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ON THE STATISTICAL PROPERTIES OF SOME ESTIMATORS OF LINEAR SYSTEM PARAMETERS IN TIME DOMAIN ANALYSIS

by

A. C. LUCIA

1970

Joint Nuclear Research Center
Ispra Establishment — Italy

Reactor Physics Department
Resarch Reactors

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ABSTRACT

This report deals with the problem of determining some of the fundamental parameters of stationary ergodic random signals (mean value, auto- and cross-correlation functions, covariance functions, probability density, etc...), using continuous estimators and working from time history analog records.

Within this context particular reference is made to the possibilities of the S.D.A. statistical analyzer, designed and built at the J.R.C. — Euratom, Ispra.

KEYWORDS

AMPLIFIERS
LIOUVILLE THEOREM
AUTO-CORRELATION
FUNCTIONS
PROBABILITY
DENSITY
MATHEMATICS
NUMERICALS

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FIGURE CAPTIONS

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ON THE STATISTICAL PROPERTIES OF SOME ESTIMATORS OF LINEAR SYSTEM PARAMETERS IN TIME DOMAIN ANALYSIS *)

INTRODUCTION

In a previous report [1] we discussed the statistical properties of continuous estimators of some functions belonging to the frequency domain (power and cross-power spectral densities and Fourier transforms) in the case of random, stationary ergodic signals.

In this report a similar sort of study has been extended to cover the correlation and probability density functions. A knowledge of these functions, like that of those mentioned in the first paragraph, is of great interest for the identification of systems and processes and for choosing the most appropriate and accurate mathematical models [2] [3].

As a particular example, the cross-correlation function has proved itself to be most helpful in studying vibration transmission pathe in installations and structures, while probability density can indicate the presence of non-linearity in the processes or systems being examined, as well as provide some informations about the nature and the origins of the noise [4] [5] of system variables.

As already in the case of spectral estimators, [1] the estimators analyzed here are also of a general type and constitute the algorithms upon which the operation of the S.D.A. statistical dynamic analyzer, built at C.C.R. - Euratom, Ispra, [6] [7] is based.

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*) Manuscript received on 21 May 1970

1.) MEAN VALUE ESTIMATION

The mean value μ_x of a random variable $x(t)$ is defined as [8] :

$$\mu_x = E(x(t)) \tag{1}$$

but in the cases where $x(t)$ is stationary and ergodic, one can substitute for relation (1) :

$$\mu_x = \underset{T \to \infty}{\mathrm{l.i.m.}} \frac{1}{T} \int_0^T x(t)dt \tag{2}$$

in which l.i.m. means limit in mean square [8].

It must be noted that for (2) to be valid, (and together with it similar expressions (25), (42), (56), (63) which are relative to the mean square value and to the correlation and covariance functions) it is not necessary that $x(t)$ and $y(t)$ are ergodic in the strict sense (strongly ergodic): it is sufficient that they are ergodic with respect to the covariance functions. Random functions are said to be strongly ergodic if the equivalence of time and ensemble averages is extended to all their statistical properties. Taking into consideration expression (2), the mean value of a stationary, ergodic random signal $x(t)$ can be estimated by using estimator:

$$\hat{\mu}_x = \frac{1}{T} \int_0^T x(t)dt \tag{3}$$

having chosen the instant at which the measurement begins as the time - axis origin.

Let us examine estimator (3), which is an unbiased estimator, as [8] [9] :

$$E(\hat{\mu}_x) = \frac{1}{T}\int_0^T E(x(t))dt = \mu_x \tag{4}$$

where we used the interchangeability property of the operation of finding the mathematical expectation and the operation of integration [10].

Because the estimator we are dealing with is unbiased, its mean-squared error is equal to its variance, which is given by:

$$\mathrm{var}(\hat{\mu}_x) = E(\hat{\mu}_x^2) - \left\{E(\hat{\mu}_x)\right\}^2 = E(\hat{\mu}_x^2) - \mu_x^2 \tag{5}$$

Substituting (3) in (5) we have:

$$\mathrm{var}(\hat{\mu}_x) = E\left[\frac{1}{T^2}\int_0^T x(t)dt \int_0^T x(\tau)d\tau\right] - \mu_x^2 =$$

$$= \frac{1}{T^2}\int_0^T\int_0^T E(x(t)x(\tau))\,dt\,d\tau - \mu_x^2 = \frac{1}{T^2}\int_0^T\int_0^T C_{xx}(t-\tau)\,dt\,d\tau \tag{6}$$

which gives at last [8]:

$$\mathrm{var}(\hat{\mu}_x) = \frac{1}{T}\int_{-T}^{T}\left(1 - \frac{|\tau|}{T}\right) C_{xx}(0)\,\rho_{xx}(\tau)\,d\tau \tag{7}$$

./.

where C_{xx} is the autovariance function of $x(t)$ and $\rho_{xx}(\tau)$ is the normalized autocovariance function, given by:

$$\rho_{xx}(\tau) = \frac{C_{xx}(\tau)}{C_{xx}(0)} \qquad (8)$$

If $x(t)$ is a real function, its covariance function is real and (7) can be reduced to the form:

$$\mathrm{var}(\hat{\mu}_x) = \frac{2}{T}\int_0^T \left(1 - \frac{\tau}{T}\right) C_{xx}(0)\, \rho_{xx}(\tau)\, d\tau \qquad (9)$$

If the autocovariance function $C_{xx}(\tau)$ satisfies the condition:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \left(1 - \frac{\tau}{T}\right) C_{xx}(\tau)\, d\tau = 0 \qquad (10)$$

we can say that the estimator (3) is consistent, as:

$$\lim_{T\to\infty} \mathrm{var}(\hat{\mu}_x) = 0 \qquad (11)$$

The condition (10) is satisfied in the case of ergodic processes [8].

It will be observed that while the property (4) is a guarantee against the appearance of a systematic error when replacing the mathematical expectation by its estimated value, the property of consistency guarantees that we can reduce the statistical errors by increasing the time T of integration.

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If the integration shown in (3) is repeated k times, and the results averaged out, the following estimator is used to determine the mean value:

$$ {}_k\hat{\mu}_x = \frac{1}{T}\,\frac{1}{k}\sum_{i=1}^{k}\int_{t_i}^{t_i+T} x(t)\,dt \qquad (12) $$

where t_i indicates the instant at which the i^{th} repetition begins; this estimator, like the preceding one, is unbiased, inso far as:

$$ E({}_k\hat{\mu}_x) = \frac{1}{k}\,\frac{1}{T}\sum_{i=1}^{k}\int_{t_i}^{t_i+T} E(x(t))\,dt = \mu_x \qquad (13) $$

as the expected value operation is a linear operator.

As far as the variance of the estimator (12) is concerned, it can be obtained simply be considering the estimations $\hat{\mu}_{x,o}$; $\hat{\mu}_{x,1}$; .... $\hat{\mu}_{x,k}$ of the mean value found in the k repetitions, as k measured values of a random process. The sample mean is therefore:

$$ {}_k\hat{\mu}_x = \frac{1}{k}\sum_{i=1}^{k} \hat{\mu}_{x,i} \qquad (14) $$

where the subscript i indicates the result of the i^{th} measurement.

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Hence we can write:

$$\mathrm{var}\left[{}_k\hat{\mu}_x\right] = E\left[{}_k\hat{\mu}_x^2\right] - \left\{E\left[{}_k\hat{\mu}_x\right]\right\}^2 = E\left[\left(\frac{1}{k}\sum_{i=1}^{k}\hat{\mu}_{x,i}\right)^2\right] - \mu_x^2 \tag{15}$$

and, by developing (15):

$$\mathrm{var}\left[{}_k\hat{\mu}_x\right] = E\left[\frac{1}{k^2}\sum_{i=1}^{k}\hat{\mu}_{x,i}^2 + \frac{1}{k^2}\sum_{\substack{i,j=1\\ i\neq j}}^{k}\hat{\mu}_{x,i}\cdot\hat{\mu}_{x,j}\right] - \mu_x^2 \tag{16}$$

Remembering that for two random variables z and w we can write [8]:

$$\begin{aligned} E(z^2) &= \sigma_z^2 + \left\{E(z)\right\}^2 \\ E(z^r w^s) &= \alpha_{rs} \\ \alpha_{11} &= R_{zw}(\tau) = E(z)\cdot E(w) + \rho_{zw}(\tau)\sqrt{C_{zz}(0)\,C_{ww}(0)} = \\ &= E(z)\cdot E(w) + \rho_{zw}(\tau)\cdot\sigma_z\cdot\sigma_w \end{aligned} \tag{17}$$

where α_{rs} is the moment of order $r+s$, expression (16) can

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be rearranged:

$$\mathrm{var}\left[{}_k\hat{\mu}_x\right] = \frac{1}{k}\left\{\mathrm{var}\left[\hat{\mu}_{x,i}\right] + \mu_x^2\right\} + \frac{k-1}{k}\mu_x^2 + \frac{1}{k^2}\,\mathrm{var}\left[\hat{\mu}_{x,i}\right]\sum_{\substack{i,j=1\\ i\neq j}}^{k}\rho_{ij}(\tau) - \mu_x^2 =$$

$$= \frac{1}{k}\,\mathrm{var}\left[\hat{\mu}_{x,i}\right]\left[1 + \frac{1}{k}\sum_{\substack{i,j=1\\ i\neq j}}^{k}\rho_{ij}(\tau)\right] \qquad (18)$$

where $\rho_{ij}(\tau)$ represents the degree of correlation existing between the values assumed by the variable $x(t)$ in the ith interval and those assumed in the jth interval.

We can reduce (18) to a simpler, though only approximate, form:

$$\mathrm{var}\left[{}_k\hat{\mu}_x\right] \cong \mathrm{var}\left[\hat{\mu}_{x,i}\right]\frac{1+(k-1)\,\rho_k(\tau)}{k} \qquad (19)$$

where $\rho_{ij}(\tau)$ is infact assumed to be constant whatever the values of i and j.

From (18), remembering that $\mathrm{var}(\hat{\mu}_{x,i})$ is actually given by the second member of (7), we have:

$$\lim_{T\to\infty}\mathrm{var}\left[{}_k\hat{\mu}_x\right] = 0 \qquad (20)$$

by virtue of the hypotheses already made for the case of estimator (3), except that now the convergence is more rapid because of the divisor k. It can once more be seen that similarly:

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$$\lim_{k \to \infty} \operatorname{var}\left[{}_k\hat{\mu}_x\right] = 0 \tag{21}$$

even if T remains a finite value.

This is all obviously valid when $\rho_{ij}(\tau)$ is less than unity. If $\rho_{ij}(\tau)$ was equal to one, however, the repetitions would not have any effect, and the result would be:

$$\operatorname{var}\left[{}_k\hat{\mu}_x\right] = \operatorname{var}\left[\hat{\mu}_x\right] \tag{22}$$

as can be imagined from the fact that a $\rho_{ij}(\tau)$ with identically unitary values would be the same thing as always making measurements over the same time interval. The most favorable situation, however, is that of completely uncorrelated measurements, for which $\rho_{ij}(\tau)$ has a value of zero, so that we have:

$$\operatorname{var}\left[{}_k\hat{\mu}_x\right] = \frac{1}{k} \operatorname{var}\left[\hat{\mu}_{x,i}\right] \tag{23}$$

In order to have $\rho_{ij}(\tau)$ values small enough, it is obviously necessary to choose the instant t_i from which the measurements begin sufficiently far apart.

In practice $\rho_k(\tau)$ will be greater than zero but less than one, because of which the variance of estimator (12) will be less than that of estimator (3).

In appendix A a direct procedure to evaluate the influence of the repetitions on the estimate variance is shown.

The S.D.A. analyzer employs estimator (12) in order to determine the mean value of a signal under examination; the estimation of the mean value is done by S.D.A. when used as a digital voltmeter (D.V.M. function): more details will be given in chap. 5.

2.) MEAN SQUARE VALUE ESTIMATION

The mean square value $\overline{x^2}$ of a random variable $x(t)$ is defined by:

$$m_{2,x} = E\left[x^2(t)\right] \tag{24}$$

where the symbol $m_{2,x}$ of the moment of the second order is introduced. If $x(t)$ is stationary and ergodic, we can write:

$$m_{2,x} = \underset{T \to \infty}{\mathrm{l.i.m.}} \frac{1}{T} \int_0^T x^2(t)\, dt \tag{25}$$

Let $x(t)$ therefore be a stationary ergodic random signal; its mean square value can be evaluated by the estimator:

$$\hat{m}_{2,x} = \frac{1}{T} \int_0^T x^2(t)\, dt \tag{26}$$

which is unbiased, since its expected value is given by:

$$E\left[\hat{m}_{2,x}\right] = \frac{1}{T} \int_0^T E\left[\, x^2(t)\, \right] dt = m_{2,x} \tag{27}$$

Let us now look at its variance (equal to its mean-squared error, owing to unbiasedness of the estimate) :

$$\mathrm{var}\left[\hat{m}_{2,x}\right] = E\left[\hat{m}^2_{2,x}\right] - \left\{E\left[\hat{m}_{2,x}\right]\right\}^2 = E\left[\hat{m}^2_{2,x}\right] - m^2_{2,x} \tag{28}$$

./.

from which, introducing (26) and carrying the mathematical expectation operation inside the integration operation, we obtain:

$$\mathrm{var}\left[\hat{m}_{2,x}\right] = \frac{1}{T^2}\int_0^T\int_0^T E\left[x^2(t)\,x^2(\theta)\right] dt\, d\theta - m_{2,x}^2 \tag{29}$$

Remembering now the expression relating the mathematical expectation of the product of four normal random variables to their correlation functions and mean values [11]):

$$E\left[x_1 x_2 x_3 x_4\right] = R_{12}R_{34} + R_{13}R_{24} + R_{14}R_{23} - 2\,\bar{x}_1\,\bar{x}_2\,\bar{x}_3\,\bar{x}_4 \tag{30}$$

we can rearrange equation (29) , so that it becomes:

$$\mathrm{var}\left[\hat{m}_{2,x}\right] = \frac{2}{T^2}\int_0^T\int_0^T R_{xx}^2(\theta-t)\, dt\, d\theta - 2\,\mu_x^4 \tag{31}$$

If we wish to express (31) in terms of the autocovariance function $C_{xx}(\tau)$ and to resolve partially the double integration, we have finally:

$$\mathrm{var}\left[\hat{m}_{2,x}\right] = \frac{2}{T}\int_{-T}^{T}\left(1 - \frac{|\tau|}{T}\right)\left[C_{xx}^2(\tau) + 2\,\mu_x^2\, C_{xx}(\tau)\right] d\tau \tag{32}$$

If $x(t)$ is real:

$$\mathrm{var}\left[\hat{m}_{2,x}\right] = \frac{4}{T}\int_0^T\left(1 - \frac{\tau}{T}\right)\left[C_{xx}^2(\tau) + 2\,\mu_x^2\, C_{xx}(\tau)\right] d\tau \tag{33}$$

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In both cases the variance tends to zero if T tends towards the infinite, always presuming that the autocovariance function and its square are absolutely integrable; if this is so, then:

$$\lim_{T\to\infty} \mathrm{var}\left[\hat{m}_{2,x}\right] = 0 \qquad (34)$$

namely, the estimator is consistent.

Where the measurement is repeated k times and the k results are averaged out, the estimator takes the form:

$$_k\hat{m}_{2,x} = \frac{1}{T}\,\frac{1}{k}\sum_{i=1}^{k}\int_{t_i}^{t_i+T} x^2(t)\,dt \qquad (35)$$

where t_i is the instant at which the i^{th} repetition commences. This estimator is also unbiased:

$$E\left[{}_k\hat{m}_{2,x}\right] = \frac{1}{T}\,\frac{1}{k}\sum_{i=1}^{k}\int_{t_i}^{t_i+T} E\left[x^2(t)\right] dt = m_{2,x} \qquad (36)$$

while as far as its variance (equal to its mean-squared error) is concerned, what we have already said about the mean value estimator is valid. One has, in fact:

$$\mathrm{var}\left[{}_k\hat{m}_{2,x}\right] = \frac{1}{k}\left[1 + \frac{1}{k}\sum_{\substack{i,j=1\\ i\neq j}}^{k} \rho_{ij}(\tau)\right]\cdot \mathrm{var}\left[\hat{m}_{2,x,i}\right] \qquad (37)$$

where $\mathrm{var}\left[\hat{m}_{2,x,i}\right]$ represents the variance of an estimate carried out by a single measurement, and is given by expression (32).

./.

By assuming $\rho_{ij}(\tau)$ to be constant whatever the values of i and j, we can write the approximate, simplified expression:

$$\mathrm{var}\left[{}_k\hat{m}_{2,x}\right] \cong \mathrm{var}\left[\hat{m}_{2,x,i}\right] \frac{1+(k-1)\rho_k(\tau)}{k} \qquad (38)$$

The less the different measurements are correlated, the greater is the efficiency of the repetitions in reducing the estimate variance.

Where $x(t)$ has a mean value of zero, (38) becomes (if $x(t)$ is real) :

$$\mathrm{var}\left[{}_k\hat{m}_{2,x}\right] = \frac{1+(k-1)\rho_k(\tau)}{k} \; \frac{4}{T^2} \; \sigma_x^4 \int_0^T (T-\tau)\, \rho^2(\tau)\, d\tau \qquad (39)$$

where the autocorrelation function $R_{xx}(\tau)$ has been expressed as:

$$R_{xx}(\tau) = \sigma_x^2 \cdot \rho(\tau) = \mathrm{var}(x) \cdot \rho(\tau) \qquad (40)$$

It will be observed that, if $x(t)$ has a mean value of zero, the moment of the second order, or mean square value $m_{2,x}$, coincides with the variance σ_x^2 .

In the S.D.A. statistical dynamic analyzer the estimation of the mean square value is done on the basis of estimator (35); more information about this will be given in chap. 6.

./.

3.) CORRELATION FUNCTION ESTIMATION

3.1.) Cross-correlation function estimation

Let us consider two random variables $x(t)$ and $y(t)$; their cross-correlation function is defined as [11] [12] :

$$R_{xy}(\tau) = E\left[x(t)\, y(t+\tau) \right] \tag{41}$$

If the variables are stationary and ergodic, the cross-correlation function may be computed by a time average, that is [13]:

$$R_{xy}(\tau) = \underset{T \to \infty}{\mathrm{l.i.m.}} \; \frac{1}{T} \int_0^T x(t)\, y(t+\tau)\, dt \tag{42}$$

Therefore, as in the case of mean square value, the estimator of the correlation function of two real ergodic stationary random functions can be provided by:

$$\hat{R}_{xy}(\tau) = \frac{1}{T} \int_0^T x(t)\, y(t+\tau)\, dt \tag{43}$$

where T is a sufficiently long period.

Let us look at the properties of this estimator. We find:

$$E\left[\hat{R}_{xy}(\tau) \right] = \frac{1}{T} \int_0^T E\left[x(t)\, y(t+\tau) \right] dt = R_{xy}(\tau) \tag{44}$$

that is, it is not biased.

Its variance (equal to its mean-squared error), defined as:

$$\mathrm{var}\left[\hat{R}_{xy}(\tau)\right] = E\left[\hat{R}^2_{xy}(\tau)\right] - \left\{E\left[\hat{R}_{xy}(\tau)\right]\right\}^2 \tag{45}$$

is given by:

$$\mathrm{var}\left[\hat{R}_{xy}(\tau)\right] = \frac{1}{T^2}\int_0^T\int_0^T E\left[x(t)\ y(t+\tau)\ x(\theta)\ y(\theta+\tau)\right]dt\ d\theta - R^2_{xy}(\tau) \tag{46}$$

Expression (46) shows that, in order to evaluate the variance of the estimator of the cross-correlation functions, knowledge of the correlation functions is not sufficient; it is also necessary to know moments of higher order.

Nevertheless, when the random processes under examination are normal processes, the moments can be expressed in terms of mathematical expectations and correlation functions. In general, for four normal variables $x_1(t)$, $x_2(t)$, $x_3(t)$ and $x_4(t)$, expression (30) has been proved.
In the present case we can write:

$$\begin{aligned} x_1(t) &= x(t)\ ; & x_2(t) &= y(t+\tau) \\ x_3(t) &= x(\theta)\ ; & x_4(t) &= y(\theta+\tau) \end{aligned} \tag{47}$$

for which (46) becomes:

./.

$$\mathrm{var}\left[\hat{R}_{xy}(\tau)\right] = \frac{1}{T^2}\int_0^T\int_0^T\Big[R^2_{xy}(\tau) + R_{xx}(\theta-t)\,R_{yy}(\theta-t) +$$

$$+ R_{xy}(\theta+\tau-t)\,R_{yx}(\theta-t-\tau) - 2\,\mu_x^2\,\mu_y^2\Big]\,dt\,d\theta - R^2_{xy}(\tau) \tag{48}$$

which, when developed, gives us the expression we are seeking:

$$\mathrm{var}\left[\hat{R}_{xy}(\tau)\right] = \frac{1}{T}\int_{-T}^{T}\left(1 - \frac{|\eta|}{T}\right)\Big[R_{xx}(\eta)\,R_{yy}(\eta) + R_{xy}(\eta+\tau)\,R_{yx}(\eta-\tau)\Big]\,d\eta +$$

$$- 2\,\mu_x^2\,\mu_y^2 \tag{49}$$

or, in terms of covariance functions:

$$\mathrm{var}\left[\hat{R}_{xy}(\tau)\right] = \frac{1}{T}\int_{-T}^{T}\left(1 - \frac{|\eta|}{T}\right)\Big[C_{xx}(\eta)\,C_{yy}(\eta) + C_{xy}(\eta+\tau)\,C_{yx}(\eta-\tau)\Big]\,d\eta +$$

$$+ \frac{1}{T}\int_{-T}^{T}\left(1 - \frac{|\eta|}{T}\right)\Big[\mu_y^2 C_{xx}(\eta) + \mu_x^2 C_{yy}(\eta) + \mu_x\mu_y C_{xy}(\eta+\tau) +$$

$$+ \mu_x\mu_y C_{yx}(\eta-\tau)\Big]\,d\eta \tag{50}$$

Therefore, where C_{xx}, C_{yy}, $C_{xx}\cdot C_{yy}$ and $C_{xy}\cdot C_{yx}$ are absolutely integrable, (50) allows us to write:

$$\lim_{T\to\infty} \mathrm{var}\left[\hat{R}_{xy}(\tau)\right] = 0 \tag{51}$$

./.

which means that estimator (43) is consistent.

Formula (50) is quite complicated and it is often rather difficult to prove that relation (51) is satisfied. For practical purposes it is therefore more convenient to observe that (51) is proved if the covariance functions C_{xx}, C_{yy}, C_{xy}, decrease indefinitely in absolute value as $|\tau|$ tends to infinity.

We repeat that stationary random variables $x(t)$ and $y(t)$ which satisfies relations (10) and (51) are said to be ergodic with respect to the covariance functions.

If we consider the estimator relative to the case of k repetitions for the cross-correlation function too, it can be expressed as:

$$ {}_k\hat{R}_{xy}(\tau) = \frac{1}{k}\,\frac{1}{T}\sum_{i=1}^{k}\int_{t_i}^{t_i+T} x(t)\, y(t+\tau)\, dt \tag{52} $$

which is unbiased, as:

$$ E\left[{}_k\hat{R}_{xy}(\tau)\right] = \frac{1}{k}\,\frac{1}{T}\sum_{i=1}^{k}\int_{t_i}^{t_i+T} E\left[x(t)\, y(t+\tau)\right] dt = R_{xy}(\tau) \tag{53} $$

while its variance can be expressed as a function of the variance of estimator (43), of the number of k repetitions, and of the degree of correlation $\rho_k(\tau)$ existing between the k measurements; the formula is the same one already seen in the case of the mean value and mean square value:

$$ \mathrm{var}\left[{}_k\hat{R}_{xy}(\tau)\right] \cong \mathrm{var}\left[\hat{R}_{xy,i}(\tau)\right] \frac{1+(k-1)\rho_k(\tau)}{k} \tag{54} $$

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the considerations already made for relations (19) and (38) beeing also valid for expression (54).

3.2.) Auto-correlation function estimation

What we have said about the cross-correlation is substantially valid also for the autocorrelation function.
The autocorrelation, defined for a random variable $x(t)$ by :

$$R_{xx}(\tau) = E\left[x(t)\, x(t+\tau)\right] \tag{55}$$

can, if $x(t)$ is stationary and ergodic, be expressed as :

$$R_{xx}(\tau) = \underset{T \to \infty}{\text{l.i.m.}} \frac{1}{T}\int_0^T x(t)\, x(t+\tau)\, dt \tag{56}$$

for which one can in practice use the estimator:

$$\hat{R}_{xx}(\tau) = \frac{1}{T}\int_0^T x(t)\, x(t+\tau)\, dt \tag{57}$$

to estimate the autocorrelation function of an ergodic random variable. This estimator, like that of the cross-correlation function, is unbiased:

$$E\left[\hat{R}_{xx}(\tau)\right] = \frac{1}{T}\int_0^T E\left[x(t)\, x(t+\tau)\right] dt = R_{xx}(\tau) \tag{58}$$

while its variance is given by the expression:

$$\operatorname{var}\left[\hat{R}_{xx}(\tau)\right] = \frac{1}{T}\int_{-T}^{T}\left(1 - \frac{|\eta|}{T}\right)\left[R^2_{xx}(\eta) + R_{xx}(\eta+\tau)\, R_{xx}(\eta-\tau)\right] d\eta - 2\,\mu_x^4 \tag{59}$$

which can be obtained rapidly from (49) by substituting the variable x for the variable y.

In terms of covariance functions, the expression of the variance of the estimate of the autocorrelation function is obviously similar to expression (50), from which it is obtainable by substituting μ_x for μ_y and C_{xx} for C_{xy} and C_{yy}.

As far as the consistency of the estimate is concerned, what we have said for the cross-correlation function is also valid; the relation which expresses this consistency:

$$\lim_{T\to\infty} \operatorname{var}\left[\hat{R}_{xx}(\tau)\right] = 0 \tag{60}$$

is proved in the case where C_{xx} and C^2_{xx} are absolutely integrable.

An estimator which takes into account k repetitions of the measurements can also be considered for the autocorrelation:

$${}_k\hat{R}_{xx}(\tau) = \frac{1}{k}\,\frac{1}{T}\sum_{i=1}^{k}\int_{t_i}^{t_i+T} x(t)\, x(t+\tau)\, dt \tag{61}$$

For the properties of this last estimator we can refer back to what was said about the cross-correlation functions.

In Chapter 6 the procedure by which the S.D.A. analyzer evaluates the correlation functions will be described. This procedure is based on estimators (52) and (61).

4.) COVARIANCE FUNCTION ESTIMATION

Let us consider the usual two random variables x(t) and y(t) ; the cross-covariance function is defined as:

$$C_{xy}(\tau) = E\left\{ \left[x(t) - \mu_x\right]\left[y(t+\tau) - \mu_y\right] \right\} \tag{62}$$

or, if the variables x(t) and y(t) are ergodic with respect to the covariance functions (weakly ergodic):

$$C_{xy}(\tau) = \underset{T \to \infty}{\text{l.i.m.}} \frac{1}{T} \int_0^T \left[x(t) - \mu_x\right]\left[y(t+\tau) - \mu_y\right] dt \tag{63}$$

The cross-covariance estimate $\hat{C}_{xy}(\tau)$ can consequently be defined by:

$$\hat{C}_{xy}(\tau) = \frac{1}{T} \int_0^T \left[x(t) - \mu_x\right]\left[y(t+\tau) - \mu_y\right] dt \tag{64}$$

In practice the effective mean values μ_x and μ_y of the variables x(t) and y(t) are replaced by the estimated mean values $\hat{\mu}_x$ and $\hat{\mu}_y$; because of this the following estimator is the one actually used:

$$\hat{C}_{xy}(\tau) = \frac{1}{T}\int_0^T x(t)\, y(t+\tau)\, dt - \hat{\mu}_x \cdot \hat{\mu}_y \tag{65}$$

which means that the cross-correlation function $\hat{R}_{xy}$ and the mean values $\hat{\mu}_x$ and $\hat{\mu}_y$ are measured separately, and from these measurements the cross-covariance is then deduced:

$$\hat{C}_{xy}(\tau) = \hat{R}_{xy}(\tau) - \hat{\mu}_x \cdot \hat{\mu}_y \tag{66}$$

./.

Let us now look at the properties of this estimator. As far as the bias is concerned, we have that the expected value for the estimate $\hat{C}_{xy}(\tau)$ is given by:

$$E\left[\hat{C}_{xy}(\tau)\right] = E\left[\hat{R}_{xy}(\tau)\right] - E\left[\hat{\mu}_x \cdot \hat{\mu}_y\right] \tag{67}$$

which, rearranged on the basis of what we said about the estimates of the correlation function and of the mean values, becomes:

$$E\left[\hat{C}_{xy}(\tau)\right] = R_{xy}(\tau) - \frac{1}{T^2}\int_0^T \int_0^T E\left[x(t)\, y(\theta)\right] dt\, d\theta \tag{68}$$

and finally gives:

$$E\left[\hat{C}_{xy}(\tau)\right] = C_{xy}(\tau) - \frac{1}{T}\int_{-T}^{T} \left(1 - \frac{|\eta|}{T}\right) C_{xy}(\eta)\, d\eta \tag{69}$$

Hence $\hat{C}_{xy}(\tau)$ is a biased estimate of the cross-covariance function $C_{xy}(\tau)$; its bias is given by:

$$\text{bias}\left[\hat{C}_{xy}(\tau)\right] = \frac{1}{T}\int_{-T}^{T} \left(1 - \frac{|\eta|}{T}\right) C_{xy}(\eta)\, d\eta \tag{70}$$

If however the cross-covariance function is absolutely integrable, the right hand member of equation (70) tends to zero when integration time T tends to infinity, so that we have:

$$\lim_{T \to \infty} \text{bias}\left[\hat{C}_{xy}(\tau)\right] = 0 \tag{71}$$

./.

which means that:

$$\lim_{T\to\infty} E\left[\hat{C}_{xy}(\tau)\right] = C_{xy}(\tau) \tag{72}$$

so that we can say of the estimator under examination that it is unbiased in the limit.

It is clear from (72) that the estimated value $\hat{C}_{xy}(\tau)$ can assume the true value $C_{xy}(\tau)$ only by taking the measurement time to infinity. But it is also obvious that if one could replace the estimated mean values $\hat{\mu}_x$ and $\hat{\mu}_y$ by the true mean values μ_x and μ_y in (65), then the estimator would be unbiased.

In order to see now whether or not this is consistent, it is necessary, on account of the bias, to evaluate the mean square error and not the variance. The mean square error (m.s.e.) is defined by:

$$\text{m.s.e.}\left[\hat{C}_{xy}(\tau)\right] = E\left\{\left[\hat{C}_{xy}(\tau) - C_{xy}(\tau)\right]^2\right\} \tag{73}$$

or, in terms of the variance and the square of the bias:

$$\text{m.s.e.}\left[\hat{C}_{xy}(\tau)\right] = \text{var}\left[\hat{C}_{xy}(\tau)\right] + \text{bias}^2\left[\hat{C}_{xy}(\tau)\right] \tag{74}$$

where the bias of the estimator has the expression we gave in equation (70), while the variance of the estimate can be evaluated on the basis of the definition:

$$\text{var}\left[\hat{C}_{xy}(\tau)\right] = E\left\{\left[\hat{C}_{xy}(\tau)\right]^2\right\} - \left\{E\left[\hat{C}_{xy}(\tau)\right]\right\}^2 \tag{75}$$

./.

The strict development of expression (74) is given in appendix B, for the sake of accuracy, but it has no effective or practical interest because the final expression is too complex. For this reason it is convenient in practice to construct a simplified form of this expression, even if it is only approximate, by replacing the mean estimated values $\hat{\mu}_x$ and $\hat{\mu}_y$ by the mean values μ_x and μ_y. This does not lead to a significant error, since the variance of the estimator of the mean value is rather small.

Ignoring thus the difference between the estimated mean values and the effective mean values, estimator (65) can be written as:

$$\hat{C}_{xy}(\tau) = \frac{1}{T}\int_0^T \left[x(t) - \mu_x\right]\left[y(t+\tau) - \mu_y\right] dt \qquad (76)$$

which gives an unbiased estimate of the cross-covariance function, so that the mean squared error of the estimate becomes:

$$\text{m.s.e.}\left[\hat{C}_{xy}(\tau)\right] = \text{var}\left[\hat{C}_{xy}(\tau)\right] = E\left\{\left[\hat{C}_{xy}(\tau)\right]^2\right\} - C^2_{xy}(\tau) \qquad (77)$$

On the other hand we know that:

$$E\left\{\left[\hat{C}_{xy}(\tau)\right]^2\right\} = \frac{1}{T^2}\int_0^T\int_0^T E\left\{\left[x(t)-\mu_x\right]\left[y(t+\tau)-\mu_y\right]\left[x(\theta)-\mu_x\right]\left[y(\theta+\tau)-\mu_y\right]\right\} dt\, d\theta =$$

$$= C^2_{xy}(\tau) + \frac{1}{T}\int_{-T}^{T}\left(1 - \frac{|\eta|}{T}\right)\left[C_{xx}(\eta)C_{yy}(\eta) + C_{xy}(\eta+\tau)C_{yx}(\eta-\tau)\right] d\eta$$

$$(78)$$

./.

for which the approximate expression of the mean square error of the cross-covariance estimator becomes:

$$\text{m.s.e.}\left[\hat{C}_{xy}(\tau)\right] \cong \frac{1}{T}\int_{-T}^{T}\left(1-\frac{|\eta|}{T}\right)\left[C_{xx}(\eta)C_{yy}(\eta) + C_{xy}(\eta+\tau)C_{yx}(\eta-\tau)\right] d\eta \tag{79}$$

which, if $x(t)$ and $y(t)$ are ergodic, tends to zero if T tends to the infinite.

As far as the autocovariance function is concerned, everything that has been said for the cross-covariance is valid.
The relative formulae can be obtained immediately from the analogs given for the cross-covariance by substituting $x(t)$ for the variable $y(t)$.

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5.) ESTIMATION OF THE MEAN VALUE AND OF THE MEAN MODULUS BY THE S.D.A. ANALYZER

The S.D.A. statistical dynamics analyzer evaluates the mean value and the mean modulus of the $x(t)$ and $y(t)$ signals being examined, on the basis of estimators:

$$\hat{\bar{x}} = \hat{\mu}_x = \frac{1}{k}\frac{1}{T}\sum_{i=1}^{k}\int_{t_i}^{t_i+T} x(t)\,dt \tag{80}$$

$$\hat{\overline{|x|}} = \hat{\mu}_{|x|} = \frac{1}{k}\frac{1}{T}\sum_{i=1}^{k}\int_{t_i}^{t_i+T} |x(t)|\,dt \tag{81}$$

and on the basis of the similar estimators relative to variable $y(t)$. The quantities α_i , β_i , γ_i and δ_i supplied by the analyzer to the computer which carries out the averaging and the normalizations on them, are given by:

$$\alpha_i = \int_{t_i}^{t_i+T} f_x(t)\,dt = \frac{1}{h}\int_{t_i}^{t_i+T} x(t)\,dt \tag{82}$$

$$\beta_i = \int_{t_i}^{t_i+T} |f_x(t)|\,dt = \frac{1}{h}\int_{t_i}^{t_i+T} |x(t)|\,dt \tag{83}$$

$$\gamma_i = \int_{t_i}^{t_i+T} f_y(t)\,dt = \frac{1}{h}\int_{t_i}^{t_i+T} y(t)\,dt \tag{84}$$

./.

$$\delta_i = \int_{t_i}^{t_i+T} |f_y(t)|\,dt = \frac{1}{h}\int_{t_i}^{t_i+T} |y(t)|\,dt \tag{85}$$

where $f_x(t)$ and $f_y(t)$ are the output frequencies of the voltage to frequency convertors of the x and y channels respectively, and $|f_x(t)|$ and $|f_y(t)|$ are the same frequencies taken always with a positive sign; the proportionality coefficient of the voltage to frequency conversion is indicated by h.
Finally the analysis time T is equal to n times the unity T_m of machine time, equal to 10 milliseconds:

$$T = n\,T_m$$

while the instant at which the ith repetition starts is indicated by t_i. From the above, and on the basis of (80), (81) and (82) .... (85), we have:

$$\hat{\mu}_x = \frac{1}{k}\,\frac{1}{n\,T_m}\,h\sum_{i=1}^{k}\alpha_i = \frac{h_v}{n\,k}\sum_{i=1}^{k}\alpha_i \tag{86}$$

$$\hat{\mu}_{|x|} = \frac{1}{k}\,\frac{1}{n\,T_m}\,h\sum_{i=1}^{k}\beta_i = \frac{h_v}{n\,k}\sum_{i=1}^{k}\beta_i \tag{87}$$

$$\hat{\mu}_y = \frac{1}{k}\,\frac{1}{n\,T_m}\,h\sum_{i=1}^{k}\gamma_i = \frac{h_v}{n\,k}\sum_{i=1}^{k}\gamma_i \tag{88}$$

$$\hat{\mu}_{|y|} = \frac{1}{k}\,\frac{1}{n\,T_m}\,h\sum_{i=1}^{k}\delta_i = \frac{h_v}{n\,k}\sum_{i=1}^{k}\delta_i \tag{89}$$

./.

The h_v factor which appears in expressions (86) to (89) is the normalization coefficient of the S.D.A. when operating as a digital voltmeter (D.V.M.) [7] and is equal to:

$$h_v = \frac{h}{T_m} \tag{90}$$

In the D.V.M. operation, the S.D.A. apparatus can be employed to obtain the static characteristics of a system.
In fact, if $x(t)$ is the perturbation signal sent to the system or process being examined, and $y(t)$ is the reply of the system itself, one can obtain, point by point, the static characteristics A of the system by giving to $x(t)$ a step by step behaviour:

$$A = \frac{Y}{X} \tag{91}$$

The averagings performed on x and y serve in this case to reduce the influence of spurious noise which is superimposed upon the useful signals. We can write:

$$\hat{A} = \frac{\hat{Y}}{\hat{X}} = \frac{\hat{\mu}_Y}{\hat{\mu}_X} = \frac{\sum_{i=1}^{k} \gamma_i}{\sum_{i=1}^{k} \alpha_i} \tag{92}$$

Fig. 1 gives the sequential operating diagram of the D.V.M. function of the S.D.A. analyzer, and Appendix C.2 describes the corresponding calculation programmes for cases where an Olivetti P102 is used as final computer.

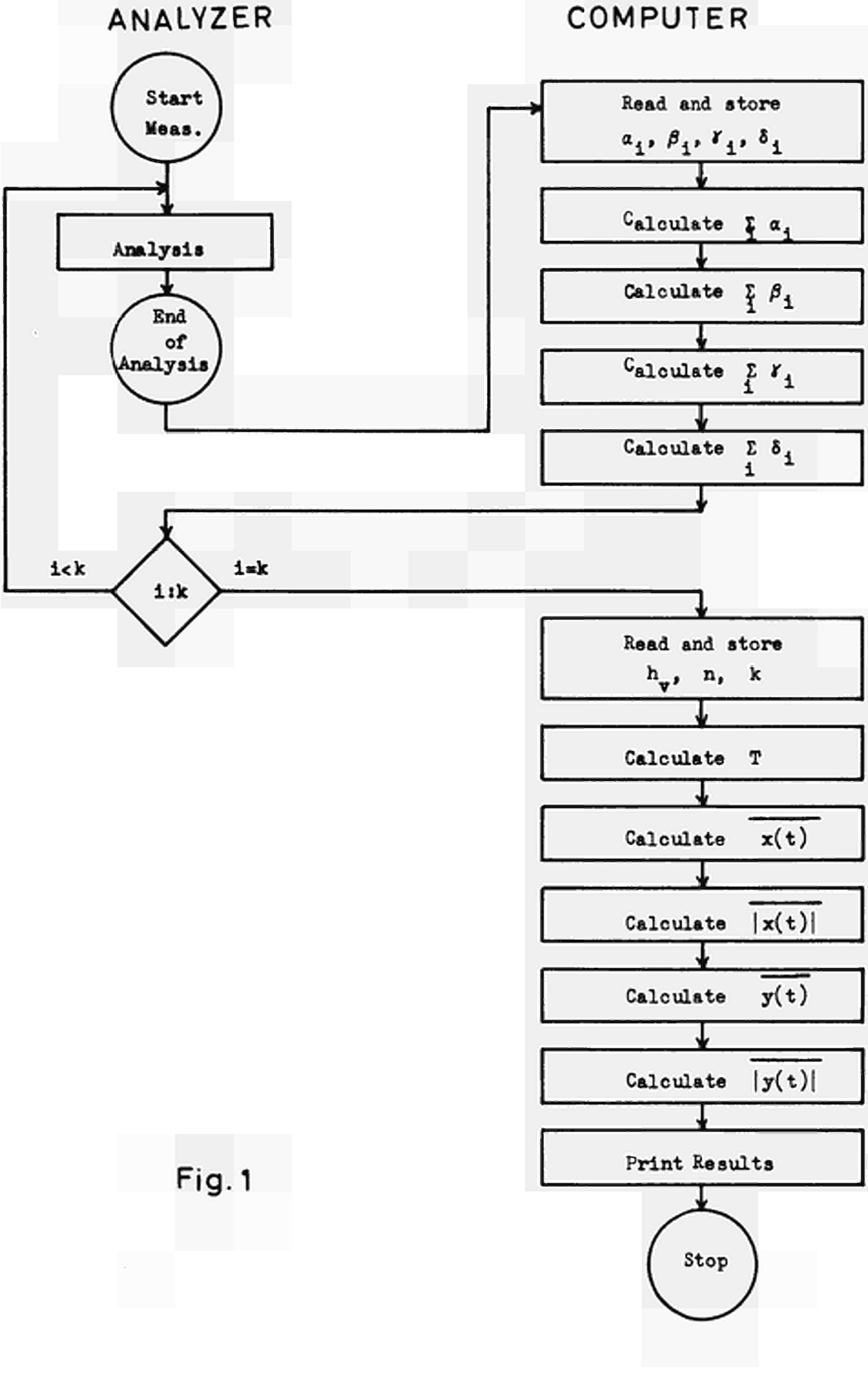
ANALYZER
COMPUTER
Start Meas.
Analysis
End of Analysis
Read and store $\alpha_i, \beta_i, \gamma_i, \delta_i$
Calculate $\sum_i \alpha_i$
Calculate $\sum_i \beta_i$
Calculate $\sum_i \gamma_i$
Calculate $\sum_i \delta_i$
i<k
i:k
i=k
Read and store h_v, n, k
Calculate T
Calculate $\overline{x(t)}$
Calculate $\overline{|x(t)|}$
Calculate $\overline{y(t)}$
Calculate $\overline{|y(t)|}$
Print Results
Stop

Fig. 1

6.) ESTIMATION OF THE CORRELATION FUNCTIONS AND THE MEAN SQUARE VALUE USING THE S.D.A. ANALYZER

The mean square value, and the auto and cross correlation functions are evaluated by the S.D.A. on the basis of estimators of the type (35), (52) and (61); in particular:

$$\hat{m}_{2,x} = \frac{1}{k}\,\frac{f}{n}\sum_{i=1}^{k}\int_{t_i}^{t_i+\frac{n}{f}} x^2(t)\,dt \tag{93}$$

$$\hat{R}_{xx}(\tau) = \frac{1}{k}\,\frac{f}{n}\sum_{i=1}^{k}\int_{t_i}^{t_i+\frac{n}{f}} x(t)\,x(t+\tau)\,dt \tag{94}$$

$$\hat{R}_{xy}(\tau) = \frac{1}{k}\,\frac{f}{n}\sum_{i=1}^{k}\int_{t_i}^{t_i+\frac{n}{f}} x(t)\,y(t+\tau)\,dt \tag{95}$$

where as usual k indicates the number of repetitions, while the analysis time T is expressed as the number of cycles of the reference generator frequency [7]):

$$T = \frac{n}{f}$$

The lag τ of the correlation functions is given by [7]):

$$\tau = \frac{1}{2f}$$

Now, in order to evaluate the normalization coefficient for the correlation, it is necessary to consider the operation of the S.D.A. more closely.

Let us take the case of the mean square value $m_{2,x}$, i.e. the autocorrelation of $x(t)$ for a lag of zero.
The input signal $x(t)$, after having been led to dynamic 2A by amplification, is sent to two level–comparators and compared with two mutually independent reference voltages (one with a saw-tooth form, and the other one randomly variable and equiprobable on 2A) , such that the probabilities $p_1(t)$ and $p_2(t)$ that $x(t)$ will be greater than the first and second comparison voltages respectively are equal [14]) . These probabilities are proportional to the amplitude of the signal $x(t)$ under examination [6]) [12]):

$$p_1(x) = p_2(x) = \frac{x(t) + A}{2A} \tag{96}$$

where 2A is the dynamic, in amplitude, of $x(t)$ after having been amplified.
The probability that a logical exclusive-or operation applied to the logical output voltages of the two comparators will give a positive result, is equal to:

$$p_p = p_1(x) \cdot p_2(x) + \left[1 - p_1(x)\right]\left[1 - p_2(x)\right] \tag{97}$$

while the probability p_n that the result of the exclusive or operation will be negative is equal to:

$$p_n = p_1(x)\left[1 - p_2(x)\right] + p_2(x)\left[1 - p_1(x)\right] \tag{98}$$

Introducing the expression given in (96) for $p_1(x)$ and $p_2(x)$, we find:

$$p_p = \frac{x^2(t) + A^2}{2A^2} \tag{99}$$

$$p_n = \frac{A^2 - x^2(t)}{2A^2} \tag{100}$$

./.

A bidirectional counter placed after the exclusive or circuit gives us a measurement of the difference between the time during which the output of the exclusive or is positive (positive count) and the time during which it is negative (negative count) .
This time is given by the difference between the two probabilities p_p and p_n :

$$p_p - p_n = \frac{x^2(t)}{A^2} \tag{101}$$

where the time is given in relative units, as a fraction of the total analysis time.
It is measured by the bidirectional counter, by means of pulse counts; if f_t is the timing frequency, we have:

$$\alpha_i = \int_{t_i}^{t_i+\frac{n}{f}} \frac{x^2(t)}{A^2} f_t \, dt = \frac{1}{A^2} \frac{n}{f} f_t \overline{\hat{x^2}} \tag{102}$$

If we consider all the bidirectional counters, their contents α_i , β_i , γ_i and δ_i are given by:

$$\alpha_i = \int_{t_i}^{t_i+n/f} \frac{x^2(t)}{A^2} f_t \, dt \tag{103}$$

$$\beta_i = \int_{t_i}^{t_i+n/f} \frac{x(t)\, x(t+\tau)}{A^2} f_t \, dt \tag{104}$$

$$\gamma_i = \int_{t_i}^{t_i+n/f} \frac{x(t)\, y(t+\tau)}{A^2} f_t \, dt \tag{105}$$

$$\delta_i = \int_{t_i}^{t_i + n/f} \frac{y^2(t)}{A^2} f_t \, dt \tag{106}$$

because of which, considering the case of k repetitions, and taking (93), (94) and (95) into account, we can write the following relations which evaluate auto and cross-correlation functions and mean square values by means of α_i , β_i , γ_i and δ_i :

$$\hat{m}_{2,x} = A^2 \frac{1}{f_t} \frac{1}{k} \frac{f}{n} \sum_{i=1}^{k} \alpha_i = \frac{1}{h_c} \frac{1}{k} \frac{f}{n} \sum_{i=1}^{k} \alpha_i \tag{107}$$

$$\hat{R}_{xx}(\tau) = A^2 \frac{1}{f_t} \frac{1}{k} \frac{f}{n} \sum_{i=1}^{k} \beta_i = \frac{1}{h_c} \frac{1}{k} \frac{f}{n} \sum_{i=1}^{k} \beta_i \tag{108}$$

$$\hat{R}_{xy}(\tau) = A^2 \frac{1}{f_t} \frac{1}{k} \frac{f}{n} \sum_{i=1}^{k} \gamma_i = \frac{1}{h_c} \frac{1}{k} \frac{f}{n} \sum_{i=1}^{k} \gamma_i \tag{109}$$

$$\hat{m}_{2,y} = A^2 \frac{1}{f_t} \frac{1}{k} \frac{f}{n} \sum_{i=1}^{k} \delta_i = \frac{1}{h_c} \frac{1}{k} \frac{f}{n} \sum_{i=1}^{k} \delta_i \tag{110}$$

The normalization coefficient h_c introduced in the right hand members of (107) ....(110) is given by:

$$h_c = \frac{A^2}{f_t}$$

where f_t varies from decade to decade, and A is equal to 10 volts.

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If we wish to obtain the normalized correlation functions, in such a way that they vary between a maximum of one and a minimum of zero, the formulae become:

$$\hat{m}_{2,x} = \frac{1}{h_c} \frac{1}{k} \frac{f}{n} \sum_{i=1}^{k} \alpha_i \tag{111}$$

$$\frac{\hat{R}_{xx}(\tau)}{\hat{R}_{xx}(0)} = \frac{\hat{R}_{xx}(\tau)}{\hat{m}_{2,x}} = \frac{\sum_{i=1}^{k} \beta_i}{\sum_{i=1}^{k} \alpha_i} \tag{112}$$

$$\frac{\hat{R}_{xy}(\tau)}{\sqrt{\hat{R}_{xx}(0).\hat{R}_{yy}(0)}} = \frac{\hat{R}_{xy}(\tau)}{\sqrt{\hat{m}_{2,x} \cdot \hat{m}_{2,y}}} = \frac{\sum_{i=1}^{k} \gamma_i}{\sqrt{\sum_{i=1}^{k} \alpha_i \cdot \sum_{i=1}^{k} \delta_i}} \tag{113}$$

$$\hat{m}_{2,y} = \frac{1}{h_c} \frac{1}{k} \frac{f}{n} \sum_{i=1}^{k} \delta_i \tag{114}$$

In fig.2 a sequential diagram of the S.D.A. analyzer operation for calculating the correlation functions is shown. The operations performed by the computer, which constitutes the final element of the SDA, are listed separately from those performed in the upper section, called the analyzer, and which carries out all the elaborations on the signals to be examined.

Appendix C.3 describes programmes usable with an Olivetti P 102 computer.

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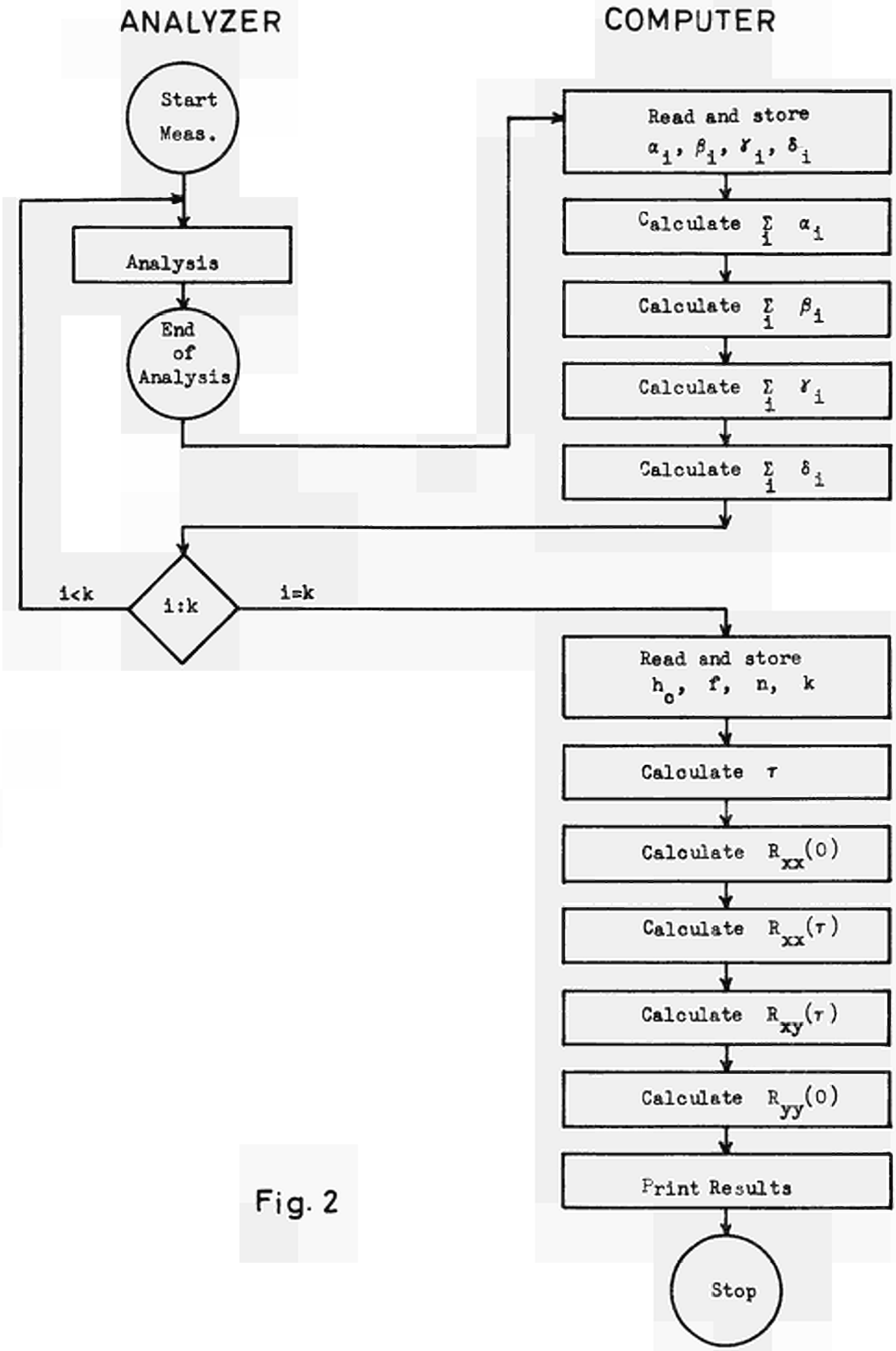
ANALYZER
COMPUTER
Start Meas.
Analysis
End of Analysis
Read and store $\alpha_i, \beta_i, \gamma_i, \delta_i$
Calculate $\sum_i \alpha_i$
Calculate $\sum_i \beta_i$
Calculate $\sum_i \gamma_i$
Calculate $\sum_i \delta_i$
i<k
i:k
i=k
Read and store h_o, f, n, k
Calculate τ
Calculate $R_{xx}(0)$
Calculate $R_{xx}(\tau)$
Calculate $R_{xy}(\tau)$
Calculate $R_{yy}(0)$
Print Results
Stop

Fig. 2

7.) PROBABILITY DENSITY ESTIMATION

7.1.) First-order probability density estimation

Given a random variable $x(t)$, the first order probability density $p_x(\xi)$ is defined as:

$$p_x(\xi) = \lim_{\substack{T\to\infty \\ \Delta x\to 0}} \frac{1}{\Delta x} \frac{1}{T} \int_0^T dT' = \lim_{\substack{T\to\infty \\ \Delta x\to 0}} \frac{1}{\Delta x} \frac{T'}{T} \tag{115}$$

where T indicates the measurement time, T' the total time during which, in the course of the measurement, the variable $x(t)$ has assumed values between $\xi - \Delta x/2$ and $\xi + \Delta x/2$, and finally Δx is the amplitude of the window centered on the value ξ under examination.

The estimator of the first order probability density can thus be the following:

$$\hat{p}_x(\xi) = \frac{1}{\Delta x} \frac{1}{T} \int_0^T dT' = \frac{1}{\Delta x} \frac{T'}{T} \tag{116}$$

The expected value for the estimate $\hat{p}_x(\xi)$ is given by:

$$E\left[\hat{p}_x(\xi)\right] = \frac{1}{\Delta x} E\left[\frac{T'}{T}\right] = \frac{1}{\Delta x} p_x\left(\xi \pm \frac{\Delta x}{2}\right) \tag{117}$$

where $p_x(\xi \pm \Delta x/2)$ indicates the pr bability that the value of $x(t)$ will be within the amplitude window Δx centered, as already said, on the value ξ:

./.

$$p_x\left(\xi \pm \frac{\Delta x}{2}\right) = \int_{\xi-\Delta x/2}^{\xi+\Delta x/2} p_x(\eta)\, d\eta \tag{118}$$

Expression (117) tells us, therefore, that generally estimator (116) is biased; the value it gives is not in fact precisely $p_x(\xi)$, but the mean of the values it assumes for values of $x(t)$ included between $\xi - \Delta x/2$ and $\xi + \Delta x/2$.
This bias can obviously be reduced by reducing the window amplitude Δx .
On the other hand, estimator:

$$\hat{p}_x\left(\xi \pm \frac{\Delta x}{2}\right) = \frac{1}{T}\int_0^T dT' = \frac{T'}{T} \tag{119}$$

is unbiased, as its expected value is:

$$E\left[\hat{p}_x\left(\xi \pm \frac{\Delta x}{2}\right)\right] = E\left[\frac{T'}{T}\right] = p_x\left(\xi \pm \frac{\Delta x}{2}\right) \tag{120}$$

i.e., it gives us precisely the integral of the probability density inside the amplitude window.

The mean square error of the determinations carried out by means of estimator (116) is not easy to evaluate, but can be expressed fairly approximately by the following relation [15]:

$$\text{m.s.e.}\left[\hat{p}_x(\xi)\right] = \frac{A^2\, p_x^2(\xi)}{B\, T\, \Delta x\, \hat{p}_x(\xi)} + \left(\frac{\Delta x^2}{24}\,\frac{\partial^2\, p_x(\xi)}{\partial \xi^2}\right)^2 \tag{121}$$

where the two terms of the right hand member represent the variance of the estimate and the error contribution due to the bias respectively.
In expression (121) , B indicates the bandwidth of signal $x(t)$ in

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examination, and T indicates the total analysis time; A represents a constant whose value is linked to the nature of signal $x(t)$ and for which the value of $1/\sqrt{2}$ can be assumed where $x(t)$ has a uniform spectrum within the frequency range B [16]).

The above mentioned expression shows that the mean square error does not cancel out when T tends to the infinite, inasmuch as the error due to the bias remains; the estimator is consequently not consistent.

The mean square error cancels out only when T tends to the infinite and Δx to zero, as long as the product $T.\Delta x$ also tends to the infinite. In practice, however, the second derivative of the probability density of the signals most often encountered is small, so that when Δx has been fixed at a sufficiently small value, the contribution due to the estimator bias can be ignored.

When the determination of the probability density is per formed on the basis of k measurements, the estimator becomes:

$$_k\hat{p}_x(\xi) = \frac{1}{k}\sum_{i=1}^{k} \frac{1}{\Delta x}\frac{T_i'}{T_i} \qquad (122)$$

Also in this case the estimator is biased :

$$E\left[{}_k\hat{p}_x(\xi)\right] = \frac{1}{k}\sum_{i=1}^{k} \frac{1}{\Delta x} E\left[\frac{T_i'}{T_i}\right] = \frac{1}{k}\sum_{i=1}^{k} \frac{1}{\Delta x}\, p_x\left(\xi \pm \frac{\Delta x}{2}\right) \qquad (123)$$

while the effect of the k repetitions on the variance of the estimate is similar to that already seen in the case of the mean value determinations or of the correlation functions.
Consequently:

$$\mathrm{var}\left[{}_k\hat{p}_x(\xi)\right] = \frac{A^2\ p_x^2(\xi)}{B\ T\ \Delta x\ \hat{p}_x(\xi)}\ \frac{1}{k}\left[1 + \frac{1}{k}\sum_{\substack{i,j=1\\ i\neq j}}^{k} \rho_{ij}(\tau)\right] \qquad (124)$$

or, if for purposes of simplicity we suppose $\rho_{ij}(\tau)$ to be constant in spite of variations of i and j :

$$\mathrm{var}\left[{}_k\hat{p}_x(\xi)\right] \cong \frac{A^2\ p_x^2(\xi)}{B\ T\ \Delta x\ \hat{p}_x(\xi)}\ \frac{(k-1)\ \rho_k(\tau) + 1}{k} \qquad (125)$$

where for $\rho_{ij}(\tau)$ and $\rho_k(\tau)$ the comments already made in previous chapters are valid.

7.2.) Joint probability density estimation

Given two random variables $x(t)$ and $y(t)$, we define the function:

$$p_{x,y}(\xi,\eta,\tau) = \lim_{\substack{T\to\infty\\ \Delta x\to o\\ \Delta y\to o}} \frac{1}{\Delta x}\ \frac{1}{\Delta y}\ \frac{1}{T}\int_o^T dT'' = \lim_{\substack{T\to\infty\\ \Delta x\to o\\ \Delta y\to o}} \frac{1}{\Delta x}\ \frac{1}{\Delta y}\ \frac{T''}{T} \qquad (126\ )$$

as joint probability density.
In relation (126) T is the total analysis time, while T'' represents the overall time during which, in the course of the measurement, the value of the variable $x(t)$ happened to fall within the voltage window Δx, centered on the value ξ, and the variable $y(t+\tau)$ fell in the window Δy centered on the value η .

As joint probability density estimator the following can thus be adopted:

$$\hat{p}_{x,y}(\xi,\eta,\tau) = \frac{1}{\Delta x}\,\frac{1}{\Delta y}\,\frac{1}{T}\int_{o}^{T} dT'' = \frac{1}{\Delta x}\,\frac{1}{\Delta y}\,\frac{T''}{T} \qquad (127)$$

which is a biased estimator, inasmuch as the operation of mathematical expecion gives:

$$E\left[\hat{p}_{x,y}(\xi,\eta,\tau)\right] = \frac{1}{\Delta x}\,\frac{1}{\Delta y}\,E\left[\frac{T''}{T}\right] = \frac{1}{\Delta x}\,\frac{1}{\Delta y}\,p_{x,y}\left(\xi \pm \frac{\Delta x}{2}\,,\ \eta \pm \frac{\Delta y}{2}\,,\ \tau\right) \qquad (128)$$

where the notation $p_{x,y}\left(\xi \pm \frac{\Delta x}{2}\,,\ \eta \pm \frac{\Delta y}{2}\,,\ \tau\right)$ represents the probability that the $x(t)$ variable is included between the value $\xi - \frac{\Delta x}{2}$ and the value $\xi + \frac{\Delta x}{2}$, and the $y(t+\tau)$ variable between the values $\eta - \frac{\Delta y}{2}$ and $\eta + \frac{\Delta y}{2}$:

$$p_{x,y}\left(\xi \pm \frac{\Delta x}{2}\,,\ \eta \pm \frac{\Delta y}{2}\,,\ \tau\right) = \int_{\xi-\Delta x/2}^{\xi+\Delta x/2}\int_{\eta-\Delta y/2}^{\eta+\Delta y/2} p_{x,y}(x,y,\tau)\,dx\,dy = \lim_{T\to\infty}\frac{T''}{T} \qquad (129)$$

The mathematical expectation of estimator (127) does not therefore give us the exact value of the joint probability density, but the mean of the values assumed by it for values of x included between $\xi - \Delta x/2$ and $\xi + \Delta x/2$, and of y included between $\eta - \Delta y/2$ and $\eta + \Delta y/2$. The bias of the estimate can therefore be reduced by adopting narrow windows Δx and Δy.

The estimator of the integral value of the joint probability density inside the above mentioned amplitude windows given by the expression:

$$\hat{p}_{x,y}\left(\xi \pm \frac{\Delta x}{2}\,,\ \eta \pm \frac{\Delta y}{2}\,,\ \tau\right) = \frac{1}{T}\int_{o}^{T} dT'' = \frac{T''}{T} \qquad (130)$$

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is, on the other hand, unbiased, as is evident from (129).
It is exceedingly difficult at this point to obtain the mean squared error of an estimate performed by estimator (127), or its variance, by strict methods. If we suppose windows Δx and Δy to be sufficiently small, the error due to the bias can be ignored, and the mean squared error and the variance can be considered to be equal; then, where $x(t)$ and $y(t)$ are both signals with limited frequency bands B and with uniform spectral behaviour within this band, we can adopt for the variance of estimator (127) the expression [15]) :

$$\mathrm{var}\left[\hat{p}_{x,y}(\xi,\eta,\tau)\right] = \frac{C^2\ p^2_{x,y}(\xi,\eta,\tau)}{B\ T\ \Delta x\ \Delta y}\ \frac{1}{\hat{p}_{x,y}(\xi,\eta,\tau)} \tag{131}$$

where C is a constant.
In the case of measurements repeated k times and averaged out, the estimator of the joint probability density becomes:

$$_k\hat{p}_{x,y}(\xi,\eta,\tau) = \frac{1}{k}\sum_{i=1}^{k}\frac{1}{\Delta x}\ \frac{1}{\Delta y}\ \frac{T''_i}{T_i} \tag{132}$$

while its variance can be expressed as:

$$\mathrm{var}\left[{}_k\hat{p}_{x,y}(\xi,\eta,\tau)\right] = \frac{C^2\ p^2_{x,y}(\xi,\eta,\tau)}{B\ T\ \Delta x\ \Delta y}\ \frac{1}{\hat{p}_{x,y}(\xi,\eta,\tau)}\ \frac{(k-1)\ \rho_k(\tau')+1}{k} \tag{133}$$

where for the $\rho_k(\tau')$ function, the considerations made in the case of the first order probability density are valid .

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8.) ESTIMATION OF THE PROBABILITY DENSITY BY THE S.D.A. ANALYZER

The first order probability density $p_x(\xi)$ and the joint probability density $p_{x,y}(\xi,\eta,\tau)$ are evaluated, in the S.D.A. analyzer, on the basis of estimators of types (122) and (132) , which are used in the form:

$$\hat{p}_x(\xi) = \frac{1}{k}\sum_{i=1}^{k} \frac{1}{\Delta x}\,\frac{f}{n}\int_{\tau_i}^{\tau_i+f/n} dT_i' \qquad (134)$$

$$\hat{p}_{x,y}(\xi,\eta,\tau) = \frac{1}{k}\sum_{i=1}^{k} \frac{1}{\Delta x}\,\frac{1}{\Delta y}\,\frac{f}{n}\int_{\tau_i}^{\tau_i+f/n} dT_i'' \qquad (135)$$

where the analysis time T_i of each of the k measurements is expressed as:

$$T_i = \frac{n}{f}$$

f being the frequency of the reference generator [17] and η the number of integration cycles which assumes here the significance of time extension factor [7].
The lag τ of the joint probability is given by [7]:

$$\tau = \frac{1}{2f}$$

Let us look now at the significance of the quantities α_i , β_i , γ_i and δ_i supplied by the analyzer [7]:

$$\alpha_i = 10^{-3}\int_{\tau_i}^{\tau_i+n/f} f_\xi \, dt \qquad (136)$$

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$$\beta_i = 10^{-3} \int_{\tau_i}^{\tau_i+n/f} f_t \, x_c(t) \, dt \qquad (137)$$

$$\gamma_i = 10^{-3} \int_{\tau_i}^{\tau_i+n/f} f_t \, x_c(t-\tau) \, y_c(t) \, dt \qquad (138)$$

$$\delta_i = 10^{-3} \int_{\tau_i}^{\tau_i+n/f} f_\eta \, dt \qquad (139)$$

where by ξ and η are indicated the voltage levels (relative to the variables $x(t)$ and $y(t)$ respectively) at which we perform the measurement, while f_ξ and f_η are the frequencies corresponding to these levels after the voltage to frequency conversion:

$$f_\xi = \frac{\xi}{h}$$

$$f_\eta = \frac{\eta}{h}$$

h being the proportionality factor for the said conversion.

Then the output of the window comparator to which the signal $x(t)$ is sent (together with the reference voltage ξ) is indicated by $x_c(t)$. Function $x_c(t)$ is defined by:

$$x_c(t) = 1 \quad \text{if} \quad \xi - \frac{\Delta x}{2} < x(t) < \xi + \frac{\Delta x}{2}$$

$$x_c(t) = 0 \quad \text{elsewhere}$$

because of which:

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$$\int_{\tau_i}^{\tau_i+n/f} x_c(t)\,dt = T_i' \tag{140}$$

Function $x_c(t)$, delayed of a time quantity τ , is indicated by $x_c(t-\tau)$, while $y_c(t)$ represents the output logical voltage of the window comparator to which signal $y(t)$ is sent together with the reference voltage η .
Function $y_c(t)$ is defined as:

$$y_c(t) = 1 \quad \text{if} \quad \eta - \frac{\Delta y}{2} < y(t) < \eta + \frac{\Delta y}{2}$$

$$y_c(t) = 0 \quad \text{elsewhere}$$

so that we have:

$$\int_{\tau_i}^{\tau_i+n/f} x_c(t-\tau)\, y_c(t)\,dt = T'' \tag{141}$$

Finally, f_t is the timing frequency according to which the times (measured by pulse counting) are measured. From everything that has been said, we obtain:

$$\alpha_i = 10^{-3} \int_{\tau_i}^{\tau_i+n/f} \frac{\xi}{h}\,dt = \frac{\xi}{h}\,\frac{n}{f}\,10^{-3} \tag{142}$$

$$\beta_i = 10^{-3} f_t \int_{\tau_i}^{\tau_i+n/f} x_c(t)\,dt = 10^{-3} f_t\, T' = 10^{-3} f_t\, \frac{n}{f}\, \hat{p}_x(\xi)\, \Delta x$$

(143)

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$$\gamma_i = 10^{-3} f_t \int_{\tau_i}^{\tau_i + n/f} x_c(t-\tau)\, y_c(t)\, dt = 10^{-3} f_t\, T'' =$$

$$= 10^{-3} f_t \frac{n}{f} \hat{p}_{x,y}(\xi,\eta,\tau)\, \Delta x\, \Delta y \qquad (144)$$

$$\delta_i = 10^{-3} \int_{\tau_i}^{\tau_i + n/f} \frac{\eta}{h}\, dt = \frac{\eta}{h} \frac{n}{f} 10^{-3} \qquad (145)$$

From these expressions we can then obtain the following relations which give the first order and joint probability densities, as well as the voltage levels at which we are performing the measurements, in terms of the well known quantities α_i , β_i , γ_i and δ_i :

$$\hat{\xi} = \frac{h\, 10^3}{k\, n} f \sum_{i=1}^{k} \alpha_i = \frac{1}{h_p} \frac{f}{k\, n} \sum_{i=1}^{k} \alpha_i \qquad (146)$$

$$\hat{p}_x(\xi) = \frac{10^3}{f_t\, k\, n\, \Delta x} f \sum_{i=1}^{k} \beta_i = \frac{1}{h_p^*} \frac{1}{\Delta x} \frac{f}{k\, n} \sum_{i=1}^{k} \beta_i \qquad (147)$$

$$\hat{p}_{x,y}(\xi,\eta,\tau) = \frac{10^3}{f_t\, k\, n\, \Delta x\, \Delta y} f \sum_{i=1}^{k} \gamma_i = \frac{1}{h_p^*} \frac{1}{\Delta x\, \Delta y} \frac{f}{k\, n} \sum_{i=1}^{k} \gamma_i \qquad (148)$$

$$\hat{\eta} = \frac{h\, 10^3}{k\, n} f \sum_{i=1}^{k} \delta_i = \frac{1}{h_p} \frac{f}{k\, n} \sum_{i=1}^{k} \delta_i \qquad (149)$$

In expressions (146) to (149) two different normalisation coeffi-

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cients appear. That of the probability densities is indicated by h_p^* and depends upon the timing frequency f_t [7]:

$$h_p^* = 10^{-3} f_t$$

while the normalisation coefficient relative to the reference voltages ξ and η depends upon the value of the proportionality factor h of the voltage to frequency conversion [7], and is indicated by h_p :

$$h_p = \frac{10^{-3}}{h}$$

For purposes of accuracy the values of the two coefficients in the six frequency decades [7] are here listed:

Decade	I	II	III	IV	V	VI
h_p	0,01	0,1	1	10	100	100
h_p^*	0,39262	3,9262	39,262	392,62	392,62	392,62

From the above table of values it can be seen that:

$$h_p^* = h_p \cdot 39,262 \qquad \text{if } h_p \leq 10$$

$$h_p^* = 10 \cdot 39,262 \qquad \text{if } h_p > 10$$

Therefore, having stored the values of h_p in the normalisation coefficient diode matrix [7] [17], it is possible to obtain the values of h_p^* from those of h_p by means of an appropriate calculation programme (see Appendix C4).
A sequential diagram of the S.D.A. operation for calculating the probability densities is shown in fig.3 .

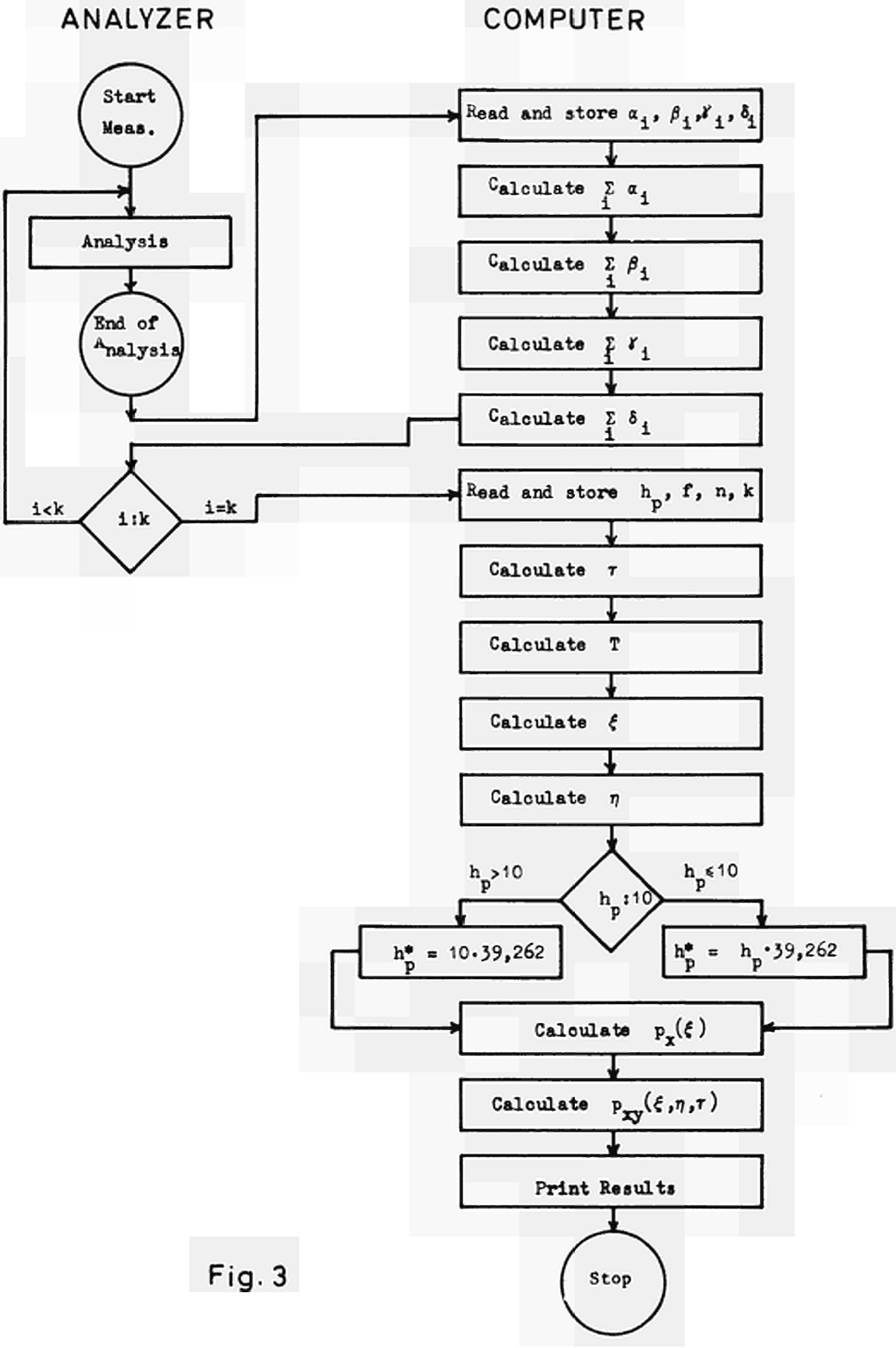
ANALYZER
COMPUTER
Start Meas.
Analysis
End of Analysis
i<k
i:k
i=k
Read and store $\alpha_i, \beta_i, \gamma_i, \delta_i$
Calculate $\sum_i \alpha_i$
Calculate $\sum_i \beta_i$
Calculate $\sum_i \gamma_i$
Calculate $\sum_i \delta_i$
Read and store h_p, f, n, k
Calculate τ
Calculate T
Calculate ξ
Calculate η
$h_p > 10$
$h_p : 10$
$h_p \leq 10$
$h_p^* = 10 \cdot 39{,}262$
$h_p^* = h_p \cdot 39{,}262$
Calculate $p_x(\xi)$
Calculate $p_{xy}(\xi, \eta, \tau)$
Print Results
Stop

Fig. 3

APPENDIX A

Given a stationary ergodic random signal $x(t)$, let us consider the estimator $\hat{\mu}_x$ of its mean value:

$$\hat{\mu}_x = \frac{1}{k} \frac{1}{T} \sum_{i=1}^{k} \int_{t_i}^{t_i+T} x(t)\, dt \qquad (A.1)\ .$$

and the variance of the estimate, defined as:

$$\mathrm{var}\left[\hat{\mu}_x\right] = E\left[\hat{\mu}_x^2\right] - \left\{ E\left[\hat{\mu}_x\right] \right\}^2 \qquad (A.2)$$

so that:

$$\mathrm{var}\left[\hat{\mu}_x\right] = E\left[\frac{1}{k^2} \sum_{i=1}^{k} \left(\frac{1}{T} \int_{t_i}^{t_i+T} x(t)\, dt \right)^2 \right] +$$

$$+ E\left[\frac{1}{k^2} \sum_{\substack{i,j=1 \\ i\neq j}}^{k} \frac{1}{T^2} \int_{t_i}^{t_i+T} x(t)\, dt \int_{t_j}^{t_j+T} x(t)\, dt \right] - \mu_x^2 \qquad (A.3)$$

If we assume, for simplicity of notation, the instant t_i of the ith repetition, to be the origin of the time axis, and if we indicate by $x_i(t)$ the variable $x(t)$ in the time interval between t_i and t_i+T , the first term of the right-hand member of (A.3) becomes:

$$E\left[\frac{1}{k^2} \sum_{i=1}^{k} \frac{1}{T^2} \left(\int_{o}^{T} x_i(t)\, dt \right)^2 \right] = \frac{1}{k^2} \sum_{i=1}^{k} \frac{1}{T^2} \int_{o}^{T} \int_{o}^{T} R_{xx}(t-\tau)\, dt\, d\tau$$

(A.4)

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where the final expression has been obtained by considering the square of the integral as a double integral and then by interchanging the operation of mathematical expectation and the operation of integration.

Remembering now that:

$$R_{xx}(\tau) = C_{xx}(\tau) + \mu_x^2 \tag{A.5}$$

and partially resolving the double integral, we finally arrive at the following expression :

$$\frac{1}{k^2} \sum_{i=1}^{k} \frac{1}{T}\int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right) C_{xx}(\tau)\, d\tau + \frac{\mu_x^2}{k} \tag{A.6}$$

for the first term of the right-hand member of (A.3).

If we develop in a similar manner the second term, it becomes:

$$\frac{1}{k^2}\sum_{\substack{i,j=1\\ i\neq j}}^{k} \frac{1}{T^2}\, E\left[\int_0^T\int_0^T x_i(t)\, x_j(\tau)\, dt\, d\tau\right] = \frac{1}{k^2}\sum_{\substack{i,j=1\\ i\neq j}}^{k} \frac{1}{T^2}\int_0^T\int_0^T R_{x_i x_j}(t-\tau)\, dt\, d\tau \tag{A.7}$$

which, on the basis of (A.5) and relation:

$$C_{xy}(\tau) = \sqrt{C_{xx}(0)\,.\,C_{yy}(0)}\,.\,\rho_{xy}(\tau) \tag{A.8}$$

becomes:

$$\frac{1}{k^2}\sum_{\substack{i,j=1\\ i\neq j}}^{k} \frac{1}{T^2}\int_0^T\int_0^T \sqrt{C_{x_i x_i}(0)\; C_{x_j x_j}(0)}\,.\,\rho_{x_i x_j}(t-\tau)\, dt\, d\tau + \frac{k-1}{k}\,\mu_x^2 =$$

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$$= \frac{1}{k^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{k} \frac{1}{T} \int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right) C_{xx}(0)\, \rho_{x_i x_j}(\tau)\, d\tau + \frac{k-1}{k}\, \mu_x^2 \tag{A.9}$$

By substituting (A.6) and (A.9) respectively for the first and second term of the second member of (A.3), we have for the variance of the estimate the expression :

$$\mathrm{var}\left[\hat{\mu}_x\right] = \frac{1}{k^2} \left[\sum_{i=1}^{k} \frac{1}{T} \int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right) C_{xx}(0)\, \rho_{xx}(\tau)\, d\tau \, + \right.$$

$$\left. + \sum_{\substack{i,j=1 \\ i \neq j}}^{k} \frac{1}{T} \int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right) C_{xx}(0)\, \rho_{x_i x_j}(\tau)\, d\tau \right] \tag{A.10}$$

In order to point out the difference between expression (18) and the more rigorous expression (A.10), it is useful to remember that, in relation (18) :

$$\rho_{ij}(\tau) = \rho_{xx}(\Delta_{ij}) \tag{A.11}$$

while in expression (A.10) :

$$\rho_{x_i x_j}(\tau) = \rho_{xx}(\Delta_{ij}+\tau) \tag{A.12}$$

Δ_{ij} being the time interval between the ith and the jth measurements.

If the k measurements are mutually indipendent (i.e. in the most favorable case), expression (A.10) becomes:

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$$\text{var}\left[\hat{\mu}_x\right] = \frac{1}{k}\,\frac{1}{T}\int_{-T}^{T}\left(1 - \frac{|\tau|}{T}\right) C_{xx}(0)\,\rho_{xx}(\tau)\,d\tau \tag{A.13}$$

as $\rho_{x_i x_j}(\tau)$ is equal to zero.

In the least favorable case, $\rho_{x_i x_j}(\tau)$ can be considered to be equal to $\rho_{xx}(\tau)$, because of which the repetitions have no effect on the variance :

$$\text{var}\left[\hat{\mu}_x\right] = \frac{1}{T}\int_{-T}^{T}\left(1 - \frac{|\tau|}{T}\right) C_{xx}(0)\,\rho_{xx}(\tau)\,d\tau \tag{A.14}$$

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APPENDIX B

Let us consider two stationary ergodic random signals, $x(t)$ and $y(t)$, and estimator (65) of their cross-covariance function. Expression (65) may be rearranged, so that:

$$\hat{C}_{xy}(\tau) = \frac{1}{T}\int_0^T x(t)\, y(t+\tau)\, dt - \frac{1}{T^2}\int_0^T\int_0^T x(t)\, y(\xi)\, dt\, d\xi =$$

$$= \frac{1}{T^2}\int_0^T\int_0^T x(t)\left[y(t+\tau) - y(\xi)\right] dt\, d\xi \tag{B.1}$$

The variance of this estimate is defined as:

$$\operatorname{var}\left[\hat{C}_{xy}(\tau)\right] = E\left\{\left[\hat{C}_{xy}(\tau)\right]^2\right\} - \left\{E\left[\hat{C}_{xy}(\tau)\right]\right\}^2 \tag{B.2}$$

where the expectation of the estimate is given by (see expression (68)):

$$E\left[\hat{C}_{xy}(\tau)\right] = C_{xy}(\tau) - \frac{1}{T^2}\int_0^T\int_0^T C_{xy}(\theta - t)\, dt\, d\theta \tag{B.3}$$

We can now evaluate the variance of the estimate by substitution of (B.1) and (B.3) in (B.2):

$$\operatorname{var}\left[\hat{C}_{xy}(\tau)\right] =$$

$$= E\left[\frac{1}{T^4}\int_0^T\int_0^T\int_0^T\int_0^T x(t)x(\theta)\left[y(t+\tau) - y(\xi)\right]\left[y(\theta+\tau) - y(\eta)\right] dt\, d\theta\, d\xi\, d\eta\right] +$$

$$- \left[C_{xy}(\tau) - \frac{1}{T^2}\int_0^T\int_0^T C_{xy}(\theta - t)\, dt\, d\theta\right]^2 \tag{B.4}$$

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To develope the operation of mathematical expectation in the first term of the right-hand member of the above formula, we apply expression (30) , letting:

$$x_1 = x(t) \qquad x_2 = x(\theta)$$
$$x_3 = y(t+\tau) - y(\xi) \qquad x_4 = y(\theta+\tau) - y(\eta) \tag{B.5}$$

In terms of covariance functions, we have:

$$\operatorname{var}\left[\hat{C}_{xy}(\tau)\right] =$$

$$= \frac{1}{T^4}\int_0^T\int_0^T\int_0^T\int_0^T \Big\{ \left[C_{xx}(\theta-t) + \mu_x^2\right]\left[C_{yy}(\theta-t) - C_{yy}(\theta+\tau-\xi) + \right.$$

$$\left. - C_{yy}(\eta-t-\tau) + C_{yy}(\eta-\xi)\right] + \left[C_{xy}(\tau) - C_{xy}(\xi-t)\right]\left[C_{xy}(\tau) - C_{xy}(\eta-\theta)\right] +$$

$$+ \left[C_{xy}(\theta+\tau-t) - C_{xy}(\eta-t)\right]\left[C_{xy}(t+\tau-\theta) - C_{xy}(\xi-\theta)\right]\Big\}\, dt\, d\theta\, d\xi\, d\eta +$$

$$- C_{xy}^2(\tau) - \frac{1}{T^4}\int_0^T\int_0^T\int_0^T\int_0^T C_{xy}(\theta-t)\, C_{xy}(\eta-\xi)\, dt\, d\theta\, d\xi\, d\eta +$$

$$+ C_{xy}(\tau)\,\frac{2}{T^2}\int_0^T\int_0^T C_{xy}(\theta-t)\, dt\, d\theta \tag{B.6}$$

Developing the products of the integrand of the first integral on the right we obtain:

$$\operatorname{var}\left[\hat{C}_{xy}(\tau)\right] =$$

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$$= \frac{1}{T^2}\int_0^T\int_0^T\left[C_{xx}(\theta-t)\,C_{yy}(\theta-t) + C_{xy}(\theta-t+\tau)\,C_{yx}(\theta-t-\tau)\right]dt\,d\theta\,+$$

$$-\frac{2}{T^3}\int_0^T\int_0^T\int_0^T\left[C_{xx}(\xi-\theta)\,C_{yy}(\theta+\tau-t) + C_{xy}(\xi-\theta)\,C_{xy}(\theta+\tau-t)\right]dt\,d\theta\,d\xi\,+$$

$$+\frac{1}{T^4}\int_0^T\int_0^T\int_0^T\int_0^T\left[C_{xx}(\theta-t)\,C_{yy}(\eta-\xi) + C_{xy}(\theta-t)\,C_{xy}(\eta-\xi)\right]dt\,d\theta\,d\xi\,d\eta\,+$$

$$+\mu_x^2\cdot\frac{1}{T^2}\int_0^T\int_0^T C_{yy}(\theta-t)\,dt\,d\theta - \mu_x^2\cdot\frac{1}{T^2}\int_0^T\int_0^T\left[C_{yy}(\theta+\tau-t)+C_{yy}(\theta-\tau-t)\right]dt\,d\theta$$

(B.7)

Expression (B.3) shows that estimator (B.1) is biased; its bias is given by:

$$\text{bias}\left[\hat{C}_{xy}(\tau)\right] = \frac{1}{T^2}\int_0^T\int_0^T C_{xy}(\theta-t)\,dt\,d\theta \tag{B.8}$$

Remembering now that:

$$\text{m.s.e.}\left[\hat{C}_{xy}(\tau)\right] = \text{var}\left[\hat{C}_{xy}(\tau)\right] + \text{bias}^2\left[\hat{C}_{xy}(\tau)\right] \tag{B.9}$$

we can, taking into account expressions (B.7) and (B.8), and partially solving the multiple integrals, arrive at the following expression of the mean squared error of the estimate:

$$\text{m.s.e.}\left[\hat{C}_{xy}(\tau)\right] =$$

./.

$$= \frac{1}{T}\int_{-T}^{T}\left(1 - \frac{|\lambda|}{T}\right)\left[C_{xx}(\lambda)\ C_{yy}(\lambda) + C_{xy}(\lambda+\tau)\ C_{yx}(\lambda-\tau)\right] d\lambda\ +$$

$$- \frac{2}{T^3}\int_{o}^{T}\int_{o}^{T}\int_{o}^{T}\left[C_{xx}(\xi-\theta)C_{yy}(\theta+\tau-t) + C_{xy}(\xi-\theta)\ C_{xy}(\theta+\tau-t)\right] dt\ d\theta\ d\xi\ +$$

$$+ \frac{1}{T^2}\int_{-T}^{T}\int_{-T}^{T}\left(1- \frac{|\lambda|}{T}\right)\left(1- \frac{|\lambda'|}{T}\right)\left[C_{xx}(\lambda)C_{yy}(\lambda')+2C_{xy}(\lambda')\,C_{xy}(\lambda)\right] d\lambda\ d\lambda'\ +$$

$$+ \frac{\mu_x^2}{T}\int_{-T}^{T}\left(1- \frac{|\lambda|}{T}\right)\left[2\ C_{yy}(\lambda) - C_{yy}(\lambda+\tau) - C_{yy}(\lambda-\tau)\right] d\lambda \tag{B.10}$$

As already said in chapter 4, this expression is too complex to have any practical interest.

APPENDIX C

C.1.) INTRODUCTION

Some S.D.A. statistical dynamics analyzer computer programmes for estimating correlation functions, probability densities and for D.V.M. operations, are described here.

These programmes require the use of an Olivetti P102 as the final element of the analyzer.

Let us remember that all the programmes have a common structure, each of them being composed of four parts [1]: the first one (from AZ to AV) designed for the reading and storage of data; the second one (from AV to Z) for the elaboration of these data, and the third part (AW ... Z) for averaging and normalizing the quantities obtained in the second part and for printing the results.
A fourth part, dependent upon a conditional jump (/V), directs the sequence of operations related to on overload of the system [1] [7].

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C.2.) PROGRAMMES FOR D.V.M. OPERATION

We saw in chapter 5 that the S.D.A. analyzer, used as a digital voltmeter, supplies the mean values and mean absolute values of the signals $x(t)$ and $y(t)$ under examination.

The appropriate programme is described in Table C.2.1.; it supplies the following data as results:

k	number of repetitions
T	analysis time, in seconds
$\hat{\overline{x(t)}}$	in volts (tenths of the full scale of the input amplifier)
$\hat{\overline{\lvert x(t)\rvert}}$	" " " " " " " " " " "
$\hat{\overline{y(t)}}$	" " " " " " " " " " "
$\hat{\overline{\lvert y(t)\rvert}}$	" " " " " " " " " " "

In Table C.2.2. a modified programme is shown, which supplies, point by point, the static characteristics of a plant or process of which $x(t)$ and $y(t)$ represent the input and output voltages respectively. The results are printed in the following order:

k	
T	in seconds
$\hat{\overline{x(t)}}$	in volts (tenths of the F.S. of the input amplifier)
$\hat{\overline{\lvert x(t)\rvert}}$	" " " " " " " " " "
$\hat{\overline{y(t)}}$ / $\hat{\overline{x(t)}}$	
$\hat{\overline{\lvert y(t)\rvert}}$ / $\hat{\overline{\lvert x(t)\rvert}}$	

./.

TABLE C.2.1.

PROGRAMME INSTRUCTIONS					
Reg. 1		Reg. 2		Reg. F	
1	A Z	25	↓	49	C ↓
2	A S	26	/ V	50	D :
3	D / ↑	27	Y	51	A ⟠
4	A S	28	A / V	52	A Y
5	D ↑	29	A ⟠	53	B / *
6	A S	30	E / x	54	B *
7	E / ↑	31	D / :	55	C / *
8	A S	32	D ↕	56	C *
9	A V	33	A / ↑	57	Z
10	↓	34	R ↓	58	
11	C +	35	R S	59	
12	C ↕	36	D / S	60	
13	D / ↓	37	↓	61	
14	B / +	38	E / x	62	
15	B / ↕	39	A ⟠	63	
16	D ↓	40	B / ↓	64	
17	B +	41	D :	65	
18	B ↕	42	A ⟠	66	
19	E / ↓	43	B ↓	67	
20	C / +	44	D :	68	
21	C / ↕	45	A ⟠	69	
22	Z	46	C / ↓	70	
23	A W	47	D :	71	
24	/ ⟠	48	A ⟠	72	

TABLE C.2.2.

PROGRAMME INSTRUCTIONS					
	Reg. 1		Reg. 2		Reg. F
1	A Z	25	↓	49	C ↓
2	A S	26	/ V	50	B :
3	D / ↑	27	Y	51	A ⇳
4	A S	28	A / V	52	A Y
5	D ↑	29	A ⇳	53	B / *
6	A S	30	E / x	54	B *
7	E / ↑	31	D / :	55	C / *
8	A S	32	D ↕	56	C *
9	A V	33	A / ↑	57	Z
10	↓	34	R ↓	58	
11	C +	35	R S	59	
12	C ↕	36	D / S	60	
13	D / ↓	37	↓	61	
14	B / +	38	E / x	62	
15	B / ↕	39	A ⇳	63	
16	D ↓	40	B / ↓	64	
17	B +	41	D :	65	
18	B ↕	42	A ⇳	66	
19	E / ↓	43	B ↓	67	
20	C / +	44	D :	68	
21	C / ↕	45	A ⇳	69	
22	Z	46	C / ↓	70	
23	A W	47	B / :	71	
24	/ ⇳	48	A ⇳	72	

C.3.) PROGRAMME FOR THE CALCULATION OF THE AUTO - AND CROSS-CORRELATION FUNCTIONS

Table C.3.1. gives the programme for evaluating the correlation functions of two signals $x(t)$ and $y(t)$; this programme allows one to calculate the autocorrelation function $R_{xx}(\tau)$ of the signal $x(t)$ and the cross-correlation function $R_{xy}(\tau)$, as well as the mean square values $R_{xx}(0)$ and $R_{yy}(0)$ of the two signals being examined.

The results are printed in the following order:

τ	lag of the correlation, in seconds
k	number of repetitions
n	
$\hat{R}_{xx}(0) = \hat{m}_{2,x}$	in squared volts
$\hat{R}_{xx}(\tau)$	" " "
$\hat{R}_{xy}(\tau)$	" " "
$\hat{R}_{yy}(0) = \hat{m}_{2,y}$	" " "

Normalised correlation functions can be evaluated with the programme listed in Table C.3.2. This gives the following results:

τ	in seconds
k	
n	
$\hat{R}_{xx}(0) = \hat{m}_{2,x}$	
$\hat{R}_{xx}(\tau) / \hat{R}_{xx}(0)$	
$\hat{R}_{xy}(\tau) / \sqrt{\hat{R}_{xx}(0) \cdot \hat{R}_{yy}(0)}$	
$\hat{R}_{yy}(0) = \hat{m}_{2,y}$	

./.

TABLE C.3.1.

PROGRAMME INSTRUCTIONS					
Reg. 1		Reg. 2		Reg. F	
1	A Z	25	E ↑	49	A ⇕
2	A S	26	D √	50	C ↓
3	D / ↑	27	:	51	D / :
4	A S	28	D :	52	A ⇕
5	D ↑	29	A ⇕	53	A Y
6	A S	30	E ↓	54	B / *
7	E / ↑	31	/ V	55	B *
8	A S	32	Y	56	C / *
9	A V	33	A / V	57	C *
10	↓	34	A ⇕	58	Z
11	C +	35	E / ⇕	59	
12	C ↕	36	E / ↓	60	
13	D / ↓	37	D :	61	
14	B / +	38	E x	62	
15	B / ↕	39	D / x	63	
16	D ↓	40	D / ↕	64	
17	B +	41	B / ↓	65	
18	B ↕	42	D / :	66	
19	E / ↓	43	A ⇕	67	
20	C / +	44	B ↓	68	
21	C / ↕	45	D / :	69	
22	Z	46	A ⇕	70	
23	A W	47	C / ↓	71	
24	/ ⇕	48	D / :	72	

TABLE C.3.2.

PROGRAMME INSTRUCTIONS					
Reg. 1		Reg. 2		Reg. F	
1	A Z	25	E ↑	49	C √
2	A S	26	D √	50	B / x
3	D / ↑	27	:	51	B / ↕
4	A S	28	D :	52	C / ↓
5	D ↑	29	A ◊	53	B / :
6	A S	30	E ↓	54	A ◊
7	E / ↑	31	/ V	55	C ↓
8	A S	32	Y	56	D / :
9	A V	33	A / V	57	A ◊
10	↓	34	A ◊	58	A Y
11	C +	35	E / ◊	59	B / *
12	C ↕	36	E / ↓	60	B *
13	D / ↓	37	D :	61	C / *
14	B / +	38	E x	62	C *
15	B / ↕	39	D / x	63	Z
16	D ↓	40	D / ↕	64	
17	B +	41	B / ↓	65	
18	B ↕	42	D / :	66	
19	E / ↓	43	A ◊	67	
20	C / +	44	B ↓	68	
21	C / ↕	45	B / :	69	
22	Z	46	A ◊	70	
23	A W	47	B / √	71	
24	/ ◊	48	B / ↕	72	

C.4.) PROGRAMMES FOR DETERMINING THE PROBABILITY DENSITIES

A programme which allows the calculation of the first order probability density is described in Table C.4.1. Results will be printed in the following order:

T analysis time, in seconds

ξ comparison level, in volts

$\hat{p}_x(\xi)$

The probability density is normalized for the case of an amplitude window Δx of 400 millivolts [7] [14] (1/50 of voltage dynamic of the input amplifiers); therefore the first order probability density $p_x(\xi)$ assumes values between 0 and 50.

If the 40 millivolt window [7] [14] is used for the probability analysis (1/500 of the amplifier dynamic), it is necessary to normalyse for the new Δx ; to do this it is sufficient to codify the constant 500 instead of the constant 50 (coded in the instructions number 45, 46, 47). The codification of 500 is given by the following instructions:

A / ↑
R / S
R S
D -

Table C.4.2. describes a programme which calculates both the first order probability density and the joint probability density.

The results are printed in the following order:

τ lag of the joint probability, in seconds

T analysis time, in seconds

ξ comparison level for the signal $x(t)$, in volts

./.

η comparison level for the signal $y(t+\tau)$, in volts

$\hat{p}_x(\xi)$

$\hat{p}_{x,y}(\xi,\eta,\tau)$

The first order probability density $p_x(\xi)$, is normalised for an amplitude window Δx equal to 1/50 of the voltage dynamic of the input amplifiers.

The joint probability density, $p_{x,y}(\xi\ \eta,\tau)$, is normalized for amplitude windows Δx and Δy equal to 1/50 and 1/32 of the voltage dynamic respectively [7] [14].

Therefore, the values assumed by $p_x(\xi)$ and $p_{x,y}(\xi,\eta,\tau)$ can vary between 0 and 50 , and 0 and 1600 respectively.

It is important to note that the programme shown in Table C.4.2. is valid only where k (number of repetitions) is equal to one: the programme has not left any memory positions free for making the necessary summations for the case where k is not equal to one.

./.

TABLE C.4.1.

PROGRAMME INSTRUCTIONS					
Reg. 1		Reg. 2		Reg. F	
1	A Z	25	A ◊	49	B :
2	A S	26	E x	50	A ◊
3	D / ↑	27	D / x	51	A Y
4	A S	28	B / ↕	52	B / *
5	D ↑	29	B / :	53	B *
6	A S	30	A ◊	54	C / *
7	E / ↑	31	A / ↑	55	C *
8	A S	32	R ↑	56	Z
9	A V	33	R x	57	A / W
10	D / ↓	34	R ↑	58	D / $\sqrt{}$
11	B / +	35	R / *	59	↕
12	B / ↕	36	D ↕	60	E ↓
13	D ↓	37	E ↑	61	:
14	B +	38	D / ↓	62	↕
15	B ↕	39	-	63	C V
16	Z	40	/ W	64	
17	A W	41	B V	65	
18	/ ◊	42	B / ↓	66	
19	E ↑	43	x	67	
20	/ V	44	B ↕	68	
21	Y	45	A / ↑	69	
22	A / V	46	R / S	70	
23	E / ↓	47	D -	71	
24	D :	48	x	72	

TABLE C.4.2.

PROGRAMME INSTRUCTIONS					
Reg. 1		Reg. 2		Reg. F	
1	A Z	25	E ↓	49	B V
2	A S	26	/ V	50	B / ↓
3	D / ↑	27	Y	51	:
4	A S	28	A / V	52	B ↕
5	D ↑	29	E / x	53	A / ↑
6	A S	30	D :	54	R / ↑
7	E / ↑	31	A ◊	55	D ↕
8	A S	32	D / x	56	:
9	A V	33	B / ↕	57	B :
10	C ↑	34	B / :	58	A ◊
11	D / ↓	35	A ◊	59	C / ↓
12	B / ↕	36	C ↓	60	B :
13	D ↓	37	B / :	61	A ◊
14	B ↕	38	A ◊	62	A Y
15	E / ↓	39	A / ↑	63	B / *
16	C / ↕	40	R ↑	64	B *
17	Z	41	R -	65	C / *
18	A W	42	R :	66	C *
19	/ ◊	43	R / S	67	Z
20	E ↑	44	D +	68	A / W
21	D $\sqrt{}$	45	E ↑	69	D / $\sqrt{}$
22	:	46	D / ↓	70	E x
23	D :	47	-	71	↕
24	A ◊	48	/ W	72	C V

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