y}(j\omega))_i \qquad (89)$$

where the subscript i indicates the ith of the k repetitions.

From the preceding formulae one obtains:

$$Re(\hat{X}(j\omega)) = \frac{h}{\pi^2} \frac{10^3}{f \tau_{ke} K_\tau} \frac{1}{k} \sum_{i=1}^{k} \beta_i = \frac{1}{h_a k} \sum_{i=1}^{k} \beta_i \qquad (90)$$

$$Im(\hat{X}(j\omega)) = - \frac{h}{\pi^2} \frac{10^3}{f \tau_{ke} K_\tau} \frac{1}{k} \sum_{i=1}^{k} \alpha_i = - \frac{1}{h_a k} \sum_{i=1}^{k} \alpha_i \qquad (91)$$

$$Re(\hat{Y}(j\omega)) = \frac{h}{\pi^2} \frac{10^3}{f \tau_{ke} K_\tau} \frac{1}{k} \sum_{i=1}^{k} \delta_i = \frac{1}{h_a k} \sum_{i=1}^{k} \delta_i \qquad (92)$$

$$Im(\hat{Y}(j\omega)) = - \frac{h}{\pi^2} \frac{10^3}{f \tau_{ke} K_\tau} \frac{1}{k} \sum_{i=1}^{k} \gamma_i = - \frac{1}{h_a k} \sum_{i=1}^{k} \gamma_i \qquad (93)$$

where h_a, called the normalization coefficient for aperiodic signals, represents the expression:

$$\frac{\pi^2 f\ \tau_{ke}\ K_T}{h\ 10^3} \qquad (94)$$

whose value is supplied directly to the final computer from the matrix of the normalization coefficients; it depends upon the frequency decade under examination, but not upon the frequency itself.

In the case where $x(t)$ and $y(t)$ are respectively the input and the output of a linear, time - invariant system, their Fourier transforms also allow the determination of the system transfer function defined by:

$$G(j\omega) = \frac{Y(j\omega)}{X(j\omega)} \qquad (95)$$

Substitution of expressions (90) to (93) in (95) allows us to write:

$$Re(\hat{G}(j\omega)) = Re\left[\frac{\hat{Y}(j\omega)}{\hat{X}(j\omega)}\right] = \frac{\sum_{i=1}^{k} \delta_i}{\sum_{i=1}^{k} \beta_i} \qquad (96)$$

$$Im(\hat{\hat{G}}(j\omega)) = Im\left[\frac{\hat{Y}(j\omega)}{\hat{X}(j\omega)}\right] = \frac{\sum_{i=1}^{k} \gamma_i}{\sum_{i=1}^{k} \alpha_i} \qquad (97)$$

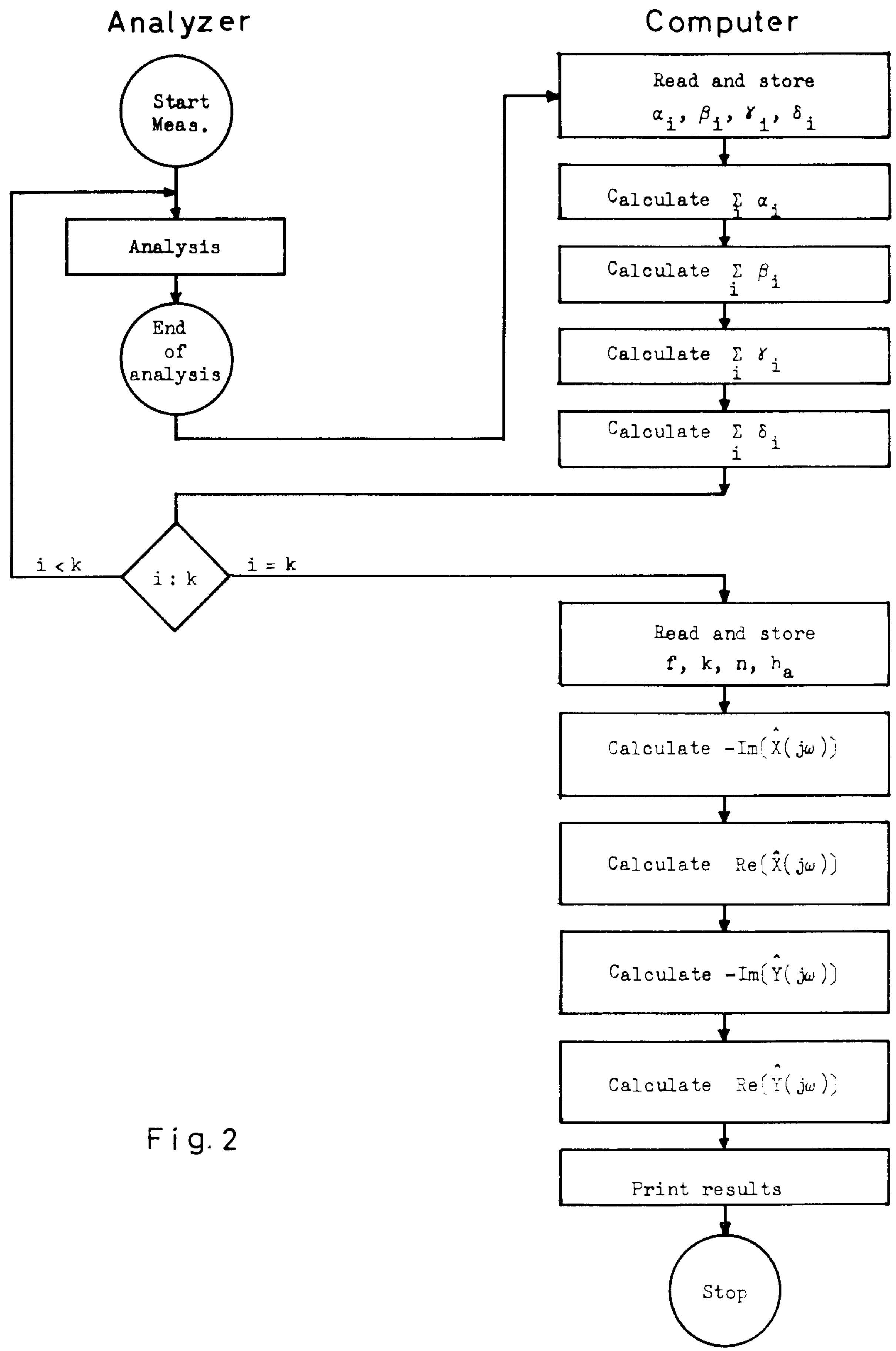

Analyzer
Computer
Start Meas.
Analysis
End of analysis
Read and store $\alpha_i, \beta_i, \gamma_i, \delta_i$
Calculate $\sum_i \alpha_i$
Calculate $\sum_i \beta_i$
Calculate $\sum_i \gamma_i$
Calculate $\sum_i \delta_i$
i < k
i : k
i = k
Read and store f, k, n, h_a
Calculate $-\mathrm{Im}(\hat{X}(j\omega))$
Calculate $\mathrm{Re}(\hat{X}(j\omega))$
Calculate $-\mathrm{Im}(\hat{Y}(j\omega))$
Calculate $\mathrm{Re}(\hat{Y}(j\omega))$
Print results
Stop

Fig. 2

In fig. 2 is given a brief sequential operating diagram for the Fourier analysis of aperiodic signals.

In Appendix A.3 the complete program for the Olivetti P102 computer is shown.

2.3.) Energy and cross-energy spectral density estimation

The energy spectral densities $\psi_{xx}(\omega)$ and $\psi_{yy}(\omega)$ of the aperiodic signals $x(t)$ and $y(t)$, and the cross-energy spectral density $\psi_{xy}(j\omega)$ can be calculated from (76) (77) and (78), by using the estimated values of $X(j\omega)$ and $Y(j\omega)$:

$$\hat{\psi}_{xx}(\omega) = 2\pi \left|\hat{X}(j\omega)\right|^2 = \frac{1}{2\pi}\left|\int_{t_o}^{t_o+T} x(t)\, e^{-j\omega t}\, dt\right|^2 \tag{98}$$

$$\hat{\psi}_{yy}(\omega) = 2\pi \left|\hat{Y}(j\omega)\right|^2 = \frac{1}{2\pi}\left|\int_{t_o}^{t_o+T} y(t)\, e^{-j\omega t}\, dt\right|^2 \tag{99}$$

$$\hat{\psi}_{xy}(j\omega) = 2\pi\, \overline{\hat{X}(j\omega)} \cdot \hat{Y}(j\omega) = \frac{1}{2\pi}\int_{t_o}^{t_o+T} x(t)\, e^{j\omega t}\, dt \cdot \int_{t_o}^{t_o+T} y(t)\, e^{-j\omega t}\, dt \tag{100}$$

It is demonstrated [7] that:

$$\lim_{T\to\infty} \hat{\psi}_{xx}(\omega) = \psi_{xx}(\omega)$$

$$\left.\begin{aligned}\lim_{T\to\infty} \hat{\psi}_{yy}(\omega) &= \psi_{yy}(\omega) \\ \lim_{T\to\infty} \hat{\psi}_{xy}(j\omega) &= \psi_{xy}(j\omega)\end{aligned}\right] \qquad (101)$$

where, as with random signals, the physical significance of an analysis time T tending to infinity is to restrict progressively the width of the spectral line around the frequency whose spectral density is to be calculated.

For the repetitions, what has already been said for the Fourier transform estimation is valid.

2.4.) Energy and cross-energy spectral density estimation with the statistical dynamics analyzer S.D.A.

The energy and cross-energy spectral densities of the aperiodic signals $x(t)$ and $y(t)$ are calculated, in the S.D.A., from the expressions:

$$\hat{\psi}_{xx}(\omega) = \frac{1}{2\pi}\,\frac{1}{k}\sum_{i=1}^{k}\left|\left[\int_{o}^{T} x(t)\, e^{-j\omega t}\, dt\right]_i\right|^2 \qquad (102)$$

$$\hat{\psi}_{yy}(\omega) = \frac{1}{2\pi}\,\frac{1}{k}\sum_{i=1}^{k}\left|\left[\int_{o}^{T} y(t)\, e^{-j\omega t}\, dt\right]_i\right|^2 \qquad (103)$$

$$\hat{\psi}_{xy}(j\omega) = \frac{1}{2\pi}\,\frac{1}{k}\sum_{i=1}^{k}\left[\int_{o}^{T} x(t)\, e^{j\omega t}\, dt\right]\left[\int_{o}^{T} y(t)\, e^{-j\omega t}\, dt\right]_i \qquad (104)$$

where, as already in (82).....(85), we have assumed the instant of the start of analysis as the origin of the axis of the times, and where the subscript i indicates the ith repetition.

Rearranging equations (102), (103) and (104) on the basis of expressions (90) to (93), we obtain the relations by which the functions we are looking for, are related to the quantities α_i, β_i, γ_i and δ_i :

$$\hat{\psi}_{xx}(\omega) = 2\pi \sum_{i=1}^{k} \left(\mathrm{Re}^2(\hat{X}(j\omega))_i + \mathrm{Im}^2(\hat{X}(j\omega))_i \right) = \frac{2\pi}{h_a^2} \frac{1}{k} \sum_{i=1}^{k} (\alpha_i^2 + \beta_i^2) \qquad (105)$$

$$\hat{\psi}_{yy}(\omega) = 2\pi \sum_{i=1}^{k} \left(\mathrm{Re}^2(\hat{Y}(j\omega))_i + \mathrm{Im}^2(\hat{Y}(j\omega))_i \right) = \frac{2\pi}{h_a^2} \frac{1}{k} \sum_{i=1}^{k} (\gamma_i^2 + \delta_i^2) \qquad (106)$$

$$\mathrm{Re}(\hat{\psi}_{xy}(j\omega)) = 2\pi \sum_{i=1}^{k} \mathrm{Re}(\overline{\hat{X}(j\omega)}.\hat{Y}(j\omega))_i = \frac{2\pi}{h_a^2} \frac{1}{k} \sum_{i=1}^{k} (\alpha_i\gamma_i + \beta_i\delta_i) \qquad (107)$$

$$\mathrm{Im}(\hat{\psi}_{xy}(j\omega)) = 2\pi \sum_{i=1}^{k} \mathrm{Im}(\overline{\hat{X}(j\omega)}.\hat{Y}(j\omega))_i = \frac{2\pi}{h_a^2} \frac{1}{k} \sum_{i=1}^{k} (\alpha_i\delta_i - \beta_i\gamma_i) \qquad (108)$$

In practice it is often more interesting to evaluate the transfer function $G(j\omega)$ between the signals $x(t)$ and $y(t)$, rather than the cross-power spectral density. For determining the real and imaginary parts of the $G(j\omega)$, relations (71) (72) and (73) supplied for the case of the random signals are valid.

The sequence of operations in the S.D.A. system for the energy spectral analysis of transient signals is in practise the same as that for random signals, which can be seen in fig.1.

Appendix A4 presents in detail the computation programmes for the Olivetti P102.

3.) SPECTRAL ANALYSIS OF PERIODIC SIGNALS

When one is dealing with periodic signals, the Fourier transforms of $x(t)$ and $y(t)$ are given by [16]).

$$X(jm\omega_o) = \frac{1}{m\,T_o} \int_{t_o - m\frac{T_o}{2}}^{t_o + m\frac{T_o}{2}} x(t)\, e^{-jm\omega_o t}\, dt \qquad (109)$$

$$Y(jm\omega_o) = \frac{1}{m\,T_o} \int_{t_o - m\frac{T_o}{2}}^{t_o + m\frac{T_o}{2}} y(t)\, e^{-jm\omega_o t}\, dt \qquad (110)$$

where T_o represents the period of the signal under examination, and m the order of the analyzed harmonic, while t_o is an arbitrary value of the time chosen to be the central instant of the measurement.

The functions $X(jm\omega_o)$ and $Y(jm\omega_o)$ are complex line spectra. Their absolute values (i.e. the amplitude spectra) are expressed in volts.

The power and cross-power spectral densities can be estimated from [16]):

$$\psi_{xx}(m\omega_o) = \left|X(jm\omega_o)\right|^2 \qquad (111)$$

$$\psi_{yy}(m\omega_o) = \left|Y(jm\omega_o)\right|^2 \qquad (112)$$

$$\psi_{xy}(jm\omega_o) = \overline{X}(jm\omega_o) \cdot Y(jm\omega_o) \qquad (113)$$

where, as already in (109) and (110), ω_o is the fundamental angular frequency of the periodic function to be analyzed.

The cross-power spectral density is a complex function; its absolute value and the power spectral densities are expressed in squared volts.

Expressions (109).....(113) give, also for finite measurement times, determinations which correspond to the theoretical values.

In practise it is better, as before in the case of aperiodic signals, to repeat the measurement several times and to average out the results in order to reduce the influence of spurious noise which may have been superimposed upon the signal to be analyzed.

3.1.) Fourier transform estimation with the statistical dynamics analyzer S.D.A.

As we have already said, the Fourier transforms of periodic signals have line spectra; therefore the analysis is only carried out on the harmonics of the fundamental frequency ω_o on the basis of the expressions:

$$\mathrm{Re}(\hat{X}(jm\omega_o)) = \frac{1}{k\ n\ m\ T_o} \sum_{i=1}^{k} \int_{t_i}^{t_i+n\ m\ T_o} x(t)\ \cos m\ \omega_o\ t\ dt \qquad (114)$$

$$\mathrm{Im}(\hat{X}(jm\omega_o)) = -\ \frac{1}{k\ n\ m\ T_o} \sum_{i=1}^{k} \int_{t_i}^{t_i+n\ m\ T_o} x(t)\ \sin m\ \omega_o\ t\ dt \qquad (115)$$

and of the similar expressions for the signal $y(t)$.

In equations (114) and (115) T_o and m represent respectively the period of the signal $x(t)$ and the order of the harmonic under examination, while n represents the number of periods of integration and k the number of repetitions.

Taking equations (57) ... (60) into account, and indicating by ω' the angular frequency $m\ \omega_o$ and by T' the period $m\ T_o$ of the harmonic of order m, the values α_i, β_i, γ_i and δ_i supplied by the S.D.A. analyzer can be expressed in the following way:

$$\alpha_i = -\ 10^{-3}\ \frac{\omega'\ \tau_{ke}\ K_\tau}{4\ h}\ n\ T'\ \mathrm{Im}(\hat{X}(j\omega')) \qquad (116)$$

$$\beta_i = 10^{-3}\ \frac{\omega'\ \tau_{ke}\ K_\tau}{4\ h}\ n\ T'\ \mathrm{Re}(\hat{X}(j\omega')) \qquad (117)$$

$$\gamma_i = -\ 10^{-3}\ \frac{\omega'\ \tau_{ke}\ K_\tau}{4\ h}\ n\ T'\ \mathrm{Im}(\hat{Y}(j\omega')) \qquad (118)$$

$$\delta_i = 10^{-3}\ \frac{\omega'\ \tau_{ke}\ K_\tau}{4\ h}\ n\ T'\ \mathrm{Re}(\hat{Y}(j\omega')) \qquad (119)$$

because of which, in the case of k repetitions, the real and imaginary parts of the Fourier transforms of the periodic signals $x(t)$ and $y(t)$ are given by:

$$\mathrm{Re}(\hat{X}(j\omega')) = \frac{1}{k}\,\frac{2\cdot 10^3\cdot h}{\pi\, n\, \tau_{ke}\, K_\tau}\sum_{i=1}^{k}\beta_i = \frac{f'}{k\, n\, h_p}\sum_{i=1}^{k}\beta_i \tag{120}$$

$$\mathrm{Im}(\hat{X}(j\omega')) = -\frac{1}{k}\,\frac{2\cdot 10^3\cdot h}{\pi\, n\, \tau_{ke}\, K_\tau}\sum_{i=1}^{k}\alpha_i = -\frac{f'}{k\, n\, h_p}\sum_{i=1}^{k}\alpha_i \tag{121}$$

$$\mathrm{Re}(\hat{Y}(j\omega')) = \frac{1}{k}\,\frac{2\cdot 10^3\cdot h}{\pi\, n\, \tau_{ke}\, K_\tau}\sum_{i=1}^{k}\delta_i = \frac{f'}{k\, n\, h_p}\sum_{i=1}^{k}\delta_i \tag{122}$$

$$\mathrm{Im}(\hat{Y}(j\omega')) = -\frac{1}{k}\,\frac{2\cdot 10^3\cdot h}{\pi\, n\, \tau_{ke}\, K_\tau}\sum_{i=1}^{k}\gamma_i = -\frac{f'}{k\, n\, h_p}\sum_{i=1}^{k}\gamma_i \tag{123}$$

where h_p represents the normalization coefficient for the spectral analysis of periodic signals, and is given by:

$$h_p = \frac{\pi\, \tau_{ke}\, K_\tau\, f'}{2\; 10^3\; h} \tag{124}$$

where the product $\tau_{ke}\cdot f'$ is independent of the frequency.

As far as the determination is concerned, through the Fourier transforms, of the transfer function of the system in which $x(t)$ and $y(t)$ are respectively the input and the output signal, the relations (96) and (97) are valid.

In Appendix A5 the programmes (for Olivetti P 102) relative to the Fourier analysis of periodic signals are shown.

3.2.) <u>Power and cross-power spectral density estimation with the statistical dynamics analyzer S.D.A.</u>

The power spectral densities $\psi_{xx}(\omega)$ and $\psi_{yy}(\omega)$ of the periodic signals $x(t)$ and $y(t)$, and their cross-power spectral density $\psi_{xy}(j\omega)$ are calculated, in the S.D.A. system, by virtue of the relations:

$$\hat{\psi}_{xx}(m\omega_o) = \frac{1}{k} \frac{1}{n^2 m^2 T_o^2} \sum_{i=1}^{k} \left| \int_{t_i}^{t_i+n\,m\,T_o} x(t)\, e^{-jm\omega_o t}\, dt \right|^2 \qquad (125)$$

$$\hat{\psi}_{yy}(m\omega_o) = \frac{1}{k} \frac{1}{n^2 m^2 T_o^2} \sum_{i=1}^{k} \left| \int_{t_i}^{t_i+n\,m\,T_o} y(t)\, e^{-jm\omega_o t}\, dt \right|^2 \qquad (126)$$

$$\hat{\psi}_{xy}(j\omega) = \frac{1}{k} \frac{1}{n^2 m^2 T_o^2} \sum_{i=1}^{k} \int_{t_i}^{t_i+n\,m\,T_o} x(t)\, e^{+jm\omega_o t}\, dt \int_{t_i}^{t_i+n\,m\,T_o} y(t)\, e^{-jm\omega_o t}\, dt \qquad (127)$$

where k, n, m, T_o and t_i have the same meaning as in equations (114) and (115).

By taking equations (120) to (123) into account, we have:

$$\hat{\psi}_{xx}(\omega') = \frac{1}{k} \sum_{i=1}^{k} \hat{\psi}_{xx,i}(\omega') = \frac{1}{k} \frac{f'^2}{n^2 h_p^2} \sum_{i=1}^{k} (\alpha_i^2 + \beta_i^2) \tag{128}$$

$$\hat{\psi}_{yy}(\omega') = \frac{1}{k} \sum_{i=1}^{k} \hat{\psi}_{yy,i}(\omega') = \frac{1}{k} \frac{f'^2}{n^2 h_p^2} \sum_{i=1}^{k} (\gamma_i^2 + \delta_i^2) \tag{129}$$

$$\mathrm{Re}(\hat{\psi}_{xy}(j\omega')) = \frac{1}{k} \sum_{i=1}^{k} \mathrm{Re}(\hat{\psi}_{xy,i}(j\omega')) = \frac{1}{k} \frac{f'^2}{n^2 h_p^2} \sum_{i=1}^{k} (\alpha_i \gamma_i + \beta_i \delta_i) \tag{130}$$

$$\mathrm{Im}(\hat{\psi}_{xy}(j\omega')) = \frac{1}{k} \sum_{i=1}^{k} \mathrm{Im}(\hat{\psi}_{xy,i}(j\omega')) = \frac{1}{k} \frac{f'^2}{n^2 h_p^2} \sum_{i=1}^{k} (\alpha_i \delta_i - \beta_i \gamma_i) \tag{131}$$

For the determination of transfer functions from power spectra, relations (71) (72) and (73) , given for random signals, are valid.

Appendix A6 gives the programmes for the determinations set out above.

APPENDIX

A.1.

Computing programmes for the computer composing the final element of the S.D.A. system all have a common structure, irrespective of the type of analysis to which they refer.

A typical programme can in fact be considered to be composed of four parts.

The first part (from AZ to AV , if an Olivetti P 102 is being used) corresponds to the introduction of data α_i , β_i , γ_i and δ_i , into the machine, and, when the repetitions are finished, of the values f, n, k and h_n, where the generic normalization coefficient is indicated by h_n .

The second part of the programme (AV.....Z) processes the data in order to obtain the functions for which one is looking.

The third part, (AW.....Z) is designed to carry out a normalization of the quantities calculated in the second part, and to print the results.

The final part of the programme performs the sequence of operations linked to an overload of the S.D.A. system. Any type of overload (for exemple in the input amplifier, or in the α , β , γ , δ , counters) in fact cancels out the number of the repetitions , which number, in normal operating conditions, is never zero. The cancellation of k triggers off this part of the programme (/V), which interrupts the calculation sequence relative to the point under examination, cancels any results relative to it which have already been obtained, and takes the apparatus forward to the following point.

Finally, a practical consideration is made. For any kind of processing (e-g. spectral analysis of random signals), and by using the data α_i , β_i , γ_i , δ_i , f , n , k and h_n supplied by the analyzer to the final computer, it would be possible to calculate all the interesting functions in one programme, if only the capacity of the computer did not place limits upon the length of programme that can be memorized.

We therefore list here, for each function of the S.D.A. , different programmes, each one of which calculates some of the functions of interest. The programmes are written for the Olivetti P 102.

A.2. - Programmes for the power spectral analysis of random signals

Table A.2.1. describes the programme which calculates the power spectral densities $\psi_{xx}(\omega)$ and $\psi_{yy}(\omega)$ of the random signals x(t) and y(t) being dealt with.

Results will be printed in the following order:

f	in Hz	(analysis frequency)
k		(repetition number)
n		(number of integration cicles)
$\hat{\psi}_{xx}(\omega)$	in v^2/Hz	
$\hat{\psi}_{yy}(\omega)$	in v^2/Hz	

Table A.2.2. shows the programme for calculating the power spectral density of the signal x(t) , and the real and imaginary parts and the modulus of the cross-power spectral density $\psi_{xy}(j\omega)$.

Results will be printed in the following order:

f	in Hz
k	
n	
$\hat{\psi}_{xx}(\omega)$	in v^2/Hz
$Re(\hat{\psi}_{xy}(j\omega))$	in v^2/Hz
$Im(\hat{\psi}_{xy}(j\omega))$	in v^2/Hz
$\lvert\hat{\psi}_{xy}(j\omega)\rvert$	in v^2 Hz

Table A.2.3. shows the programme concerned with the calculation of the transfer function $G(j\omega)$ of the system of which $x(t)$ and $y(t)$ represent respectively the input and the output signal.

Results will be printed in the following order:

f in Hz

k

n

$\mathrm{Re}(\hat{G}(j\omega))$

$\mathrm{Im}(\hat{G}(j\omega))$

$|\hat{G}(j\omega)|$

$$tg\varphi = \frac{\mathrm{Im}(\hat{G}(j\omega))}{\mathrm{Re}(\hat{G}(j\omega))}$$

This last programme is the same also for periodic and aperiodic signals, since for the calculation of the $G(j\omega)$ one uses power spectra ratios in which there are no normalizations.

In effect, the values of the power and cross-power spectral densities are half of the real values, because they are supplied as bilateral spectra (i.e. also for negative frequencies).
The transfer function is calculated as a ratio of spectra and is therefore in effective values.

TABLE A.2.1.

PROGRAMME INSTRUCTIONS					
Reg. 1		Reg. 2		Reg. F	
1	A Z	25	B +	49	E / :
2	A S	26	B ↕	50	A ⟠
3	D / ↑	27	Z	51	A Y
4	A S	28	A W	52	B / *
5	D ↑	29	/ ⟠	53	B *
6	A S	30	D ⟠	54	C / *
7	E / ↑	31	↓	55	C *
8	A S	32	/ V	56	Z
9	A V	33	Y	57	
10	E ↑	34	A / V	58	
11	D / ↓	35	A ⟠	59	
12	A x	36	E /⟠	60	
13	B / +	37	E / x	61	
14	B / ↕	38	E / ↕	62	
15	D ↓	39	D / ↓	63	
16	A x	40	D / x	64	
17	B / +	41	D :	65	
18	B / ↕	42	D / ↕	66	
19	E / ↓	43	B / ↓	67	
20	A x	44	D / :	68	
21	B +	45	E / :	69	
22	B ↕	46	A ⟠	70	
23	E ↓	47	B ↓	71	
24	A x	48	D / :	72	

TABLE A.2.2.

PROGRAMME INSTRUCTIONS					
Reg. 1		Reg. 2		Reg. F	
1	A Z	25	C / +	49	B / :
2	A S	26	C / ↕	50	D / :
3	D / ↑	27	D ↓	51	A ⇕
4	A S	28	E / x	52	C / ↓
5	D ↑	29	D / ↕	53	B / :
6	A S	30	E x	54	D / :
7	E / ↑	31	D / -	55	A ⇕
8	A S	32	C +	56	A x
9	A V	33	C ↕	57	C ↕
10	E ↑	34	Z	58	B / :
11	D / ↓	35	A W	59	D / :
12	A x	36	/ ⇕	60	A ⇕
13	B / +	37	D ⇕	61	A x
14	B / ↕	38	↓	62	C +
15	D ↓	39	/ V	63	A √
16	A x	40	Y	64	A ⇕
17	B / +	41	A / V	65	A Y
18	B / ↕	42	A ⇕	66	B / *
19	D / ↓	43	E / ⇕	67	B *
20	E / x	44	E / x	68	C / *
21	C / +	45	D / ↕	69	C *
22	C / ↕	46	A x	70	Z
23	D ↓	47	D :	71	
24	E x	48	B / ↕	72	

TABLE A.2.3.

PROGRAMME INSTRUCTIONS					
Reg. 1		Reg. 2		Reg. F	
1	A Z	25	C / +	49	C ▾
2	A S	26	C / ↕	50	B / :
3	D / ↑	27	D ↓	51	A ◊
4	A S	28	E / x	52	A x
5	D ▴	29	D / ↕	53	E / ÷
6	A S	30	E x	54	A √
7	E / ▴	31	D / -	55	A ◊
8	A S	32	C +	56	C ▾
9	A V	33	C ↕	57	C / :
10	E ↑	34	Z	58	A ÷
11	D / ↓	35	A W	59	A Y
12	A x	36	/ ◊	60	B / *
13	B / +	37	D ◊	61	B *
14	B / ↕	38	↓	62	C / *
15	D ↓	39	/ V	63	C *
16	A x	40	Y	64	Z
17	B / +	41	A / V	65	
18	B / ↕	42	A ◊	66	
19	D / ↓	43	E / ◊	67	
20	E / x	44	C / ↓	68	
21	C / +	45	B / :	69	
22	C / ↕	46	A ◊	70	
23	D ↓	47	A x	71	
24	E x	48	E / ↕	72	

A.3. - Programme for computation of the Fourier transforms of aperiodic signals

Table A.3.1. shows the programme which calculates the real and imaginary parts of the Fourier transforms $X(j\omega)$ and $Y(j\omega)$ of the two aperiodic signals $x(t)$ and $y(t)$.

Results will be printed in the following order:

f	in Hz
k	
T	in sec. (analysis time)
$-\mathrm{Im}(\hat{X}(j\omega))$	
$\mathrm{Re}(\hat{X}(j\omega))$	
$-\mathrm{Im}(\hat{Y}(j\omega))$	
$\mathrm{Re}(\hat{Y}(j\omega))$	

TABLE A.3.1.

PROGRAMME INSTRUCTIONS					
Reg. 1		Reg. 2		Reg. F	
1	A Z	25	D ⇕	49	C ↓
2	A S	26	↓	50	D :
3	D / ↑	27	/ V	51	A ⇕
4	A S	28	Y	52	A Y
5	D ↑	29	A / V	53	B / *
6	A S	30	A ⇕	54	B *
7	E / ↑	31	D / x	55	C / *
8	A S	32	D ↕	56	C *
9	A V	33	A / ↑	57	Z
10	↓	34	R ↓	58	
11	C +	35	R S	59	
12	C ↕	36	D / S	60	
13	D / ↓	37	↓	61	
14	B / +	38	E / x	62	
15	B / ↕	39	A ⇕	63	
16	D ↓	40	B / ↓	64	
17	B +	41	D :	65	
18	B ↕	42	A ⇕	66	
19	E / ↓	43	B ↓	67	
20	C / +	44	D :	68	
21	C / ↕	45	A ⇕	69	
22	Z	46	C / ↓	70	
23	A W	47	D :	71	
24	/ ⇕	48	A ⇕	72	

A.4. - Programmes for the energy spectral analysis of aperiodic signals

In Table A.4.1. we list the programme for calculating the energy spectral densities $\psi_{xx}(\omega)$ and $\psi_{yy}(\omega)$ of the transient signals $x(t)$ and $y(t)$.

The results obtained by the programme are:

f	in Hz
k	
T	in sec (analysis time)
$\hat{\psi}_{xx}(\omega)$	in volt squared-second per hertz
$\hat{\psi}_{yy}(\omega)$	" " " " " "

Table A.4.2. shows a programme which evaluates the energy spectral density $\psi_{xx}(\omega)$ and the real part, the imaginary part and the modulus of the cross-energy spectral density $\psi_{xy}(j\omega)$.

The following quantities will be printed out as computing results:

f	in Hz
k	
T	in sec.
$\hat{\psi}_{xx}(\omega)$	in volt squared-second per hertz
$\mathrm{Re}(\hat{\psi}_{xy}(j\omega))$	
$\mathrm{Im}(\hat{\psi}_{xy}(j\omega))$	
$\lvert\hat{\psi}_{xy}(j\omega)\rvert$	

TABLE A.4.1.

PROGRAMME INSTRUCTIONS					
Reg. 1		Reg. 2		Reg. F	
1	A Z	25	B +	49	B ↓
2	A S	26	B ↕	50	D / :
3	D / ↑	27	Z	51	A ⇳
4	A S	28	A W	52	A Y
5	D ↑	29	/ ⇳	53	B / *
6	A S	30	D ⇳	54	B *
7	E / ↑	31	↓	55	C / *
8	A S	32	/ V	56	C *
9	A V	33	Y	57	Z
10	E ↑	34	A / V	58	
11	D / ↓	35	A ⇳	59	
12	A x	36	D / x	60	
13	B / +	37	D / x	61	
14	B / ↕	38	D / ↕	62	
15	D ↓	39	A / ↑	63	
16	A x	40	R ↓	64	
17	B / +	41	R S	65	
18	B / ↕	42	D / S	66	
19	E / ↓	43	↓	67	
20	A x	44	E / x	68	
21	B +	45	A ⇳	69	
22	B ↕	46	B / ↓	70	
23	E ↓	47	D / :	71	
24	A x	48	A ⇳	72	

TABLE A.4.2.

PROGRAMME INTRODUCTIONS					
Reg. 1		Reg. 2		Reg. F	
1	A Z	25	C / +	49	D / S
2	A S	26	C / ↕	50	↓
3	D / ↑	27	D ↓	51	E / x
4	A S	28	E / x	52	A θ
5	D ↑	29	D / ↕	53	B / ↓
6	A S	30	E x	54	D / :
7	E / ↑	31	D / -	55	A θ
8	A S	32	C +	56	C / ↓
9	A V	33	C ↕	57	D / :
10	E ↑	34	Z	58	A θ
11	D / ↓	35	A W	59	A x
12	A x	36	/ θ	60	C ↕
13	B / +	37	D θ	61	D / :
14	B / ↕	38	↓	62	A θ
15	D ↓	39	/ V	63	A x
16	A x	40	Y	64	C +
17	B / +	41	A / V	65	A √
18	B / ↕	42	A θ	66	A θ
19	D / ↓	43	D / x	67	A Y
20	E / x	44	D / x	68	B / *
21	C / +	45	D / ↕	69	B *
22	C / ↕	46	A / ↑	70	C / *
23	D ↓	47	R ↓	71	C *
24	E x	48	R S	72	Z

A.5. - Programme for computation of the Fourier transforms of periodic signals

In Table A.5.1. we show the programme which calculates the real and imaginary parts of the Fourier transforms $X(j\omega)$ and $Y(j\omega)$ of two periodic signals $x(t)$ and $y(t)$.

Results will be printed in the following order:

f in Hz

k

n

$-\operatorname{Im}(\hat{X}(j\omega))$

$\operatorname{Re}(\hat{X}(j\omega))$

$-\operatorname{Im}(\hat{Y}(j\omega))$

$\operatorname{Re}(\hat{Y}(j\omega))$

TABLE A.5.1.

PROGRAMME INSTRUCTIONS					
Reg. 1		Reg. 2		Reg. F	
1	A Z	25	D ⟠	49	A Y
2	A S	26	↓	50	B / *
3	D / ↑	27	/ V	51	B *
4	A S	28	Y	52	C / *
5	D ↑	29	A / V	53	C *
6	A S	30	A ⟠	54	Z
7	E / ↑	31	E / ⟠	55	
8	A S	32	E / x	56	
9	A V	33	D / x	57	
10	⇣	34	D ↕	58	
11	C +	35	D :	59	
12	C ↕	36	D / ↕	60	
13	D / ⇣	37	D / ↓	61	
14	B / +	38	B / x	62	
15	B / ↕	39.	A ⟠	63	
16	D ⇣	40	B ↓	64	
17	B +	41	D / x	65	
18	B ↕	42	A ⟠	66	
19	E / ⇣	43	C / ↓	67	
20	C / +	44	D / x	68	
21	C / ↕	45	A ⟠	69	
22	Z	46	C ↓	70	
23	A W	47	D / x	71	
24	/ ⟠	48	A ⟠	72	

A.6. - Programmes for the power spectral analysis of periodic signals

Table A.6.1. shows the programme for calculation of the power spectral densities $\psi_{xx}(\omega)$ and $\psi_{yy}(\omega)$ of the periodic signals x(t) and y(t) under examination.

The following quantities will be printed as computing results, at every analysis frequency:

f in Hz

k

n

$\hat{\psi}_{xx}(\omega)$

$\hat{\psi}_{yy}(\omega)$

Table A.6.2. shows the programme which deals with the calculation of the power spectral density of the signal x(t) and of the real part, the imaginary part and the modulus of the cross-power spectral density of the signals x(t) and y(t).

Results will be printed in the following order:

f in Hz

k

n

$\hat{\psi}_{xx}(\omega)$

$Re(\hat{\psi}_{xy}(j\omega))$

$Im(\hat{\psi}_{xy}(j\omega))$

$|\hat{\psi}_{xy}(j\omega)|$

As far as the calculation (starting from $\psi_{xx}(\omega)$ and $\psi_{xy}(j\omega)$) of the transfer function between the signal x(t) and y(t) is concerned, the programme written for random signals, shown in Table A.2.3. , remains valid.

TABLE A.6.1.

PROGRAMME INSTRUCTIONS					
Reg. 1		Reg. 2		Reg. F	
1	A Z	25	B +	49	E / :
2	A S	26	B ↕	50	A ⇳
3	D / ↑	27	Z	51	A Y
4	A S	28	A W	52	B / *
5	D ↑	29	/ ⇳	53	B *
6	A S	30	D ⇳	54	C / *
7	E / ↑	31	↓	55	C *
8	A S	32	/ V	56	Z
9	A V	33	Y	57	
10	E ↑	34	A / V	58	
11	D / ↓	35	A ⇳	59	
12	A x	36	E / ⇳	60	
13	B / +	37	E / x	61	
14	B / ↕	38	E / x	62	
15	D ↓	39	D / ↕	63	
16	A x	40	D :	64	
17	B / +	41	A x	65	
18	B / ↕	42	E / ↕	66	
19	E / ↓	43	B / ↓	67	
20	A x	44	D / :	68	
21	B +	45	E / :	69	
22	B ↕	46	A ⇳	70	
23	E ↓	47	B ↓	71	
24	A x	48	D / :	72	

TABLE A.6.2.

PROGRAMME INTRODUCTIONS					
Reg. 1		Reg. 2		Reg. F	
1	A Z	25	C / +	49	B / ↕
2	A S	26	C / ↕	50	B / :
3	D / ↑	27	D ↓	51	D / :
4	A S	28	E / x	52	A ⇕
5	D ↑	29	D / ↕	53	C / ↓
6	A S	30	E x	54	B / :
7	E / ↑	31	D / -	55	D / :
8	A S	32	C +	56	A ⇕
9	A V	33	C ↕	57	A x
10	E ↑	34	Z	58	C ↕
11	D / ↓	35	A W	59	B / :
12	A x	36	/ ⇕	60	D / :
13	B / +	37	D ⇕	61	A ⇕
14	B / ↕	38	↓	62	A x
15	D ↓	39	/ V	63	C −
16	A x	40	Y	64	A √
17	B / +	41	A / V	65	A ⇕
18	B / ↕	42	A ⇕	66	A Y
19	D / ↓	43	E / ⇕	67	B / *
20	E / x	44	E / x	68	B *
21	C / +	45	E / x	69	C / *
22	C / ↕	46	D / ↕	70	C *
23	D ↓	47	D :	71	Z
24	E x	48	A x	72	

FIGURE CAPTIONS

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