PLASMA PHYSICS

EURATOM

1969

4268

PLASMA PHYSICS

1969

PLASMA PHYSICS

BY

J. G. LINHART

EURATOM/CNEN Association

Frascati (Rome) Italy

1969

Published by
the European Atomic Energy Community — EURATOM
Brussels (Belgium), OCTOBER 1969

EUR 4268 e

PREFACE TO THE THIRD EDITION

During the years preceding 1960 plasma physics developped rapidly and the pioneering mood of those days influenced many, including the author of this book, to come out with articles and books in which a fermenting, partial and somewhat hasty spirit prevails. The times have changed and nowadays it is possible to attempt to write a text-book rather than a work demonstrating novel or revolutionary features of the subject. This edition is meant for graduate and post-graduate students; wherever possible the treatment insists on physical insight rather than on mathematical rigour. I have tried to treat the subject in all its aspects without indulging in details and specialities (for which exist now several excellent books) and nevertheless aiming at students having a good background in physics and even having some acquaintance with the concepts of physics of ionized gases. This should put the present edition in a bracket between elementary books on plasma physics and those dealing thoroughly with special domains, some of which will be found among the recommended literature on p. 332.

Frascati, spring 1967.

Acknowledgment

I would like to thank those of my colleagues from Laboratorio Gas Ionizzati who have helped me so generously with the manuscript of the 3rd edition.

CONTENTS

CHAPTER 5: WAVES AND INSTABILITIES IN PLASMA

CHAPTER 6: SHOCK WAVES IN PLASMA

CHAPTER 7: PLASMA DYNAMICS

The numbering of the equations in each chapter starts from one. When reference is made to equations in other chapters, the number of the chapter is added first (e.g., Equation (3.62) means Equation (62) in Chapter 3).

INTRODUCTION

Plasma physics is concerned with the behaviour of systems of many free electrons and ionised atoms where the mutual Coulomb interactions cannot be disregarded. In a restricted sense, such systems of particles consist of nearly equal numbers of positive and negative charges. Systems of this type are examples of a medium known as plasma which in many respects behaves differently from the solid, liquid and gaseous state of matter.

All states of matter represent different degrees of organization, to which there correspond certain values of binding energy. Thus, in the solid state the important quantity is the binding energy of molecules in a crystal; in fact, a crystal could be considered as a macro- or super-molecule. If the average kinetic energy per molecule W exceeds the binding energy U (a fraction of an eV) the crystal structure breaks up, either into a liquid or directly into a gas. A similar law operates in the case of liquids, and in order to change a liquid into a gas, a certain minimum kinetic energy per molecule is required to break the bonds of the van der Waals forces. Matter can exist as plasma, i.e., in its fourth state, when the kinetic energy W per plasma particle exceeds the ionising potential of atoms which is usually a few eV. Thus the average kinetic energy per particle determines the state in which matter exists. A precise mathematical statement of this theorem is an equation of the Saha type. However, a simple criterion can be written as

$$U_n < W < U_{n+1} \tag{1}$$

where U_n, U_{n+1} are the respective binding energies, expressing that matter exists in the $(n + 1)$st state.

The plasma state will correspond to an order-of-magnitude relationship

$$1 < W_4 < 10^6 \qquad \text{(eV)}.$$

Extrapolating this principle to higher states of matter, so far unexplored, one may define the fifth state of matter as one in which

$$1 < W_5 < 500 \qquad \text{(MeV)}.$$

This will be a gas of free nucleons and electrons — a “nugas”. The sixth state would be, consequently, defined as

$$\tfrac{1}{2} < W_6 < 10 \qquad \text{(GeV)}$$

and would contain free mesons, nucleons in various states of excitation and electrons. The fifth and sixth states of matter can be expected to exhibit an even greater variety of behaviour than a plasma owing to the action of short range internucleon forces in addition to long range Coulomb forces.

On the other hand, according to eq. (1) for W, plasma spans a broader energy band than any other state of matter; it encompasses about 20 octaves on the kinetic energy scale. This width of the kinetic energy spectrum of the plasma state is the reason for much common ground between plasma physics and many other fields of physics, such as the dynamics of single charged particles (in which many-particle interactions are not considered), or the physics of electrical discharges in gases (in which interaction between charged particles and neutral atoms and molecules is of great importance), whereas some methods of description and analysis used in plasma physics belong to the subject of hydrodynamics, particularly magneto-hydrodynamics. Another physical discipline, indispensable for the theory of a plasma, is statistical mechanics and there are yet other fields from which plasma physics draws its mathematical formulation and its terminology.

Although probably more than 99.9 % of matter in our Universe is ionised and therefore in the plasma state, on our planet plasma has to be generated by special physical processes and under special conditions. These processes are the subject of the physics of electrical discharges in gases and this is the reason for the parental relationship between the latter and plasma physics.

Using an anthropomorphic analogy one may say that whereas the physics of electrical discharges is more specifically concerned with the birth and metabolism of plasma, plasma physics concentrates mostly on the anatomy and motion of plasma.

On our planet the medium which often resembles an ideal plasma is a partially ionized gas. This medium enters in the experience of prehistoric humanity in three forms; as fire, as lightning and as Aurora Borealis. In this connection it is curious to note that a number of greek philosophers, starting with Empedocles of Agrigentum (about 490-430 B.C.), held that the material Universe is built of four "roots": earth, water, air and fire. This, in modern terminology, may be compared with four states of matter, solid, liquid, gaseous and the plasma state. The privilege of identifying the medium created in electrical discharges in gases as the fourth state of matter belongs to W. Crookes who writes (1879): "The phenomena in these exhausted tubes reveal to physical science a new world, a world where matter may exist in a fourth state...". At about this time it became obvious that this newly discovered state of matter is not very much at home on our dense and cold planet and that special conditions must be realized in order to generate a plasma-like medium in the laboratory. Investigation of these conditions were the subject of the physics of electrical discharges in gases. It was only when electrical and vacuum techniques developed to the point when long-lived and relatively stable electrical discharges were available that plasma physics emerged as a separate field of study.

Around 1923 I. Langmuir developed the appropriate basic theory of an ionised gas and gave the medium the name "plasma" *. During the period 1923-1938 the subject developed further due to the efforts of L. Tonks, R. Seeliger, B. Klarfeld, M. Steenbeck, A. v. Engel, L.B. Loeb, W. Bennett, F.M. Penning, J. Townsend, W. Rogowski and many others.

At the beginning of this century astrophysicists became aware of the importance played by ionised matter in the processes in outer space and subsequently some of the finest contributions to plasma physics came from their ranks. Here one may mention the work of M.N. Saha, S. Chapman, T.G. Cowling, V.C. Ferraro, S. Chandrasekhar, L. Spitzer, H. Alfvén and the german astrophysical school at the Max-Planck Institute.

In 1929 F. Houtermanns and R. Atkinson suggested that the main source of energy in stars is the fusion reactions among the nuclei of the light elements. After 1945 a similar mechanism was exploited in the construction of hydrogen bombs and at the same time some physicists became interested in a controlled release of fusion energy.

However, it was appreciated that the energy output from fusion reactions depends critically on the kinetic energy of the colliding nuclei and that fusion outputs of practical interest depend on one's ability to produce temperatures of at least several million degrees Kelvin. If explosions are to be avoided, then the pressure of matter at this temperature must be balanced by external forces. This is within the power of our engineers only if the density of the nuclear fuel is substantially less than the density of our atmosphere. The search for a mechanism of a controlled release of fusion energy in the 1950's became, therefore, synonymous with the study of high temperature, low density plasmas. However, it should not be forgotten that the notion of controlled fusion is not inconsistent with controlled explosions as will be mentioned on p. 308. In accord with such an extension of the scope of

* The word plasma occurs first in the term protoplasma which was originally introduced into scientific terminology in 1839 by the czech biologist J. Purkynie for the jelly-like medium interspersed by numerous particles which constitutes the body of cells.

controlled fusion is also the recent extension of our interest to very high density plasmas.

The prospect of nuclear fusion gave a new lease of life to plasma physics which was becoming rather unfashionable and as one of my friends put it, regarded by most other physicists as a rather charming subject, full of small, colourful experiments, where there was little left to discover and whose only real justification was the amusement of those who bothered to waste their time on it. With the goal of a fusion reactor as an incentive, plasma physics became a subject of interest to many physicists and engineers. More recently many other applications of plasma physics have appeared, such as plasma rockets, direct conversion of thermal energy into electrical energy, transmission of radio and television signals through ionosphere and others. These are more than able to sustain the interest of physicists and engineers in plasmas.

When the first edition of this book was written in 1959 only very few experiments on plasma had been carried out and those that had been done were useful only for a general orientation and could not be compared with clear and precise experiments in other branches of physics. In five years this situation has changed considerably and there exist now some "classical" experiments on waves, shocks, diffusion and dynamics of plasma. I have attempted, therefore, to illustrate at least some of the theoretical statements by means of related experiments. Owing to the interplay of theory and experiment in the last decade it was possible to gain a feeling for the plasma medium, appreciate its many aspects which predominate according to the values assumed by the density and by the temperature of the plasma and also according to whether or not there is a magnetic field in the plasma. In order to transmit some of this feeling the book starts with a rather lengthy chapter on the general properties of the different types of plasma.

As in many problems dealing with a large ensemble of individuals (e.g., star clusters), plasma physics uses two complementary modes of description: the analysis of the movement of a single particle and the fluid model. These two treatments are the subjects of chapters 2 and 3.

These modes of description are subsequently applied to equilibrium configurations, i.e., to plasma statics (chapter 4), to wave-motion and instabilities in plasma (chapter 5) and to shocks in plasma (chapter 6). This brings us to consider the dynamics of a plasma (chapter 7).

In order to complete our description of plasma it is important to know how an equilibrium configuration is established. This problem can be solved only if one can find a suitable description of the various collision, diffusion and radiation processes that are operative in arriving at an equilibrium. This is the aim of chapter 8.

The eight chapters provide us with models of plasma processes which are used in chapter 9 to describe some of the applications of plasma physics to the research on the controlled fusion of light nuclei, and in chapter 10 to electronics and to other problems in applied physics and in engineering. Those who will be using this book as a text book may not want to get involved with some of the more complicated mathematical arguments. In such a case it may be advisable to only gloss over paragraphs marked by an asterisk. The c.g.s. system of units will be used unless specified otherwise.

CHAPTER I

1.1. Plasma State

Let us first restrict the applications of the word plasma to systems in which the positive ions are not bound in any lattice in space. This will exclude systems in which the positive ions belong to a conducting or semi-conducting solid body, it will also exclude liquid conductors and electrolytes in spite of the fact that the lattice in these latter cases is ever changing. Such a restriction amounts to the requirement hat the density of the kinetic energy of positive ions be much higher than the density of binding energy corresponding to a lattice. By making this restriction no offence is meant to thus excluded types of plasma, the behaviour of electron plasma in conductors (ref. 1, 2) and semi-conductors (ref. 3, 4) is of considerable theoretical and experimental interest, however, it is more fitting to discuss such systems in another book entitled perhaps "Electron plasmas in solids". For similar reasons it is advisable not to mix the physics of electron and of ion beams with that of plasma physics proper, besides several books have already been written on that subject (ref. 5, 6).

At this point it may be better not to go on deciding what is not a plasma, instead we shall study some important properties of a system consisting of many free electrons and ions and decide which are the parameters corresponding to a typical plasma *.

Let us first consider a special case in which the system is of infinite extension, with no fields of force imposed from the outside and the velocity vectors of the particles randomly distributed both with respect to their direction and their amplitude. Let the average density of either type of particle be n. Let us try to find out how the lines of force e of the electric field E of an electron Q are distributed in space. Obviously most of these lines will be attached to the nearest positive ions $P_1 \dots P_3$, some go to the more distant ones such as P_4, P_5 and some (e') leak out and travel far before they, too, get attached to positive charges (fig. 1). It is easy to see that no e' lines would exist in a perfectly ordered lattice. There all the lines emanating from the charge Q finish on the oppositely charged nearest neighbours. Evidently the

* Lists of symbols are given at the end of each chapter.

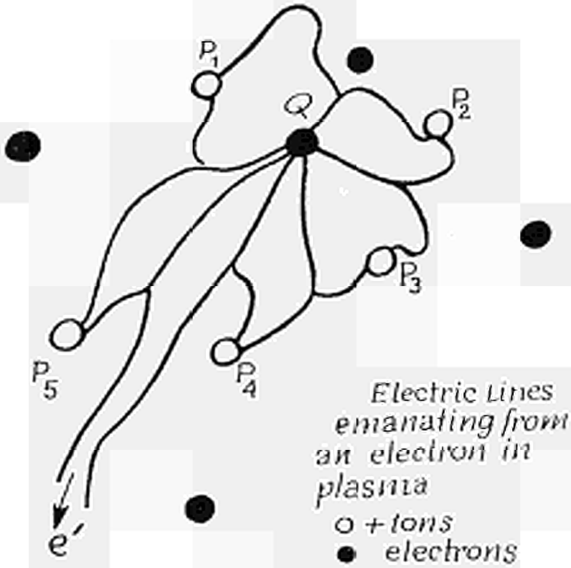


Fig. 1.

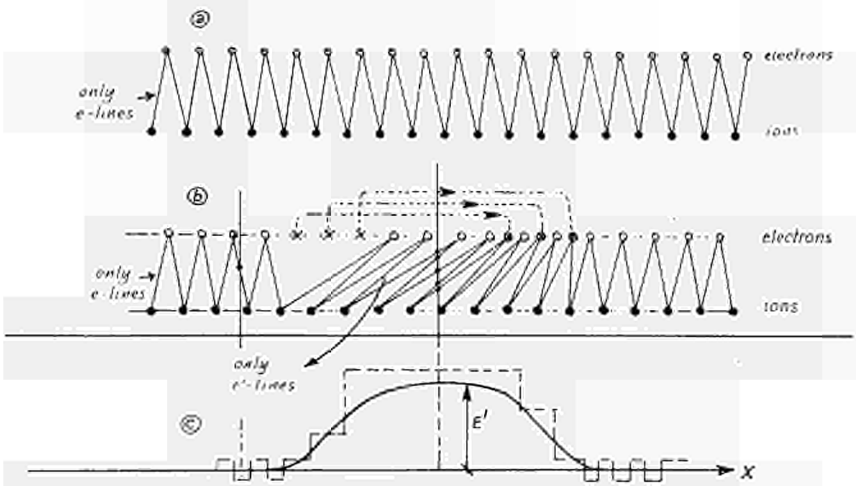


Fig. 2.

a) perfect lattice.

b) e-lines transformed into e'-lines as a result of disturbance in the electron lattice resulting from the shift of 3 electrons.

c) The intensity of the electric field in the disturbed lattice, equivalent to the number of electric lines at any position χ.

reaching out of e' lines is due only to the imperfections of a lattice, i.e. to a local accumulation of charge of one sign *. This is shown graphically in fig. 2.

Such fluctuations in plasma are limited as can be seen from the following argument. Let a layer of electrons escape from a layer of positive ions (fig. 3). The field E in the separated layers can be calculated from the Poisson's law

$$\frac{\partial E}{\partial x} = \pm 4 \pi n e \tag{1}$$

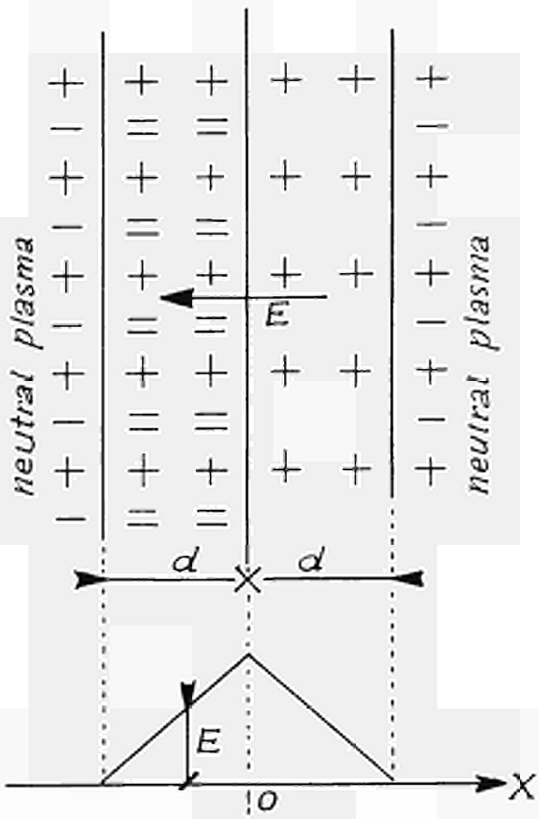


Fig. 3.

* It is also due to the $\frac{1}{r^2}$ dependence of electrical forces. A short-range force would cause the group of charges to behave more like a gas of neutral atoms.

The sign $\left\}\begin{array}{ll} + \text{ valid for} & x > 0 \\ - \text{ valid for} & x < 0 \end{array}\right.$

which gives for uniform n: $E = -4\,\pi\,n\,e\,(d - |x|)$.

The electrical energy stored in this field per cm^2 of the surface of the layer is

$$W_e = 2\int_0^d \frac{E^2}{8\pi}\,dx = \frac{4}{3}\,\pi\,n^2\,e^2\,d^3 \qquad (\mathrm{erg/cm^2}) \tag{2}$$

This energy could have been generated only by drawing on the random kinetic energy of the electron-layer. Since each electron has 3 degrees of freedom, each of which should be endowed with an energy equal to 1/2 kT, we get for the available kinetic energy W_t capable of being converted into W_e.

$$W_t = \frac{1}{2}\,n\,d\,k\,T \qquad (\mathrm{erg/cm^2}) \tag{3}$$

Putting $$W_e \leqslant W_t$$

we get

$$\frac{1}{2}\,kT \geqq \frac{4}{3}\,\pi\,n\,e^2\,d^2$$

or

$$d \leqq \sqrt{\frac{3}{2}}\sqrt{\frac{kT}{4\,\pi\,n\,e^2}} \tag{4}$$

In the following we shall take for this critical distance, known as the Debye distance, the expression

$$d = \sqrt{\frac{kT}{4\,\pi\,e^2\,n_e}} \tag{4a}$$

the numerical factor $\sqrt{3/2}$ having been introduced by the oversimplified nature of our analysis.

The maximum electric field corresponds to *

$$E_{max} = \sqrt{4\pi nkT} \tag{5}$$

* Defining the energy densities $w_e = \dfrac{E^2{}_{max}}{8\pi}$ and $w_t = \dfrac{3}{2}\,nkT$ respectively, the equation (5) can also be written $w_e = \dfrac{1}{3}\,w_t$.

Considering a spherical rather than plane geometry we obtain using similar arguments a critical distance $d_{sph} \sim d$ and $E_{\max\mathrm{sph}} \sim E_{\max}$. The E field is composed only of the e' lines and the distance d is then the longest distance to which the field of a charge in plasma can penetrate before being screened.

These ideas are similar to those used by Debye and Hückel in the theory of electrolytes (ref. 7)) and the distance d is called the Debye distance. In their theory, Debye and Hückel have shown that the distribution of electric field around a fixed charge q in an electrolyte corresponds to a "screened potential" $\phi = \frac{q}{r} \exp\left(-\frac{r}{d}\right)$. Using a simplified argument we shall show how such a potential distribution arises around a fixed charge q in a plasma of density n and temperature T. In absence of charge q the charge density of the electronic and positive ion fluid is

$$\bar{n}_e = \bar{n}_i = n \tag{6}$$

Introducing the charge $+q$ creates a spherical, positive potential wall $\phi(r)$, and the electric field $E = -\frac{\partial \phi}{\partial r}$ will tend to bend the electron trajectories towards the charge and deflect the ion trajectories. In a spherical geometry the Poisson's equation for ϕ, n_e and n_i reads

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right) = 4\pi(n_e - n_i)\,e \tag{7}$$

According to a well-known theorem of statistical mechanics (ref. 8) the distribution of electrons (temperature T) in thermodynamic equilibrium in such a potential well is given by

$$n_e\,(r) = n_e \cdot \exp\left[\frac{+e\phi(r)}{kT}\right] \tag{8a}$$

and of ions $$n_i\,(r) = n_i \cdot \exp\left[\frac{-e\phi(r)}{kT}\right] \tag{8b}$$

For radii r so large that $\frac{e\phi(r)}{kT} \ll 1$, the electron and ion distribution will not be greatly perturbed and we can write

$$n_{e,i}\,(r) = n\left[1 \pm \frac{e\phi(r)}{kT}\right] \tag{8c}$$

The potential ϕ can now be determined from

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = \frac{8\pi ne^2}{kT}\phi \tag{9}$$

The expression $\dfrac{kT}{8\pi ne^2}$ must have a dimension of (length)^2. From equation (4a) it is clear that this length d' is

$$d' = \frac{1}{\sqrt{2}}\, d = \sqrt{\frac{kT}{8\pi ne^2}} \tag{10}$$

The solution of equation (9) must satisfy two boundary conditions. At $r = \infty$ the potential $\phi = 0$ and for $r \to 0$ the solution must converge to that of a point charge q in vacuum, i.e., $\phi\ (r \to 0) = \dfrac{q}{r}$.

Such a solution is

$$\phi = \frac{q}{r}\exp\left(-\frac{r}{d'}\right). \tag{11}$$

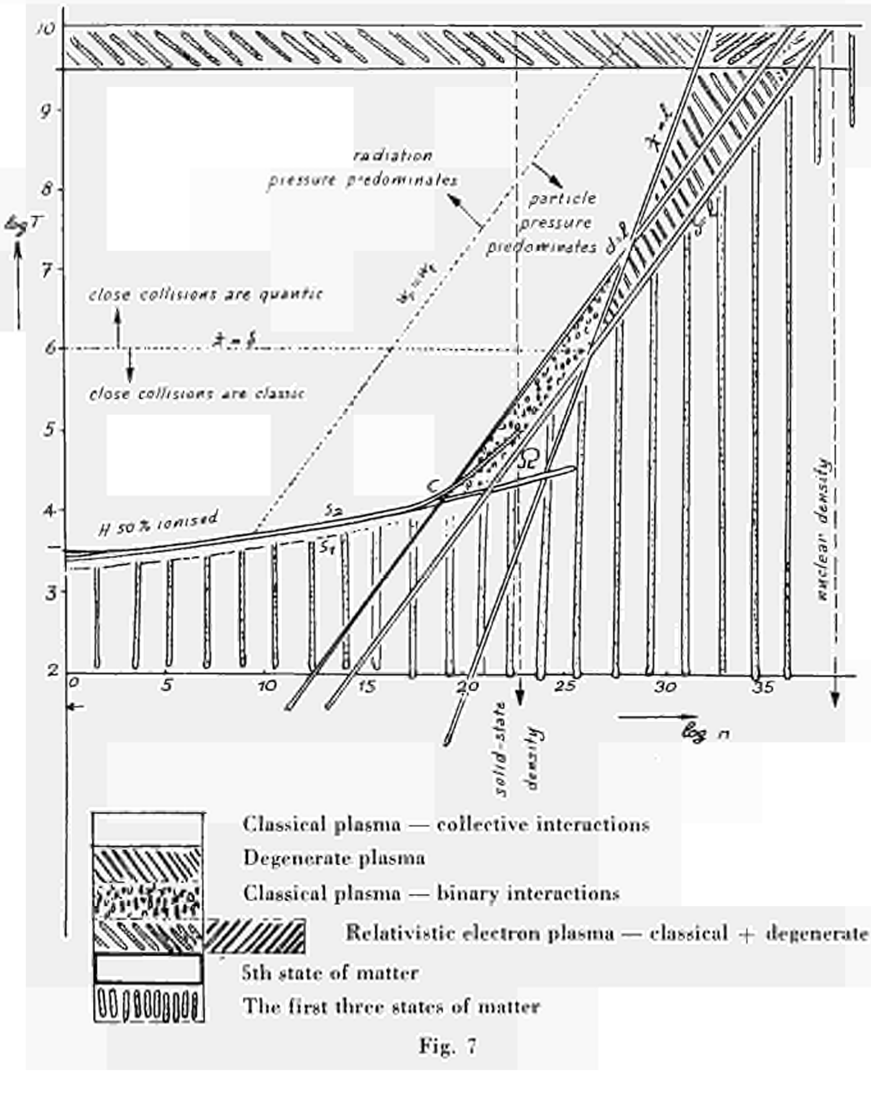
If the ions owing to their inertia connot reach thermodynamic equilibrium and thus cannot distribute themselves in the potential well according to equation (8b) one finds in the solution (11) the Debye length d instead of d'.

As long as the charge q is much larger than the elementary charge e a potential distribution of the type given by equation (11) will develop and is encountered in experiments using spherical probes in a plasma in which the Debye length is larger than the radius of the probe (ref. 9). When the analysis is extrapolated to the problem of potential distribution around a positive ion, i.e. for $q = +e$, it ceases to be valid. Each ion creates and carries its own potential well and, as the mean distance between the ions is $n^{-1/3}$, the formula (11) cannot be applied to $r < n^{-1/3}$, as the mean number of electrons within a volume $\dfrac{1}{n}$ capable of modifying the $\dfrac{e}{r}$ part of ϕ cannot exceed unity. On the other hand for $r \gg n^{-1/3}$ the field of this ion is relatively weak and the formula (11) is, therefore, of little interest.

Consequently the only chance for potentials of the type of ϕ being generated in a plasma is when local accumulations of ions arise, as has been already mentioned. Only then the lines of force of a particle in plasma may reach up to the Debye distance and one must, therefore, admit that in such a case all particles within a sphere whose radius is equal to d can mutually interact. In this respect plasma differs from gases in which the trajectories of the molecules are influenced only by binary collisions — in plasma the trajectories of an electron or an ion are dictated by long-distance Coulomb forces produced, in absence of external fields, by at most N_d particles, where

$$N_d = 2 \cdot 4/3\, \pi d^3 \cdot n = \frac{1}{3\sqrt{n\pi}} \left(\frac{kT}{e^2} \right)^{3/2} \tag{12}$$

i.e., all the particles within the Debye sphere. Only when $N_d \leqslant 2$ do we recover binary interactions. The curve $N_d = 2$ is plotted in the n,T diagramme (fig. 7) separating thus the region belonging to plasmas



Classical plasma — collective interactions
Degenerate plasma
Classical plasma — binary interactions
Relativistic electron plasma — classical + degenerate
5th state of matter
The first three states of matter

Fig. 7

exhibiting many-body interactions from that corresponding to binary collisions *.

Let us estimate the probability that electrons within a Debye sphere or a Debye layer will all have their velocity vectors pointed in a special direction. The probability that the velocity vector of a particle lies in the $v_x > 0$ hemisphere of the velocity space (fig. 4) is 1/2. The probability that all the $\frac{N_d}{2}$ electrons belong to the $V_x > O$ hemisphere is, therefore,

$$p_d = (\tfrac{1}{2})^{1/2 N_d} \tag{13}$$

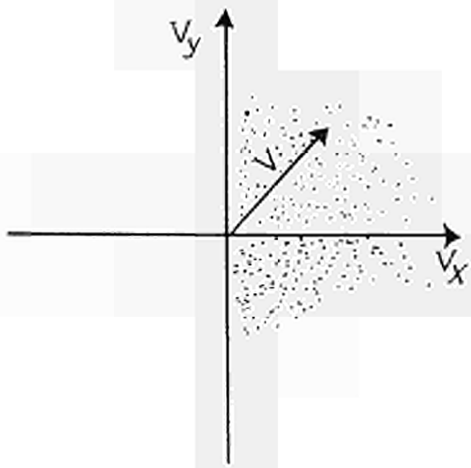


Fig. 4

provided the velocities of the particles are not correlated. Although the probability of finding such an extreme velocity distribution is small, once such a distribution is created it may reappear periodically many times afterwards. The reason is that the electron layer (fig. 3) having transformed its kinetic energy into the energy of the electric field E will be pulled back into its original kinetic energy, corresponding to a velocity in the $-x$ direction. The process is obviously limited only to these electron and positive ion layers as all the electric field (e' lines) appears only in the space between these layers (see also fig. 2)

* The condition $N_d = 2$ is equivalent approximately to $d = l$, where $l = n^{-1/3}$.

and consequently the oscillations are not transmitted to the rest of the plasma.

Later on (chapter 5.1.6) we will see that such a process can propagate owing to the randomness of v and that it is damped; nevertheless, its persistence suggests that once an oscillating electric field is created in the plasma it may be difficult to get rid of it and that there may be a tendency in a plasma to establish an equilibrium between the stored electric energy in oscillating fields and the energy of the random motion of particles.

The frequency corresponding to the electron oscilations resulting from a departure in the electric neutrality of a plasma can be obtained from the motion of an electron in the field $E_{max} . \sin\omega t$.
Thus

$$\ddot{x} = - \frac{eE_{max}}{m} \sin \omega t \tag{14}$$

from which follows after double integration and putting $x_{max} = d$ that

$$\omega^2 = \frac{eE_{max}}{m\,d} \tag{15}$$

From equations (4) and (5)

$$\frac{E_{max}}{d} = 4\pi ne$$

This substituted into eq. (15) gives for ω (which we shall call the plasma frequency ω_p)

$$\omega_p = \sqrt{\frac{4\pi ne^2}{m}} \tag{16}$$

This is quite generally the frequency with which any charge accumulation in a plasma of density n will be neutralised by the inflow of oppositely charged particles.

Having discussed the upper limit of the Coulombian interaction of a charge in a plasma, let us say a few words about the shortest distance δ corresponding to such an interaction. A natural choice for δ is the de Broglie's wave length $\lambda_B = \dfrac{\hbar}{mv}$.

In a plasma where the ion gas has the same kinetic energy of random motion (thermal energy) as the electron gas, the average momentum of the electrons will be much smaller than that of the ions and consequently their λ_B will limit the particle interactions at short distances.

Expressing

$$m^2v^2 = 2m\,(\tfrac{1}{2}\,mv^2) = 3mkT$$

we get

$$\delta = \lambda\!\!\!^{-}_B = \frac{h}{\sqrt{3mk}\cdot 2\pi}\cdot\frac{1}{\sqrt{T}} \tag{17}$$

If the density of a plasma is such that $n\lambda\!\!\!^{-}_B{}^3 \geqslant 1$, the interactions have to be treated by quantum methods — let us call such a plasma a quantum plasma or a degenerate plasma. The degenerate plasmas can be described statistically by the Fermi-Dirac statistics — whereas the non-degenerate plasmas are subject to the Boltzmann statistics. The above mentioned inequality can be written as

$$\frac{h}{2\pi\,(2mk)^{1/2}}\,\frac{n^{1/3}}{T^{1/2}} \geqslant 1 \tag{18}$$

This relation is also plotted in the n,T diagramme (fig. 7), giving us thus a boundary between degenerate and Boltzmannian plasmas. It is often necessary that the Coulomb interaction has to be truncated at distances larger than $\lambda\!\!\!^{-}_B$. When an electron encounters another in a head-on collision the minimum distance between them is given by energy consideration. Thus when their kinetic energy is spent the electric potential energy is equal to $\dfrac{e^2}{\delta}$. Taking for the kinetic energy the mean thermal energy for two particles in one degree of freedom, i.e., kT we get *

$$\delta_L = \frac{e^2}{kT} \tag{19}$$

Much the same argument could be made about p-p and e-p collisions.

In the latter case the minimum distance δ corresponds to a deviation of the electron trajectory by an angle $\pi/2$.

Let us find a criterion for the classical and quantum close-interaction. Obviously when $\delta_L > \lambda\!\!\!^{-}$ the close interaction will be described by classical mechanics, in the opposite case by quantum mechanics. The limit is decribed by $\lambda\!\!\!^{-} = \delta$ giving $T < (2\pi)^2 \cdot (3\,e^4m/kh^2) \approx 10^6$ (°K) for classical interaction.

Let us observe a region of space in plasma whose volume is equal to that of the Debye sphere. Let us consider the case in which $N_d \ll 1$, which is also equivalent to $N_d{}^{2/3} \gg 1$. From the latter we get using eq. (12).

* Also called the "Landau distance".

$$1 \ll \frac{kT}{(9\pi)^{1/3} n^{1/3} e^2} < \frac{3/2\, kT}{\dfrac{e^2}{n^{-1/3}}} \tag{20}$$

which shows that in this case the mean kinetic energy per particle is much larger than the mean interparticle potential energy evaluated for the mean interparticle distance.

The inequality $N_d \gg 1$ can be also transformed into

$$N_d^{4/3} \gg 1 \quad \text{i.e.,} \quad \left(\frac{kT}{e^2}\right)^2 \cdot \left(\frac{1}{3}\right)^{4/3} \quad \frac{1}{n^{2/3}} \gg 1$$

This is approximately

$$\frac{n^{1/3}}{\sigma_{ei}\, n} = \frac{\lambda_{ei}}{l} \gg 1 \tag{21}$$

where $\sigma_{ei} = \pi\, \delta_L^2$ and λ_{ei} is the electron-ion (close-collision) mean free path. The close collisions are, therefore, rare phenomena on a stretch equal to the mean interparticle distance (fig. 5).

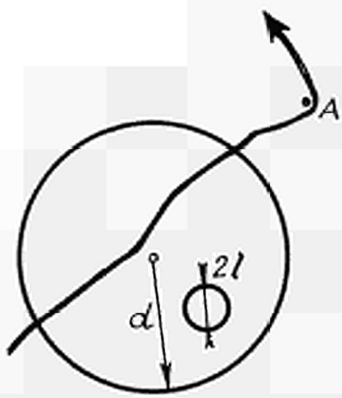


Fig. 5. *A* marks a close-collision event.

1.1.1. DEGREE OF IONIZATION — SAHA'S EQUATION

In gases not all the electrons are always bound to the atomic nuclei. The outer electrons may receive enough energy in collisions with other particles or with photons to overcome the potential well eV_i of the atom and become free *. A reverse process, the recombination, is responsible

* In degenerate gases the outer electrons are "squeezed out" of the potential well (ref. 11).

for the capture of free electrons by ions. The relative frequencies of these two processes: the ionization and the recombination, determine to what degree the medium is ionized, i.e., to what extent it resembles an ideal plasma. The corresponding relation for a medium in thermodynamic equilibrium is known as the Saha equation and it reads (ref. 10)

$$\frac{n_e n_i}{n} = \left(\frac{\sqrt{2\pi mk}}{h}\right)^3 T^{3/2} \exp\left(\frac{-eV_i}{kT}\right) \tag{22}$$

where n_e, n_i and n are respectively the densities of electrons, ions and neutral atoms and V_i is the ionization potential. When $kT > eV_i$, then the exponential term is almost equal to unity and since for once ionized atoms $n_e = n_i$ we have

$$\frac{n_e}{n} = \left(\frac{\sqrt{2\pi mk}}{h}\right)^{3/2} n^{-1/2}\ T^{3/4}. \tag{23}$$

In order that $kT > eV_i$ the temperature must be of the order of 10^5 and it follows that the ratio $\frac{n_e}{n}$ is very high as long as $n < 10^{22}$. The dependence $T^{3/4}$ is to be expected, the higher the temperature, the higher is the electron velocity and the higher the frequency of ionizing collisions.

In the other extreme, i.e., $kT < eV_i$ the degree of ionization is dictated mainly by the exponential term. The ionization can be accomplished only by the relatively few energetic electrons in the tail of the Maxwellian distribution.

The dividing line between plasmas and gases can be fixed as corresponding to a certain degree α of ionization. If, e.g., we take $\alpha = \frac{n_e}{n} = 1$ for hydrogen we get a relation

$$T^{3/2} \exp\left(\frac{-158\,000}{T}\right) = 1.45 \cdot 10^{-18} \cdot n. \tag{23a}$$

Another and more precise criterion for deciding whether a medium is plasma or only a weakly ionized gas is provided by considering the electron-ion and electron-neutral atom collision-frequencies (ν_{ei} and ν_{en}). One may then define plasma by

$$\nu_{ei} \geqslant \nu_{en}$$

where (using eq. (21) and eq. (8-56))

$$\nu_{ei} = \frac{\nu_e}{\lambda_{ei}} = \sqrt{2}\,\pi \cdot \frac{e^4}{\sqrt{k^3 m}} \cdot \frac{n_i}{T^{3/2}}\, ln\, \Lambda$$

and

$$\nu_{en} = \sqrt{\frac{2k}{m}} \frac{n}{\sigma_{en}} T^{1/2}$$

where σ_{en} is the collision cross-section between electrons and neutral atoms.

Substituting into these expressions for n_i and n from eq. (22) we get a function of T and $(n + n_i)$ which is plotted in fig. 7 (curve C) and represents the above mentioned criterion.

1.1.2. ELECTRIC FIELDS IN PLASMA

It is possible now to visualize the type of electric field an observer will register inside a plasma in which collective interactions predominate. This is shown in fig. 6 where we have chosen a typical plasma having $n = 10^{18}$ and $T = 10^6$.

When external fields penetrate into a plasma the motion of individual particles does not change appreciably. This is due to the external fields being usually weaker than the microfields E_a, E_b and E_c that are always present in a plasma. This can be appreciated from the example shown in fig. 7 and noting that in laboratory it is difficult to generate electric fields much higher than 10^5 V/cm. The same argument is valid, though to a lesser extent, for magnetic fields. In laboratory it is feasible to generate fields of the order of 100 KGauss. An electron moving in a magnetic field sees an equivalent electric field

$$E = \frac{v}{c} \text{B}.300 \text{ (V/cm, Gauss).} \tag{24}$$

For a typical plasma-electron this is

$$E = 3.7 \cdot 10^{-3} B\sqrt{T} \quad \text{(V/cm).} \tag{24a}$$

Thus for $T = 10^6$ (°K) and $B = 10^5$ Gauss we get $E = 3.7 \cdot 10^5$ V/cm, again not a very high field compared to E_a or E_b in fig. 6.

1.1.3. RADIATION IN PLASMA

In any medium in thermodynamic equilibrium there should be a radiation field whose photons are distributed in frequency according to the Planck's law

$$w_r(f) = \frac{8\ hf^3}{c^3\ (e^{hf/kT} - 1)} \tag{25}$$

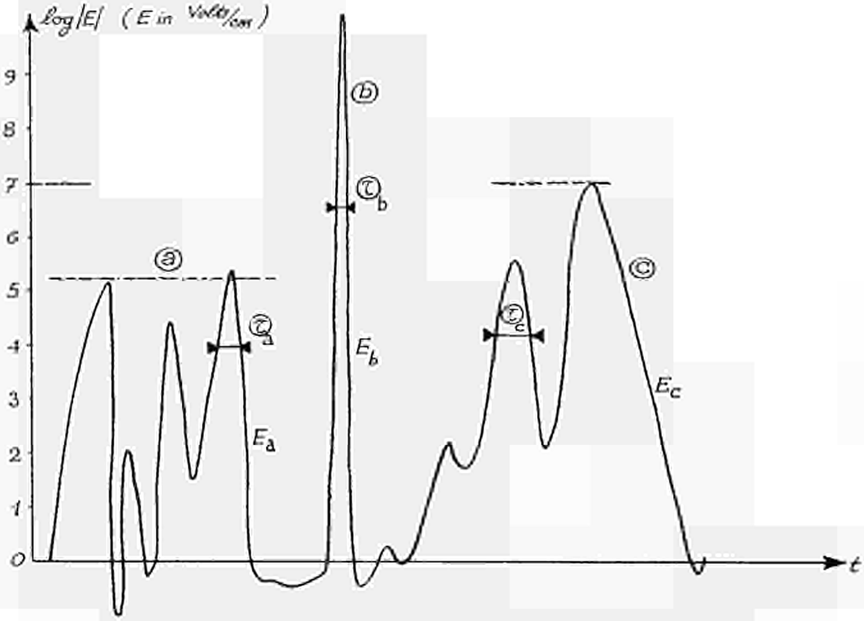


Fig. 6. Three typical electric fields records in a plasma with $n = 10^{18}$, T = 10^6 (°K) in which $N_d = 2800$.

a) The most usual behaviour: irregular oscillations due to passage of electrons. Mean amplitude $E_a = en^{2/3} \cong 150.000$ V/cm, mean interval $\tau_a = n^{-1/3}\left(\dfrac{kT}{m}\right)^{-1/2} \cong 10^{-15}$.

b) A rare event: electron passing closest possible to an ion, i.e., at a distance $\delta = \dfrac{e^2}{kT}$ away. Amplitude $E_b = \dfrac{e}{\delta^2} = \dfrac{(kT)^2}{e^3} = 2 \cdot 10^{11}$ V/cm. The mean interval of such an event is

$$\tau_b = \frac{\delta}{\sqrt{\dfrac{kT}{m}}} = \frac{e^2 m^{1/2}}{(kT)^{3/2}} \cong 10^{-18}.$$

c) Another event: oscillations of a Debye sphere. Amplitude $E_c = \sqrt{4\pi nkT} = 10^7$ V/cm. Frequency of occurence lower than E_a mean interval $\tau_c = \dfrac{2\pi}{\omega_p} \cong 1/3 \cdot 10^{-13}$.

where w_ν is the energy density in an unit interval of frequencies. This is the photon-gas analogy to the Maxwell distribution of velocities among particles in thermal equilibrium. In order that any velocity distribution can, within a finite time, relax into a Maxwellian one a certain, be it even very weak, interaction (coupling) between the particles is required. Since photon-photon interaction is for all practical purposes non-existant, one requires a charged particle, e.g., an electron, in order that an arbitrary frequency distribution of photons could relax into a Planck's distribution. This condition for coupling is satisfied in a plasma and consequently in an infinite plasma one will find apart from thermal energy density $W_t = 3nkT$ also radiation energy density $W_r = \int_0^\infty w_r \cdot df$ which is the well known

$$W_r = \frac{2\pi\sigma}{c} T^4 \tag{26}$$

The behaviour of a plasma in which $W_r > W_t$ will be largely dictated by the contained radiation fields. This will occur, therefore, when

$$\frac{2\pi\sigma}{3\,ck} \frac{T^3}{n} > 1 \tag{27}$$

The line corresponding to this boundary is also found in the n,T diagramme (fig. 7).

In most laboratory plasmas, however, no such equilibrium can occur. This is due to the small optical dimensions of such plasmas, i.e. to their transparency. Thus most of the radiation generated by various emissive processes in a plasma has no chance of being reabsorbed or trapped and is lost out of the system. Let us find the minimum radius R of a plasma sphere, which could approach the state of radiation equilibrium. Such a sphere will tend to radiate as a black body, in which case the radiation loss will amount to $4\pi R^2 \sigma T^4$. This must be continually replaced by generation of radiation within the sphere. Let us put $g(T,n)$ equal to the rate of radiation energy generation and neglect the absorption of this radiation. Then

$$4\pi R^2\, \sigma\, T^4 \leqq 4\pi \int_0^R r^2 \cdot g \cdot dr$$

Let us assume for simplicity that n and T are constant within the sphere. This will yield an order of magnitude value of R. We get

$$(3\,\sigma\, T^4)\, /\, g < R \tag{28}$$

One of the most important radiation-emission processes in a fully ionized plasma is that corresponding to the radiation emitted by electrons during their collisions with positive ions, known as bremsstrahlung. The ion deflects the electron and, therefore, changes its velocity. The electron radiates at a rate $\frac{e}{c}\,(\dot{v})^2$.

It will be shown later (p. 70) that, averaging over all the electrons, this process gives

$$g(n,T) = 1.42 \cdot 10^{-27}\, n_e^2\, Z\, \sqrt{T} \qquad (\text{ergs/cm}^3, \text{sec})$$

Substituting into eq. (28) we get

$$R > \frac{10^{23}}{Z} \frac{T^{3.5}}{n_e^2}. \tag{29}$$

Example: $Z = 1$, $T = 10^5$, $n_e = 10^{17}$ (a typical laboratory plasma), then $R > 30$ Km.

It is to be understood that since the absorption of radiation has been neglected, the criterion for R is valid only as long as $R/\lambda_r \not\gg 1$ where λ_r is the mean free path of a photon which in a completely ionized hydrogen is

$$\lambda_r = \frac{1}{n\,\sigma_T} \text{ where } \sigma_T = 6.65 \cdot 10^{-25}\ (\text{cm}^2).$$

Plasma whose dimension is smaller than R can still radiate as a black body, but only over that part of frequency spectrum in which either $g(n,T)$ is higher than that given by the bremsstrahlung process or the photons can be conserved an appreciable time in the plasma.

1.1.4. CLASSIFICATION

It is now possible to discuss the n,T diagramme (fig. 7) which permits us to divide plasmas into different types.

The realm of plasmas is limited on the low T side by the Saha's equation (eq. (22)). Whether one agrees that the degree of ionization corresponding to a gaseous state of matter is 50 % or 10^{-2} % does not make much difference to the position of this low T boundary (see the two curves S_1, S_2 corresponding to the above-mentioned degrees of ionization).

On the low n side the limitation is somewhat arbitrary, we shall take $n = 1$ as the limit since this value corresponds to the mean interstellar density and it is, therefore, unlikely that one will encounter in our galaxy a plasma density lower than 1 particle/cm^3.

On the high T side, an order of magnitude limitation is obtained by making a distinction between plasma state and the fifth state of matter. In this new state of matter nucleons are free. The maximum binding energy of nucleons being 8 MeV/nucleon it follows that the transition between pure plasma and the higher state will be around 1 MeV/nucleon, i.e., corresponding to a temperature of the order of 10^{10} (° K). Just below this temperature is the realm of plasma in which the electron gas must be treated by relativistic mechanics as 3/2 kT $\geqslant m_0c^2$, from which $T_{\text{rel}} = 3.9 \cdot 10^9$ (° K).

On the high density side there is first the boundary $\lambda_B = l$ defined by eq. (18) which can be written as

$$n^{1/3} \cong 2\pi \cdot 10^5\, T^{1/2}.$$

For n higher than that which follows from this equation, there will be more than 1 particle in a cube whose dimension is equal to the Broglie wave length and the interaction of the particle will be governed by quantum mechanics. The region to the right of the $\lambda_B = l$ corresponds, therefore, to quantum plasmas. These are often called degenerate plasmas since the relevant Fermi-Dirac description of such plasmas uses the notion of degenerate degrees of freedom for the particles. An interesting corner between the curve S_2 and the boundary $\lambda_B = l$ corresponds to a plasma in which $l \gtreqless d$. In such a plasma collective interactions, such as the oscillation of a Debye layer or sphere, are impossible as the corresponding volumes contain less than one electron. The particles thus interact mainly by means of binary collisions.

To $\lambda_B < l < \delta$ (region Ω), does not correspond any plasma. One can show easily that in such a case $\dfrac{e^2}{l} > kT$ and, therefore, the binding energy of an ion $\dfrac{e^2}{l}$ is larger than the mean kinetic energy kT in one degree of freedom which means that the electrons are no longer free.

The greatest part of the following chapters will be devoted almost exclusively to the physics of classical, i.e., Boltzmannian plasmas, which correspond to the clear area in fig. 4.

1.2. Plasma in Nature and in Laboratory

The most conspicuous condensations of matter in the Universe are the stars. The stellar material is mostly in the plasma-state. In order to represent the stellar plasmas in the n,T diagramme it is necessary to discuss the types of stars and determine to which physical conditions these correspond.

1.2.1. STARS AND INTERSTELLAR SPACE — THE DIAGRAMME OF HERTZSPRUNG AND RUSSEL

A convenient mode of representation of different types of stars is the diagramme of Hertzsprung and Russel (ref. 12). The two parameters used are the absolute magnitude (corresponds to the light-output of the star) and the spectral type (related to the superficial temperature). There are three main agglomerations — the stars of the main sequence, white dwarfs and red giants (fig. 8).

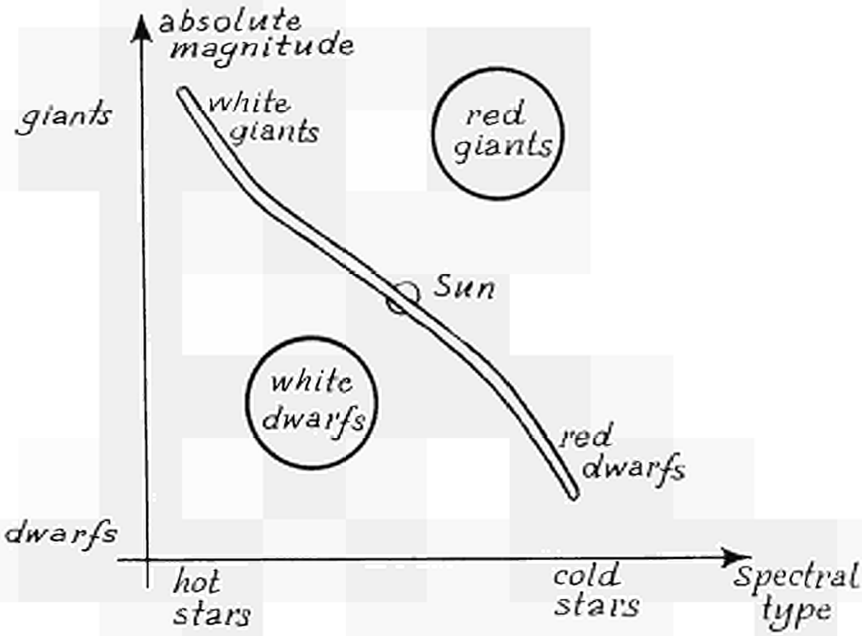


Fig. 8.

The hottest stars of the main sequence are the white giants whose superficial temperature is typically about 10^5 (° K). Their structure should be sensibly different from other stars, the temperature of the centre may be about 10^7 (° K), however, this estimate depends very much on the model one uses for their internal composition.

In the middle of the belt representing the main sequence is our sun and the stars resembling it. Their superficial temperature is of the order of 10^4 (° K), the central temperature should be about 20-30 . 10^6 (° K). The ion density in the centre is probably between 10^{24} and 10^{25} ions/cm^3.

In the lower tail of the main sequence are red dwarfs whose surface and central temperatures are well below those of our sun.

The red giants correspond to a spectrum of relatively low temperatures and densities, whereas the white dwarfs correspond to high temperatures, and above all, to high densities. The central densities of some white dwarfs may reach 10^{30} ions/cm^3.

Apart from the stars mentioned sofar, all of whom are in equilibrium, there exist also pulsating stars and exploding stars. Some of these may reach a state in which the central part of their mass reaches such temperatures and densities that an almost complete collapse of the star occurs; the densities in the center of such a collapsing mass may reach 10^{38}-10^{39} ions/cm^3. This corresponds to the density of nucleons in an atomic nucleus and such stars are called neutron-stars.

All the stars possess an atmosphere, thus e.g., our sun's atmosphere consists of a so called reversing layer, then a layer known as the chromosphere and finally the outermost - the corona. The temperature ranges from 10^4 to 10^6, the densities from 10^{18}-10^6 ions/cm^3.

Most stars, especially the unstable ones, emit streams of plasma in the interplanetary and interstellar space. The temperature and the density of these ejected plasmas diminish as the stream propagates away from the star and as it expands, until in the interstellar space the density drops to about 1 ion/cm^3 and the temperature to an order of one thousand degrees. The zones corresponding to all these states of matter are plotted in fig. 9.

1.2.2. PLANETS

The plasma state is represented relatively poorly on planets. Those planets possessing their own magnetic field are capable to reflect and deviate the solar streams. The plasma free cavity is known in the case of our earth as the geomagnetic cavity. Very energetic particles can penetrate deeply into this cavity and create ionization in upper layers of earths ionosphere. Important and interesting regions filled with such plasma resembling onion-shells have been discovered recently and are known as the Van Allen belts.

Electric fields can be generated in planetary atmosphere giving rise to storms — during which electrical discharges occur. The gas in the inner core of these discharges may reach very high temperatures ($T \sim 10^4$ ($^\circ$K)) and must be, therefore, highly ionized. This is the nearest, short-lived sample of a plasma-state in nature accessible to human experience in the past.

1.2.3. PLASMA PRODUCED BY MAN

We are able to produce plasma either thermically or in electrical discharges.

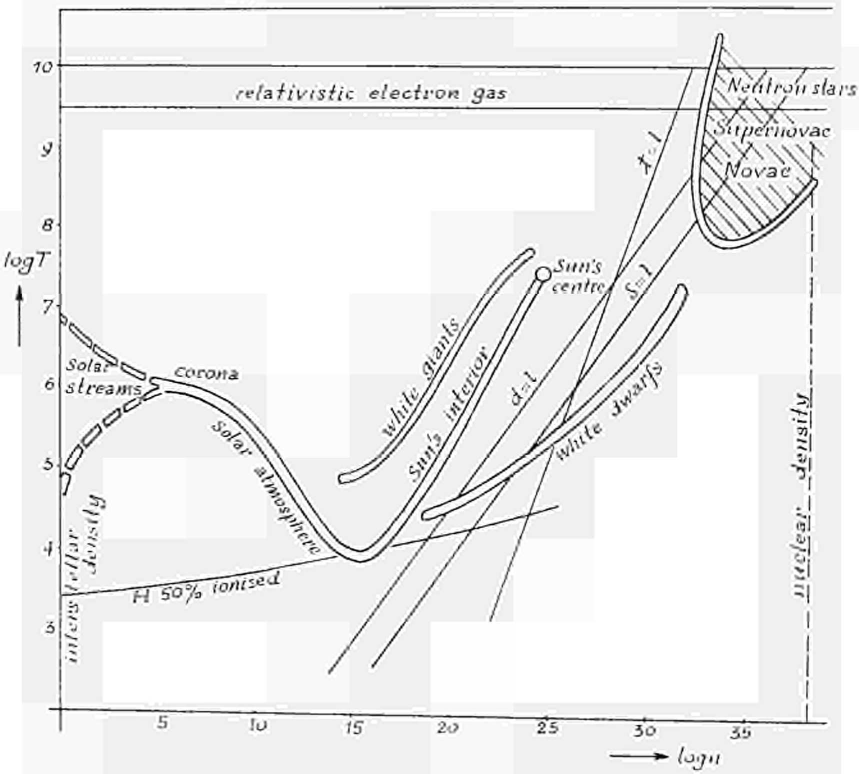


Fig. 9. Occurrence of plasma in the Universe.

Explosions of atomic and hydrogen bombs produce dense ($n > 10^{22}$ ions/cm^3) and hot, short-lived plasmas. A thermal method for the generation of stationary plasmas uses high-temperature ovens ($T >$ 2000° K) in which vapors of caesium are admitted. The caesium atoms are ionized on contact with the incandescent walls of the oven. The density and temperature of such plasmas are low.

Pulsed electrical discharges are used to produce plasmas for experiments on controlled nuclear fusion. The temperature of such plasmas may be as high as 10^8 (° K), the density may range from 10^8 to 10^{17} ions/cm^3. The life-time of such plasmas is usually limited to microseconds (the low density plasma to milliseconds), consequently in most cases such plasmas are not in thermal equilibrium and some of the observations we made for a plasma in thermal equilibrium do not apply (e.g., the $W_r = W_t$ line has no sense).

Experiments on plasmas produced by stationary electrical discharges were among the first classical plasma-experiments. A large class of glow discharges corresponds to very low ionic temperature, electron temperature of the order of 10^4 (°K) and to densities $n < 10^{13}$. As the power input of the electrical discharge increases, the ionic temperature approaches the electron temperature, the density increases and one obtains an arc. The density in arcs may be as high as 10^{18} ions/cm³, their temperature may reach 10^5 (°K).

The zones corresponding to these plasmas are outlined in the n,T diagramme in fig. 10.

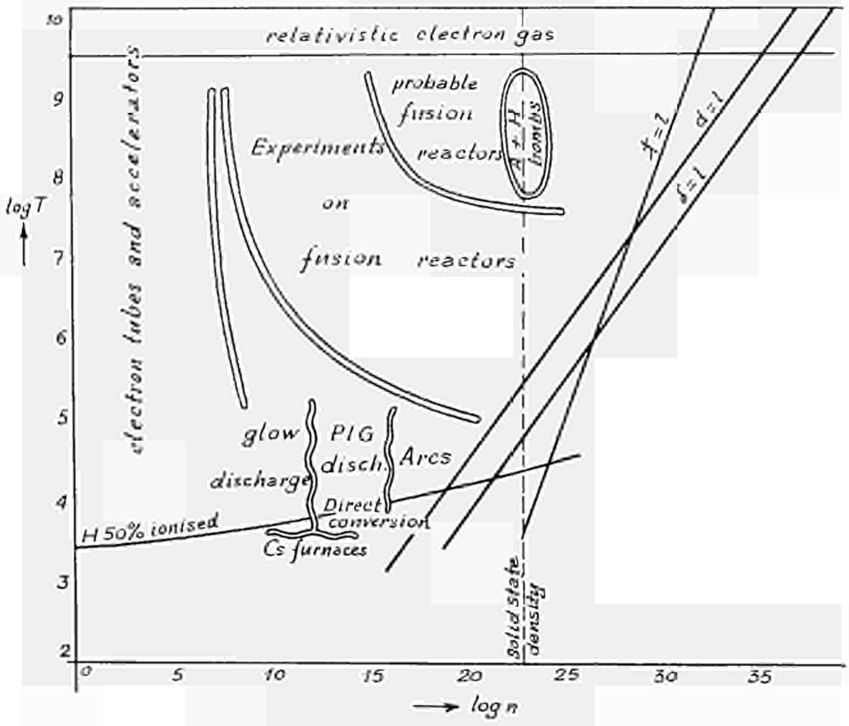


Fig. 10. Plasmas produced by man.

REFERENCES TO CHAPTER 1

1) L. Brillouin: Electrons dans les métaux, Hermann et Cie. Ed., Paris 1934.
2) D. Pines: Elementary excitations in solids, W.A. Benjamin Inc., 1963.
R. A. Ferrell, Univ. Maryland Tech. Report 101 (1958).
3) Semiconductors and semimetals, *1*, Acad. Press, New York.
4) 7th Int. Conf. on physics of semiconductors, **2** (plasma effects in solids) Paris-Dunod, 1965.
5) A. H. Beck: Space-charge waves, Pergamon Press, 1958.
6) E. Okress (editor): Crossed field microwave devices, *1*, pp. 179-357, 1961.

7) P. Debye and G. Hückel, *Phys. Zeit.*, ***24***, p. 305 (1923).
G. Joos: Theoretical Physics, pp. 548-550, Blackie and Son Ltd (1960 Ed.).
8) G. Joos: *op. cit.*, pp. 586-589.
9) Guthrie and Wakerling (editors): Characteristics of electrical discharges in magnetic fields, pp. 77-86 (D. Bohm), Mc Graw-Hill Co., 1949.
10) M. N. Saha, *Phil. Mag.*, ***40***, p. 472 (1920).
G. Joos: *op. cit.*, pp. 539-542 and p. 862.
11) G. Lehner: Reports Laboratorio Gas Ionizzati, L.G.I. 65/2 - L.G.I. 66/8.
A. A. Vedenov, Proc. conf. phys. quiescent plasmas, p. 107, Frascati 1967.
12) M. Schwarzschild: Structure and evolution of stars, p. 1, Princeton Univ. Press, 1958.

List of symbols used in Chapter 1

B	magnetic field strength	x, r	coordinate
c	velocity of light	v	velocity
d	Debye distance	V_i	ionization potential
e	elementary charge	W	energy density
E	electric field strength	Z	atomic number
f	frequency	δ_L	Landau distance
h	Planck's constant	λ	mean free path
k	Boltzmann's constant	λ_B	De Broglie length
l	mean distance between particles	σ	Stefan's constant
m	electron mass	$\sigma_{e,i}$	electron-ion cross-section
n	particle density	σ_T	Thompson cross-section
q	charge	ϕ	potential
R	radius	ω_p	plasma frequency
T	temperature		

CHAPTER 2

MOTION OF ELECTRONS AND IONS IN ELECTRIC AND MAGNETIC FIELDS

Introduction

This chapter is concerned with the study of the motion of charged particles in the various fields of force and in combinations of such fields, that are of interest in the study of plasmas. The most important of these motions is the interaction of charged particles with various magnetic fields. The last section of this chapter will be devoted to radiation emitted by single charges.

2.1. Motion in an Electrostatic Field

If the field is irrotational it may be described by a potential V, and a particle trajectory is constructed in the general case from the equation for the radius of curvature R. Thus (fig. 11)

$$m\frac{v^2}{R} = eE_\perp \qquad (1)$$

and provided that $v = 0$ for $V = 0$ one may write (for non-relativistic energies)

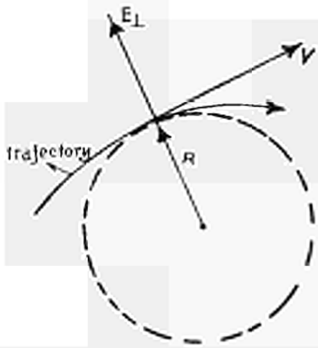


Fig. 11. The projection of a trajectory of a charged particle in an electrostatic field on its osculating plane.

$$\frac{2eV}{R} = e \left(\frac{\partial V}{\partial r}\right)_{\perp}$$

and therefore,

$$R = \frac{2V}{\left(\dfrac{\partial V}{\partial r}\right)_{\perp}} \quad \text{(cm)}. \tag{2}$$

This formula can be used very simply in a step by step graphical plotting of the trajectory. It is also used in automatic trajectory tracing in an electrolytic tank (ref. 1).

Motion of this type is of great interest in electron optics; in plasma physics it enters only in connection with the classical description of Coulomb scattering.

Let us give a brief treatment of this last problem, since it is connected with many important phenomena such as particle diffusion and emission of the bremsstrahlung.

Let us consider first the motion of a particle having charge e and mass m in the electric field of a relatively heavy particle of mass M and charge Z.e. The motion is determined by the laws of conservation of energy and of angular momentum. Let the velocity of the light particle at infinity be v_0 and let the distance of M from the line determined by $\boldsymbol{v}_0$ be p_0, the so called collision-parameter (fig. 12).

The vectors $\boldsymbol{v}_0$ and $\boldsymbol{p}_0$ determine the plane of motion. Using polar coordinates in this plane, the two conservation laws can be written

$$\dot{r}^2 + r^2 \dot{\Theta}^2 \pm 2 \frac{Ze^2}{mr} = v_o^2 \tag{3}$$

$$p_o v_o = r^2 \dot{\Theta} \tag{4}$$

the $+$ is valid for charges of the same sign, the $-$ for oppositely charged particles.

The time derivatives can be eliminated by means of

$$\frac{d}{dt} = \dot{\Theta} \frac{d}{d\Theta}$$

and substituting for $\dot{\Theta}$ from equation (4) we get

$$\left(\frac{dr}{d\Theta}\right)^2 \left(\frac{p_o v_o}{r^2}\right)^2 + \left(\frac{p_o v_o}{r}\right)^2 \pm \frac{2\, Ze^2}{mr} = v_o^2 \tag{5}$$

Separating the variables there is

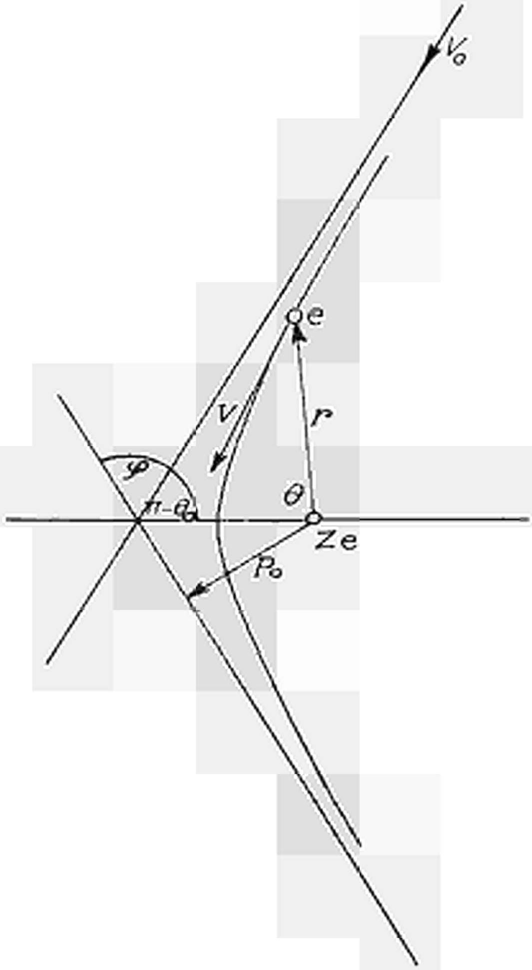


Fig. 12. Geometry of eletron-positive ion scattering.

$$d\Theta = \frac{p_o v_o}{r^2\left[v_o^2 - \left(\frac{p_o v_o}{r}\right)^2 \pm \frac{2\ Ze^2}{mr}\right]^{1/2}} \cdot dr$$

which can be integrated giving

$$r = \frac{(p_o v_o/\alpha)}{1 \pm \frac{A}{\alpha} \cdot \cos\Theta} \tag{6}$$

where

$$\alpha = \frac{Ze^2}{m\, p_o v_o}, \qquad A = \sqrt{v_o^2 + \alpha^2}.$$

Equation (6) is an expression for a hyperbolic or eliptic trajectory. In the case of charge e starting its motion from infinity the motion can be only hyperbolic. Considering the event as a collision of two particles it is customary to talk of strong and weak collisions. In order that a collision be considered strong it is necessary that the total deviation of v_0 is equal or larger than 90°, i.e., $\pi - \theta_0 \gtreqless \pi/4$. The angle $\theta_0 = \pi/4$ corresponds to the position $r = \infty$ and can be realized only if in equation (6) the denominator is equal to zero. Thus the criterion for strong collisions becomes

$$1 - \frac{A}{\alpha} \cos \frac{\pi}{4} = 0$$

or

$$\frac{m\, p_o v_o^2}{Ze^2} = 1\,. \tag{7}$$

All particles, having velocity v_o, whose collision parameter is smaller than $p_o = \dfrac{Ze^2}{mv_o^2}$ will effect strong collisions. The cross-section represented by the field of Ze corresponding to such strong collisions is, therefore,

$$\sigma_o = \pi p_o^2 = \frac{\pi Z^2 e^4}{m^2 v_o^4}\,. \tag{8}$$

2.2. Motion in a Magnetostatic Field

In a homogeneous magnetostatic field B a charged particle moves on a helical trajectory (fig. 13) with an angular frequency

$$\omega_c = \frac{e}{mc} B. \tag{9}$$

The projection of the trajectory on a plane perpendicular to B is a circle whose radius

$$\rho = \frac{m(v \wedge B)}{\dfrac{e}{c} B^2} \tag{10}$$

is called the radius of gyration.

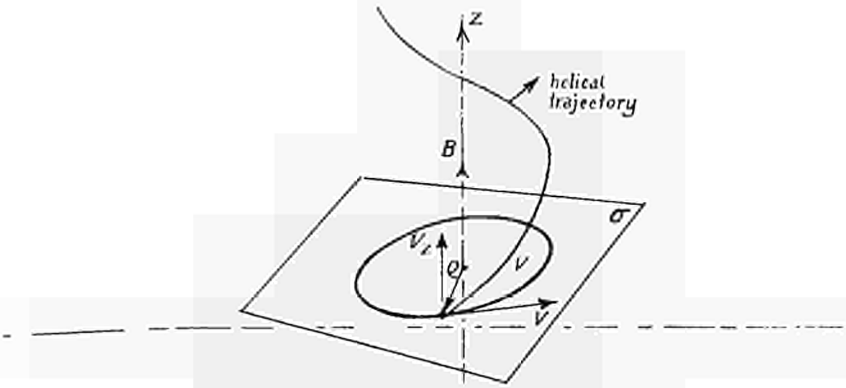


Fig. 13. Trajectory of a charged particle in a uniform magnetostatic field.

For non-relativistic electrons whose kinetic energy is W

$$\omega_c = 1.76 \times 10^7\, B \text{ (rad/sec, gauss)}, \; \rho = \frac{3.37\sqrt{W}}{B} \text{ (cm, eV, gauss)}.$$

For nuclei of atomic mass number A and charge Z

$$\omega_c = 0.957 \times 10^4 \frac{Z}{A} B \qquad \text{(rad/sec, gauss)},$$

$$\rho = \frac{145\sqrt{A}}{Z}\,\frac{\sqrt{W}}{B} \qquad \text{(cm, eV, gauss)}.$$

The motion of the particle parallel to the magnetic field vector $\boldsymbol{B}$ is independent of the magnetic field.

Let us now consider the effect of magnetic field non-uniformities on the motion of a charged particle.

In order to appreciate clearly the effects of such non-uniformities let us first consider a simple example, treated by means of a simplified analysis.

Let us have a magnetic field B_z which only increases with increasing x (fig. 14). Thus $\frac{\delta B_z}{\delta y} = \frac{\delta B_z}{\delta z} = 0$, $\frac{\delta B_z}{\delta x} > 0$. A particle whose velocity vector lies in the xy plane will move on an orbit whose mean radius of curvature below the y axis will be larger by Δr than that corresponding to the orbit above the y axis. After one complete turn, i.e. after a time $\tau\, \frac{2\pi}{\omega}$ the particle will advance in the $-y$ direction (fig. 14) by $2\,\Delta r$. As

$$r + \Delta r = \frac{mv}{\frac{e}{c} B \left(1 - \frac{\Delta B}{B}\right)}$$

we get

$$2\,\Delta r = \frac{2\,mv}{\frac{e}{c} B^2} \Delta B \tag{11}$$

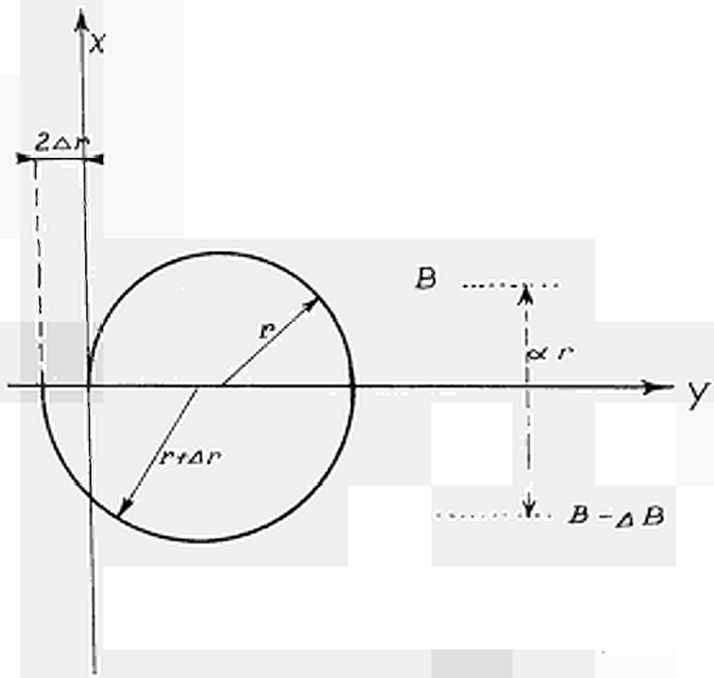


Fig. 14. Motion of a charged particle in a non-uniform magnetic field.

where $\Delta B = \alpha \cdot r \cdot \frac{\delta B}{\delta x}$ and α is a factor of the order of 1 which can be evaluated averaging B over the orbit and later will be shown to be equal to $\frac{\pi}{2}$.

The particle drifts in the -y direction with a "drift-speed"

$$U = \frac{2\,\Delta r}{\tau} = \frac{1}{2} \frac{mv^2}{\frac{e}{c} B^2} \quad \text{grad } B \tag{12}$$

Let us now derive a general formula for the drift motion of a charged particle in a non-uniform field.

If the non-uniformities cause only small perturbations in the angular speed and radius of gyration during one rotation, it is more convenient to follow the curve traced by the instantaneous centre of gyration of the particle rather than the particle itself. We shall now derive the equation of motion of the centre of gyration.

At a point $\mathrm{P}(r)$ the particle possesses a velocity v (fig. 15). The corresponding centre of gyration is at $\mathrm{G}(r + \rho)$ where

$$\rho = \frac{c}{e}\,\frac{\boldsymbol{p} \wedge \boldsymbol{B}}{B^2}. \tag{13}$$

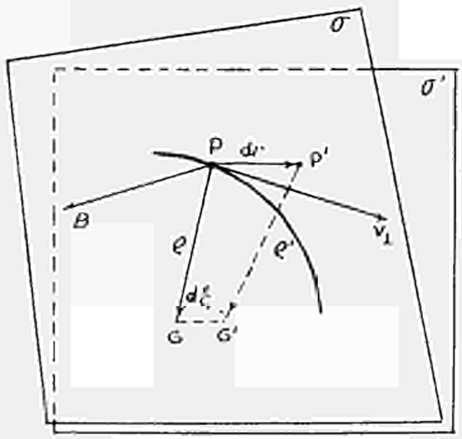


Fig. 15. The drift of the centre of gyration in a non-uniform magnetostatic field.

The force acting on the particle is

$$\boldsymbol{F} = \frac{e}{c}\,\boldsymbol{v} \wedge \boldsymbol{B} \tag{14}$$

and the change in the momentum during a time dt is

$$\mathrm{d}\boldsymbol{p} = \boldsymbol{F}\,\mathrm{d}t. \tag{15}$$

The radius vector of gyration changes during the same time into

$$\rho' = \frac{c}{e}\left\{\frac{\boldsymbol{p} \wedge \boldsymbol{B}}{B^2} + \frac{\mathrm{d}\boldsymbol{p} \wedge \boldsymbol{B}}{B^2} + \frac{\boldsymbol{p} \wedge \mathrm{d}\boldsymbol{B}}{B^2} - 2\,\frac{\boldsymbol{p} \wedge \boldsymbol{B}}{B^3}\,\mathrm{d}B\right\}. \tag{16}$$

The translation of the centre of gyration is, therefore,

$$GG' \equiv \mathrm{d}\xi = \mathrm{d}\boldsymbol{r} + \rho' - \rho$$

$$\mathrm{d}\xi = \mathrm{d}\boldsymbol{r} + \frac{e}{e}\left\{\frac{\mathrm{d}\boldsymbol{p}\wedge\boldsymbol{B}}{B^2} + \frac{\boldsymbol{p}\wedge\mathrm{d}\boldsymbol{B}}{B^2} - 2\frac{\boldsymbol{p}\wedge\boldsymbol{B}}{B^3}\,\mathrm{d}B\right\} \tag{17}$$

where

$$\mathrm{d}\boldsymbol{r} = \boldsymbol{v}\,\mathrm{d}t$$

and

$$\frac{\mathrm{d}\boldsymbol{p}\wedge\boldsymbol{B}}{B^2} = \frac{\left(\frac{e}{c}\,\boldsymbol{v}\wedge\boldsymbol{B}\right)\wedge\boldsymbol{B}}{B^2}\,\mathrm{d}t = -\frac{e}{c}\,\boldsymbol{v}\,\mathrm{d}t + \frac{e}{c}\,\frac{\boldsymbol{B}(\boldsymbol{v}\cdot\boldsymbol{B})}{B^2}\,\mathrm{d}t.$$

The equation for $d\xi$ becomes

$$d\xi = \frac{\boldsymbol{B}(\boldsymbol{v}\cdot\boldsymbol{B})}{B^2}\,\mathrm{d}t + \frac{c}{e}\left\{\frac{\boldsymbol{p}\wedge\mathrm{d}\boldsymbol{B}}{B^2} - 2\,\frac{\boldsymbol{p}\wedge\boldsymbol{B}}{B^3}\,\mathrm{d}B\right\} \tag{18}$$

from which the velocity of the centre of gyration, often called the drift velocity, is

$$\boldsymbol{u} \equiv \dot{\xi} = \boldsymbol{v}_{\parallel} + \frac{c}{e}\left\{\frac{\boldsymbol{p}\wedge\boldsymbol{v}\ \mathbf{grad}\cdot\boldsymbol{B}}{B^2} - 2\,\frac{\boldsymbol{p}\wedge\boldsymbol{B}}{B^3}\,\boldsymbol{v}\cdot\operatorname{grad}B\right\}. \tag{18a}$$

If the character of the particle motion is determined mainly by the magnetic field $\boldsymbol{B}$ and the influence of $\boldsymbol{v}\ \mathbf{grad}\cdot\boldsymbol{B}$, grad B and $\partial\boldsymbol{B}/\partial T$ can be treated as small perturbations one may expand the velocity vector $\boldsymbol{v}$ into three parts

$$\boldsymbol{v} = \boldsymbol{v}_c + \boldsymbol{v}_{\parallel} + \boldsymbol{u}_c \tag{19}$$

where $\boldsymbol{v}_c$ is the velocity vector of the cyclotron motion which rotates in a plane σ_c perpendicular to $\boldsymbol{B}$ with the angular speed ω_c and $\boldsymbol{v}_{\parallel}$ is the component parallel to $\boldsymbol{B}$ and $\boldsymbol{u}_c$ is the component of $\boldsymbol{u}$ in σ_c. According to our assumption $\boldsymbol{u}_c$ is small compared with v_c and $v_{\parallel}$ (fig. 16).

The vectors $\boldsymbol{v}\ \mathbf{grad}\cdot\boldsymbol{B}$ and grad B can be similarly considered as having one component parallel to $\boldsymbol{B}$ and a second one lying in the σ_c plane. Thus, for example,

$$\operatorname{grad} B = \operatorname{grad}_{\parallel} B + \operatorname{grad}_c B. \tag{20}$$

Let us now introduce in the plane σ_c orthogonal cartesian coordinates x, y, with unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$. The coordinate axis z and the corresponding unit vector $\boldsymbol{k}$ are then parallel to $\boldsymbol{B}$. Let us also write * $\boldsymbol{v}_c = \boldsymbol{g}\cdot v_c$.

* $g = ig_x + jg_y$, $g_x = \sin\omega_c t$, $g_y = \cos\omega_c t$.

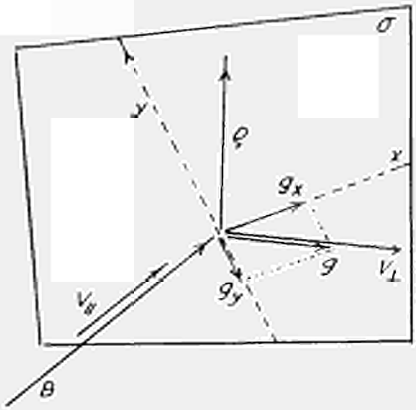


Fig. 16. The component vectors representing $\boldsymbol{p}$ and $\boldsymbol{v}$.

We have

$$(\boldsymbol{v}\,\mathbf{grad}\cdot\boldsymbol{B})_\perp = \boldsymbol{i}\left(\frac{\partial B_x}{\partial x} g_x v_\perp + \frac{\partial B_x}{\partial y} g_y v_\perp + \frac{\partial B_x}{\partial z} v_\parallel\right)$$

$$+\ \boldsymbol{j}\left(\frac{\partial B_y}{\partial x} g_x v_\perp + \frac{\partial B_y}{\partial y} g_y v_\perp + \frac{\partial B_y}{\partial z} v_\parallel\right) \tag{21a}$$

$$(\boldsymbol{v}\,\mathbf{grad}\cdot\boldsymbol{B})_\parallel = \boldsymbol{k}\left(\frac{\partial B_z}{\partial x} g_x v_\perp + \frac{\partial B_z}{\partial y} g_y v_\perp + \frac{\partial B_z}{\partial z} v_\parallel\right). \tag{21b}$$

Introducing eqs. (20), (21a, b) into eq. (18a) we have

$$\frac{B^2 e}{mc}\boldsymbol{u}_\perp = (\boldsymbol{g}\wedge\boldsymbol{k})v_\perp\left(\frac{\partial B_z}{\partial x} g_x v_\perp + \frac{\partial B_z}{\partial y} g_y v_\perp + \frac{\partial B_z}{\partial z} v_\parallel\right)$$

$$+\ \boldsymbol{k}v_\parallel\wedge\left[\ \boldsymbol{i}\left(\frac{\partial B_x}{\partial x} g_x v_\perp + \frac{\partial B_x}{\partial y} g_y v_\perp + \frac{\partial B_x}{\partial z} v_\parallel\right)\right.$$

$$\left.+\ \boldsymbol{j}\left(\frac{\partial B_y}{\partial x} g_x v_\perp + \frac{\partial B_y}{\partial y} g_y v_\perp + \frac{\partial B_y}{\partial z} v_\parallel\right)\right]$$

$$-\ 2v_\perp(\boldsymbol{g}\wedge\boldsymbol{k})\left(v_\perp g_x\frac{\partial B_z}{\partial x} + v_\perp g_y\frac{\partial B_z}{\partial y} + v_\parallel\frac{\partial B_z}{\partial z}\right), \tag{22a}$$

$$\frac{B^2 e}{mc}(u_\parallel - v_\parallel) = v_\perp g_x\left(\frac{\partial B_y}{\partial x} g_x v_\perp + \frac{\partial B_y}{\partial y} g_y v_\perp + \frac{\partial B_y}{\partial z} v_\parallel\right)$$

$$-\ v_\perp g_y\left(\frac{\partial B_x}{\partial x} g_x v_\perp + \frac{\partial B_x}{\partial y} g_y v_\perp + \frac{\partial B_x}{\partial z} v_\parallel\right). \tag{22b}$$

Let us neglect purely cscillatory terms in eqs. (22a, b) and consider only $\bar{u}_c$ and $\bar{u}_{\parallel}$. The oscillatory terms occur in connection with coefficients such as g_x, g_y, $g_x g_y$. The mean value of terms g_x^2, g_y^2 is ½.

We also note that the vector

$$g \wedge k = -ig_y + jg_x.$$

Putting $eB/mc = \omega_c$ we get for $\bar{u}_c$ and $\bar{u}_{\parallel}$

$$B\omega_c\bar{u}_c = i\left\{\frac{1}{2}\frac{\partial B_z}{\partial y}v_c^2 + \frac{\partial B_y}{\partial z}v_{\parallel}^2\right\} - j\left\{\frac{1}{2}\frac{\partial B_z}{\partial x}v_c^2 + \frac{\partial B_x}{\partial z}v_{\parallel}^2\right\} \quad (23a)$$

$$B\omega_c\bar{u}_{\parallel} = B\omega_c v_{\parallel} + \frac{1}{2}v_c^2\frac{\partial B_y}{\partial x} - \frac{1}{2}v_c^2\frac{\partial B_x}{\partial y}. \quad (23b)$$

Assuming that volume currents can be neglected, the second Maxwell equation gives

$$\text{curl}\, B = 0$$

and therefore,

$$\frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y},\ \frac{\partial B_y}{\partial z} = \frac{\partial B_z}{\partial y},\ \frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x}.$$

Substituting these relationships into eqs. (23a, b) we obtain

$$\bar{u}_c = \frac{B \wedge \text{grad}\, B_z}{B^2\omega_c}(\tfrac{1}{2}v_c^2 + v_{\parallel}^2) \quad (24a)$$

and

$$\bar{u}_{\parallel} = v_{\parallel}. \quad (24b)$$

We shall use these formulae to investigate the motion of charged particles in three different magnetic field configurations (in the following sections $\bar{u}$ will be denoted simply by u).

2.2.1. MOTION OF CHARGED PARTICLES IN A TOROIDAL MAGNETIC FIELD

Let us consider a magnetic field which in cylindrical coordinates has only a component B_φ and which is uniform in the φ- and z-direction (fig. 17). In the absence of volume currents it follows from the 2nd Maxwell equation that

$$\frac{\partial B_\varphi}{\partial r} = -\frac{B_\varphi}{r}. \quad (25)$$

In this coordinate system the velocity of the particle can be represented as

$$\boldsymbol{v}_c = \boldsymbol{v}_r + \boldsymbol{v}_z,\ v_\parallel = v_\varphi.$$

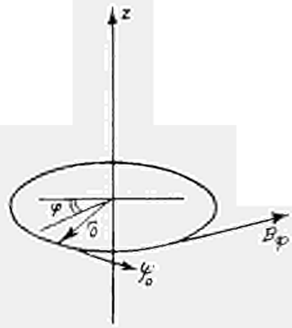


Fig. 17. The geometry of a toroidal magnetostatic field ($\boldsymbol{r}_0$ and φ_0 are unit vectors).

The drift velocity of the centre of gyration of a charged particle follows from eqs. (24a, b). Thus

$$\boldsymbol{u}_c = -\varphi_0 \wedge \boldsymbol{r}_0 \frac{1}{r \dfrac{e}{mc} B_\varphi} \left(\tfrac{1}{2} v_c^2 + v_\varphi^2\right) \tag{26a}$$

$$u_\parallel = v_\varphi. \tag{26b}$$

As eq. (26a) involves the value of the charge e carried by the particle, it follows that the u_c drift for negatively charged particles is in a direction opposite to the drift of positively charged particles.

It is interesting to note that the component of the drift associated with the speed v_c arises from the radial non-uniformity of B_φ, whereas the component associated with v_φ derives from the centrifugal force mv_φ^2/r of the circulating particle. This latter component can be also calculated directly from a force-balance equation

$$\boldsymbol{r}_0 \frac{mv_\varphi^2}{r} = \frac{e}{c} \boldsymbol{u} \wedge \boldsymbol{B} \tag{27}$$

which gives

$$u = \frac{mc}{e} \frac{v_\varphi^2}{rB_\varphi} \quad \text{(cm/sec)}. \tag{27a}$$

A beam of particles injected tangentially into the field B_φ will spiral around the axis, whilst individual particles will also spiral on a tube of flux (fig. 18). The trajectory of individual particles is, therefore, a double helix. The problem of how the injection speed $v_{\varphi 0}$ is divided into v_c and v_φ of spiral is outside the scope of this chapter.

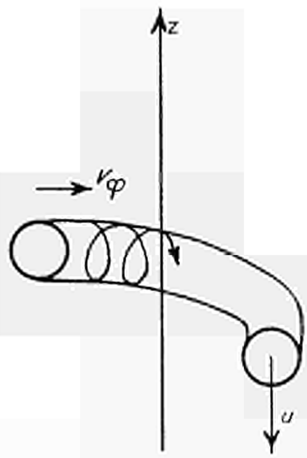


Fig. 18. Drift of particles in a toroidal magnetostatic field.

Spiral-staircase electron beams have found an interesting application in plasma physics as a means for an efficient ionization of low density gas in a toroidal chamber (ref. 2).

2.2.2. MOTION OF CHARGED PARTICLES IN THE FIELD OF A MAGNETIC LENS

The motion of charged particles in the field of a simple cylindrical magnetic lens can be considered from two extreme points of view. One is that of electron optics, in which it is usually assumed that the lens changes the momentum of the particles only by a small amount. The other view is found in plasma physics, where one usually assumes that the lens is so strong that the Larmor radius of particles is much smaller than the dimensions of the lens and that most of the particles to be studied have their velocities oriented at random at least in two dimensions. Somewhere between these extremes is the subject of cosmic ray interactions with the magnetic fields of cosmic clouds, of stars and of planets (ref. 3).

In this section we shall deal with the motion for which

$$\rho \equiv \frac{mv}{\frac{e}{c}B} \ll d$$

where d is some characteristic dimension of the lens. The description of the motion of such a particle can be based on the drift-velocity formulae. However, in order to apply these equations, information is required on v_c and $v_{\parallel}$. This can be derived from the invariance of the total kinetic energy of the particle, which is

$$W = \tfrac{1}{2}m(v_c^2 + v_{\parallel}^2) = \text{const.} \tag{28}$$

and from the adiabatic invariance of the magnetic moment μ. Magnetic moment of a gyrating charged particle is

$$\mu = \pi\rho^2 \cdot i$$

where i is the circulating current. As $i = \dfrac{e}{c} \cdot \dfrac{v_c}{2\pi\rho}$, $\rho = \dfrac{mv_c \cdot c}{eB}$ we have

$$\mu = \tfrac{1}{2}\,\rho \cdot \frac{e}{c}\,v_c = \frac{\tfrac{1}{2}\,mv_c^2}{B}. \tag{29}$$

It is evident that $\mu \propto \dfrac{W_c}{\omega_c}$ where W_c is the energy of the cyclotronic motion. It has been shown by Ehrenfest that the ratio of kinetic energy W of an oscillator and its frequency ω is an adiabatic invariant (ref. 4) and therefore, in the case of a gyrating particle

$$\mu = \text{invariant.}$$

The magnetic moment μ is only approximately invariant and its variation $\Delta\mu$ is critically dependent on the ratio

$$\frac{v_{\parallel}/L}{eB_0/2\pi mc} = \frac{\tau_c}{\tau_t}, \tag{30}$$

where τ_c is the period of the cyclotron motion and τ_t is the transit time of the particle through a non-uniformity whose dimension is L. It can be shown that

$$\frac{\Delta\mu}{\mu} = \exp\left(a\,\frac{\tau_c}{\tau_t}\right) \tag{31}$$

and a is a factor depending on the relative amplitude $\Delta B/B_0$ of the non-uniformity (ref. 5).

The consequence of assuming that the magnetic moment is constant is that, apart from drift motion of the centre of gyration, a charged particle is bound to a surface of a tube of constant flux. This follows from eqs. (29) and (10). Thus

$$\frac{\frac{1}{2}\, m^2 v_c^2}{B^2} \frac{B}{m} = \text{const.} \qquad \text{or } \rho^2 B = \text{const.}$$

and as $\pi\rho^2 B = \phi$ where ϕ is the magnetic flux, it follows that the orbit of the particle links a constant amount of magnetic flux.

Let us now consider the motion of a charged particle on a converging tube of flux of rotational symmetry (fig. 19). Here eq. (24a) for u_c gives $u_c = 0$ owing to $\boldsymbol{B} \wedge \text{grad}\, B_z = 0$.

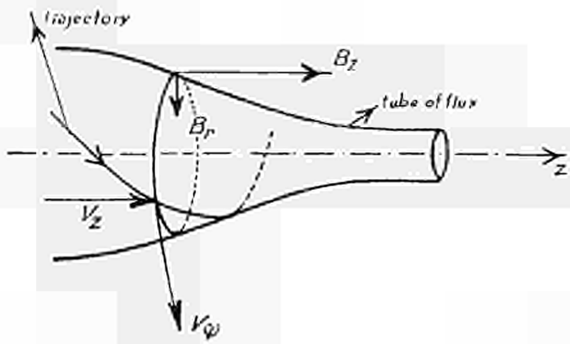


Fig. 19. Motion of a particle on a converging tube of flux.

The equation for $u_\parallel$ does not contain any information as $u_\parallel = v_\parallel$. It is, therefore, necessary to find an equation for $v_\parallel$. This is

$$\frac{\mathrm{d}}{\mathrm{d}t} v_\parallel = \frac{F_\parallel}{m} \tag{32}$$

where $F_\parallel$ is a force on the particle in the direction of the vector $\boldsymbol{B}$. This force is the component of the Lorentz force in this direction. Thus

$$F_\parallel = \left| \frac{e}{c} \boldsymbol{v} \wedge \boldsymbol{B} \right|_\parallel$$

$$= \frac{e}{c} v_c B_r. \tag{33}$$

The component B_r is related to B_z through div $\boldsymbol{B} = 0$. Thus

$$\frac{\partial(rB_r)}{\partial r} = -r\frac{\partial B_z}{\partial z}.$$

Assuming that for $r \sim \rho$, $\partial B_z/\partial z$ is independent of r one obtains on integrating this equation with respect to r:

$$B_r = -\frac{1}{2}\rho\frac{\partial B_z}{\partial z}.$$

Substituting this into eq. (33) yields

$$F_{\|} = -\frac{1}{2}\frac{e}{c}v_c\rho\frac{\partial B_z}{\partial z}$$

$$F_{\|} = -\frac{\frac{1}{2}mv_c^2}{B}\frac{\partial B_z}{\partial z} \tag{33a}$$

or expressed in a vector form

$$F_{\|} = -\mu \cdot \text{grad } B. \tag{33b}$$

The equation of motion in the B-direction becomes

$$\dot{v}_{\|} = -\frac{\mu}{m} \cdot \text{grad } B_z. \tag{34}$$

This equation shows that charged particles, incident on a region of strong magnetic field experience deceleration in the direction of B and in some cases are reflected. However, as their total kinetic energy remains constant, it follows that as $v_{\|}$ decreases v_c increases and vice versa. This can be expressed mathematically using eq. (28) as

$$v_c^2 = \frac{2W}{m} - v_{\|}^2. \tag{35}$$

The condition of reflection is that $v_{\|} = 0$. When this is so v_c reaches a maximum v_{cM}

$$v_{cM} = \sqrt{\frac{2W}{m}}. \tag{36}$$

Assuming the magnetic moment to be invariant it follows that

$$\mu = \frac{\frac{1}{2}mv_{cM}^2}{B_M} = \frac{W}{B_M}. \tag{37}$$

It is clear that the smaller the magnetic moment, for a given total energy W, the further will the charged particle penetrate along the converging flux tube *.

* In this connection one defines $J = \int mv_{\|}dl$, which can be shown to be invariant in some situations (longitudinal invariant, ref. 6).

A more general problem is represented by the motion of a charged particle in a magnetostatic field whose flux tubes form a converging and bent bundle (fig. 20). This is the configuration off axis of a cylindrical lens.

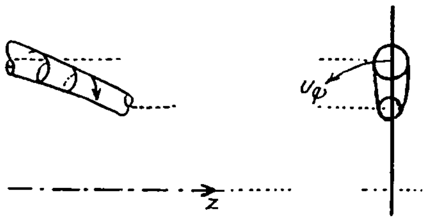


Fig. 20. Drift motion in the field of a magnetic mirror.

If B varies only slowly with respect to z one has $\boldsymbol{B} \wedge \operatorname{grad} B_z \approx \mathrm{B} \cdot \partial B_z/\partial r$ and eqs. (24a), (34) become

$$u_\varphi = \left[\left(\frac{\partial B_z}{\partial r}\right) \Big/ \left(\frac{eB^2}{mc}\right)\right] (\tfrac{1}{2}v_c^2 + v_\|^2) \tag{38}$$

$$\dot{v}_\| = \tfrac{1}{2}v_c^2 B_z^{-1} \frac{\partial B_z}{\partial z}. \tag{39}$$

The reflection condition remains approximately the same as formulated in eq. (37). However, the particle does not return after the reflection along the same tube of flux but precesses in the φ-direction during the reflection process.

Let us study this precession in the case of a magnetic mirror (fig. 21). This is a particular case of a lens geometry in which

$$B = B_0 \text{ for } z < a, \quad B = B_1 \text{ for } z > b$$

where $B_1 > B_0$ and $b > a$.

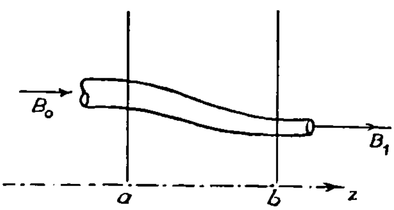


Fig. 21. Flux-tube in a magnetic mirror.

It follows from eq. (37) that particles for which

$$\mu > \frac{W}{B_1}$$

will be reflected. Let us divide this inequality by $(v_\parallel^2)_0$ where $(v_\parallel)_0 = v_\parallel$ for $z < a$. Then

$$\frac{\left(\dfrac{v_c}{v_\parallel}\right)_0^2}{B_0} > \frac{\left(\dfrac{v_c^2}{v_\parallel^2}\right)_0 + 1}{B_1}.$$

Putting

$$\left(\frac{v_c}{v_\parallel}\right)_0 = \tan\theta$$

where θ is the angle between the vector v_0 and the vector $\boldsymbol{B}_0$, one can write the condition for reflection as

$$\sin^2\theta > \frac{B_0}{B_1}. \tag{40}$$

We shall follow a typical particle which satisfies this inequality. The angle of precession φ of such a particle will be

$$\begin{aligned} \varphi &= 2\int_0^\tau \frac{u_\varphi}{r}\,\mathrm{d}t \\ &= 2\frac{c}{e}\int_0^\tau \frac{\partial B_r/\partial z}{rB}\left(\frac{2W}{B} - \mu\right)\mathrm{d}t. \end{aligned} \tag{41}$$

This can be written as

$$\varphi = 2\frac{c}{e}\int_a^{z-a} \frac{\partial B_z/\partial r}{rB}\left(\frac{2W}{B} - \mu\right)\frac{\mathrm{d}z}{v_\parallel}.$$

From eq. (39) it follows that

$$v_\parallel \dot{v}_\parallel = \frac{\mu}{m}\frac{\partial B_z}{\partial t}$$

or

$$v_\parallel = \left[2\frac{\mu}{m}(B_z - B_0) + v_{\parallel 0}^2\right]^{1/2}. \tag{42}$$

Also near the z-axis (i.e., for $r < b - a$) one can expand B_z as

$$B_z(r, z) \cong B_A - \tfrac{1}{4}r^2 B''_A$$

where

$$B_\Lambda = B_\Lambda (0, z), B''_\Lambda = \frac{\partial^2}{\partial z^2} B_\Lambda.$$

Using this expression and eq. (42) one obtains

$$\varphi \cong -\frac{c}{e} \int_a^{z-a} \frac{B''_\Lambda \left(\frac{2W}{B_z} - \mu \right) \mathrm{d}z}{B_z \left[2 \frac{\mu}{m} (B_z - B_0) + v_{\parallel 0}{}^2 \right]^{1/2}} \tag{43}$$

which can be easily evaluated for a specific field geometry. It can be appreciated that owing to B''_Λ changing sign in the interval (a, b) the sign of φ may be either positive or negative.

The field of a magnetic dipole can be also regarded as a magnetic lens of the type discussed in this section. However, as the field strength of a dipole depends on the distance r from the centre of the dipole as $1/r^3$, the motion of a charged particle in this field cannot be always correctly represented by the drift motion of its centre of gyration. In particular, the theorem of conservation of the magnetic moment μ of the particle breaks down when the radius of gyration ρ becomes larger than the distance r.

A very elegant experiment on particle trajectories in mirror-fields has been based on the production of positrons and the tracking of their excursions in the magnetic field (ref. 7).

*

2.2.3. MOTION OF CHARGED PARTICLES IN A HELICAL MAGNETIC FIELD

Let us consider the field of a cylindrical lens, to which a B_φ component has been added. The flux tubes of such a field possess a certain twist around the axis of symmetry z. This twist can be specified by the angle ψ between the osculating plane of the field line and the axis z (fig. 22).

It is evident that as the angle ψ increases, the angle ψ' between the axis z and the drift-velocity vector $\boldsymbol{u}$ increases also. In order that the progress of the particle in the z-direction be arrested one must have

$$\psi' = \psi + \tan^{-1} \left(\frac{u_c}{u_\parallel} \right) = \tfrac{1}{2}\pi$$

or

$$u_c = u_\parallel \cdot \tan (\tfrac{1}{2}\pi - \psi). \tag{44}$$

From this it follows that the drift discussed on p. 38 inherent in purely toroidal fields can be compensated for or even reversed by the addition of a B_r, B_z field of a cylindrical lens. As in most cases of

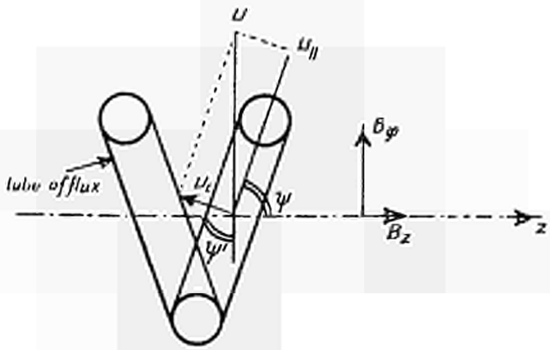


Fig. 22. Drift motion in a helical magnetostatic field.

interest $u_c/u_\parallel \ll 1$ it follows from eq. (44) that $1/2\pi - \psi \ll 1/2\pi$ and, therefore, the value of u_c can be obtained directly from eq. (26a). The value of the B_z field which will compensate the axial component of this drift is

$$B_z = B_\varphi \operatorname{cotg} \psi = B_\varphi \frac{u_c}{u_\parallel} \,. \tag{45}$$

and using eqs. (26a, b) one obtains

$$B_z = \frac{mc}{e} \frac{1}{r} \left(\frac{\frac{1}{2} v_c^2}{v_\varphi} + v_\varphi \right) . \tag{45a}$$

It is clear that this formula gives the correct field as $v_c \to 0$. The particle is then constrained to a circular trajectory of radius

$$r \equiv \rho = \frac{mv_\varphi}{\frac{e}{c} B_z} \,.$$

A similar mechanism is used for the confinement of hot plasma in a toroidal magnetic field (ref. 8); see also chapter 9 of this book.

*

2.2.4. SUPERIMPOSED TOROIDAL MAGNETIC FIELD AND BETATRON MAGNETIC FIELD

As an example of the way in which formula (24a) may be applied let us study the motion of a particle in a betatron-type field with a superimposed toroidal magnetic field (fig. 23). In this case we can compare the results of an analysis based on conventional equations of motion and that using the concept of drift-motion.

The equations of motion of a charged particle in this helical field are

$$\ddot{r} - \frac{v_\varphi^2}{r} - \frac{e}{mc} v_\varphi B_z = \frac{-e}{mc} \dot{z} B_\varphi \tag{46}$$

$$\ddot{z} + \frac{e}{mc} v_\varphi B_r = \frac{e}{mc} \dot{r} B_\varphi . \tag{47}$$

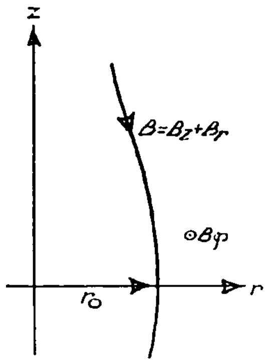


Fig. 23. Geometry of a betatron field.

Let us expand r, B_z, B_r and B_φ about their respective values at $r = r_0$, $z = 0$. Thus

$$x = r - r_0$$

$$B_z = B_{0z} + x \frac{\partial B_z}{\partial x} + z \frac{\partial B_z}{\partial z} \tag{48a}$$

$$B_r = x \frac{\partial B_r}{\partial x} + z \frac{\partial B_r}{\partial z} \tag{48b}$$

$$B_\varphi = B_{0\varphi} + x \frac{\partial B_\varphi}{\partial x} . \tag{48c}$$

As the betatron field is produced by external currents only

$$\operatorname{curl} \boldsymbol{B} = 0, \ \operatorname{div} \boldsymbol{B} = 0$$

and we get

$$\frac{\partial B_z}{\partial x} = \frac{\partial B_r}{\partial z}, \quad \frac{\partial B_z}{\partial z} = -\frac{1}{r} \frac{\partial (r B_r)}{\partial r} .$$

As $B_r = 0$ for $z = 0$ it follows from the last of these and from (48b) that

$$\left|\frac{\partial B_z}{\partial x}\right|_{z=0} = 0, \quad \left|\frac{\partial B_r}{\partial x}\right|_{z=0} = 0.$$

Eqs. (48 a, b, c) can, therefore, be written

$$B_z = B_{0z} + x\,\frac{\partial B_z}{\partial x} \tag{49a}$$

$$B_r = z\,\frac{\partial B_z}{\partial x} \tag{49b}$$

$$B_\varphi = B_{0\varphi} - x\,\frac{B_{0\varphi}}{r_0}\,. \tag{49c}$$

The value B_{0z} can be taken to be the field corresponding to the steady state solution of eqs. (46) and (47) for which the centrifugal force of the rotating particle is exactly balanced by the magnetic force. Thus

$$m\,\frac{v_\varphi^2}{r_0} = -\frac{e}{c}\,v_\varphi B_{0z}. \tag{50}$$

With the help of eqs. (49a, b, c) and (50) the equations of motion can be written:

$$\ddot{x} + \frac{v_\varphi^2}{r_0^2}\,x - \frac{e}{mc}\,v_\varphi\,\frac{\partial B_z}{\partial x}\,x = -\frac{e}{mc}\,\dot{z}B_{0\varphi}\left(1 - \frac{x}{r_0}\right) \tag{51}$$

$$\ddot{z} + \frac{e}{mc}\,v_\varphi\,\frac{\partial B_z}{\partial x}\,z = \frac{e}{mc}\,\dot{x}B_{0\varphi}\left(1 - \frac{x}{r_0}\right). \tag{52}$$

Let us define the parameters

$$-\frac{\left|r\,\dfrac{\partial B_z}{\partial r}\right|_{r=0}}{B_{0z}} = n \tag{53}$$

$$\frac{eB_{0z}}{mc} = \omega_0, \quad \frac{eB_{0\varphi}}{mc} = \omega_c. \tag{54a,b}$$

Neglecting all non-linear terms in x and z, the equations (51) and (52) become

$$\ddot{x} + \omega_0^2(1 - n)x = -\omega_c\dot{z} \tag{55}$$

$$z + \omega_0^2 n z = \omega_c x. \tag{56}$$

These equations are of the same form as those for two coupled harmonic oscillators and have two solutions of the form

$$x = \xi e^{j\omega t} \qquad z = \eta e^{j\omega t}. \tag{57a,b}$$

Substituting these into eqs. (55) and (56) one obtains the characteristic equations:

$$\xi[-\omega^2 + (1-n)\omega_0^2] = -\mathrm{j}\,\omega\,\omega_c\eta \tag{58}$$

$$\eta[-\omega^2 + n\omega_0^2] = \mathrm{j}\,\omega\,\omega_c\xi. \tag{59}$$

The determinant of these two homogeneous equations must be zero, which gives

$$\left(\frac{\omega}{\omega_0}\right)^2 = \frac{1}{2}\left[1 + \frac{\omega_c^2}{\omega_0^2}\right] \pm \frac{1}{2}\left[\left(1 + \frac{\omega_c^2}{\omega_0^2}\right)^2 - 4n(1-n)\right]^{1/2} \tag{60}$$

Let us plot the function

$$\frac{\omega}{\omega_0} = f\left(\frac{\omega_c}{\omega_0}\right)$$

for particular values of n. In fig. 24 ω/ω_0 is plotted for $n = 1/2$ and $n = 1/3$. For $\omega_c = 0$ the two proper frequencies ω_{10} and ω_{20} represent

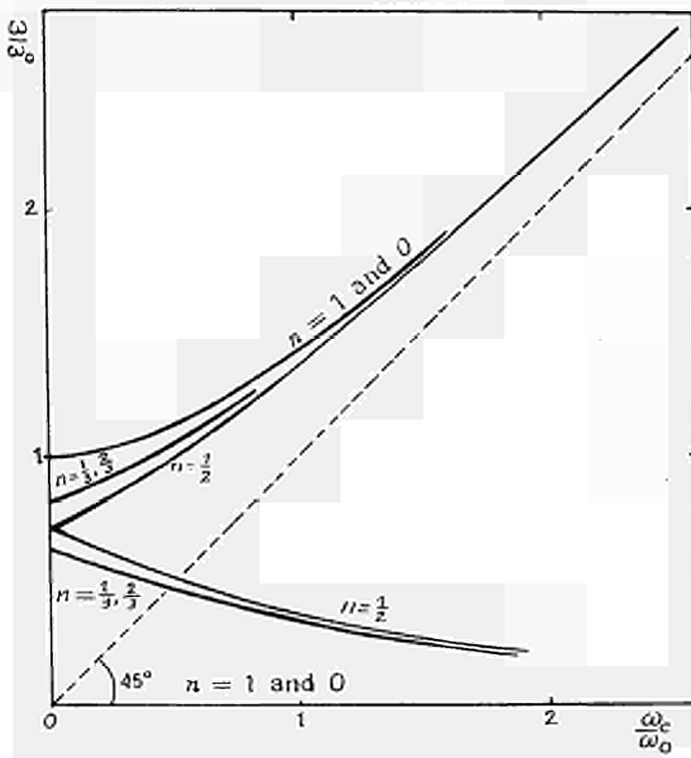


Fig. 24. The two betatron frequencies for a betatron with a super-imposed B_θ field.

two independent betatron oscillations of the particle, one in the radial and the other in the axial direction.

In the presence of the toroidal field these two oscillations are coupled and one must distinguish four different amplitudes

$$\xi_1 \eta_1 \text{ and } \xi_2 \eta_2.$$

From eq. (59) one obtains

$$\frac{\xi_k}{\eta_k} = -\mathrm{j}\,\frac{n\omega_0^2 - \omega_k^2}{\omega_k \omega_c} \quad \text{where } k = 1, 2. \tag{61}$$

For a toroidal field much stronger than the betatron field, i.e., for

$$\omega_c \gg \omega_0$$

one has from eq. (60)

$$\frac{\omega_1}{\omega_0} = \frac{\omega_c}{\omega_0}\left[1 - \frac{1}{2}\left(\frac{\omega_0}{\omega_c}\right)^2\right] \tag{62a}$$

$$\frac{\omega_2}{\omega_0} = \frac{\omega_0}{\omega_c}\sqrt{n(1-n)}\left(1 - \frac{1}{2}\,\frac{\omega_0^2}{\omega_c^2}\right). \tag{62b}$$

Substituting these into eq. (61) one has

$$\frac{\xi_1}{\eta_1} = -\mathrm{j}\left[1 - (n + 1/2)\,\frac{\omega_0^2}{\omega_c^2}\right] \tag{63a}$$

and

$$\frac{\xi_2}{\eta_2} = -\mathrm{j}\sqrt{\frac{n}{1-n}}\left[1 - (1-n)\,(n - 1/2)\,\frac{\omega_0^2}{\omega_c^2}\right]. \tag{63b}$$

The movement is, therefore, composed of two elementary motions (fig. 25), one being nearly a cyclotron motion with angular frequency

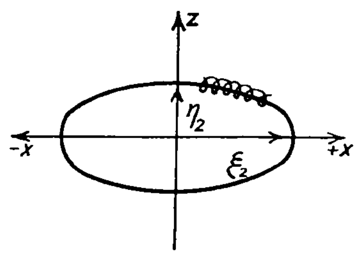


Fig. 25. Drift in a meridian plane of a betatron with superimposed toroidal magnetostatic field.

$\omega_1 \approx \omega_c$, the other a slow elliptic motion whose angular frequency is

$$\omega_2 \approx \sqrt{n(1-n)}\,\frac{\omega_0^2}{\omega_c} \tag{64}$$

and for which the ratio of the major axis to the minor axis of the ellipse is

$$\frac{\xi_2}{\eta_2} = \sqrt{\frac{n}{1-n}}\,. \tag{65}$$

In order to treat the case of $B_\varphi \gg B_{0z}$ by means of drift-velocity formula (24a) we must define the force $\boldsymbol{F}$ and grad B. In a betatron field this is the restoring force that the particle of speed v_φ experiences. This force has two components, which were already formulated in eqs. (51) and (52)

$$F_r = \frac{e}{c}\,v_\varphi\,\frac{\partial B_z}{\partial x}\,x - \frac{e}{c}\,v_\varphi\,\frac{B_{0z}}{r_0}\,x \tag{66a}$$

$$F_z = \frac{e}{c}\,v_\varphi\,\frac{\partial B_z}{\partial x}\,z. \tag{66b}$$

In the expression for $\boldsymbol{B} \wedge$ grad B we shall neglect contributions of second order in B_z and B_r. Substituting into formula (24a) and assuming that $v_c \ll v_\parallel$ we get

$$u_x = \frac{c}{e}\,\frac{\partial B_z}{\partial x}\,\frac{v_\varphi}{B_\varphi}\left(z + \frac{mv_\varphi}{B_\varphi}\right) \tag{67}$$

$$u_z = v_\varphi\frac{\partial B_z}{\partial x}\,\frac{x}{B_\varphi} + \frac{mc}{e}\,\frac{v_\varphi^2}{r_0 B_\varphi}\left(1 - \frac{3}{2}\,\frac{B_z^2}{B_\varphi^2} - \frac{x}{r_0}\right) + \frac{mc}{e}\,\frac{\partial B_z}{\partial x}\,\frac{v_\varphi^2}{B_\varphi^2}\,. \tag{68}$$

Remembering that $B_\varphi \gg B_{0z}$ and using eq. (53) we obtain

$$\dot{x} = u_x = n\,\frac{B_{0z}}{B_{0\varphi}}\,\frac{z}{r_0}\,v_\varphi \tag{69}$$

$$\dot{z} = u_z = \frac{B_{0z}}{B_{0\varphi}}\,\frac{x}{r_0}\,(n-1)v_\varphi. \tag{70}$$

These are the equations of motion of the centre of gyration in the meridian plane of the betatron. Differentiating eq. (69) and substituting $\dot{z}$ from eq. (70) we get

$$\ddot{x} = n(n-1)\left(\frac{B_{0z}}{B_{0\varphi}}\right)^2\frac{v_\varphi^2}{r_0^2}\,x. \tag{71}$$

The solution is

$$x = \xi e^{j\omega t}. \tag{72a}$$

Similarly we find

$$z = \eta e^{j\omega t}. \tag{72b}$$

The corresponding characteristic equation gives

$$\omega = \sqrt{n(1-n)}\,\frac{\omega_0^2}{\omega_c} \tag{73}$$

which is the same as eq. (64).

The ratio ξ/η follows from either eq. (69) or eq. (70).

$$\frac{\xi}{\eta} = -j\sqrt{\frac{n}{1-n}} \tag{74}$$

which agrees with eq. (65).

The comparison of the two analyses shows that the drift-velocity description is less detailed than the full analysis of the equations of motion; however, it is generally shorter and often easier to interpret than the latter.

2.3. Motion of Charged Particles in Crossed Electric and Magnetic Fields

In a plasma confined by a magnetic field the force of the electric field $e\boldsymbol{E}$ on an electron is small compared with the Lorentz force $(e/c)\boldsymbol{v} \wedge \boldsymbol{B}$, even though the velocity v is often a small fraction of c. This is not always true for the positive ions.

In this section we shall study the motion of charged particles in combined electric and magnetic field, restricting our attention to the situation mentioned above, i.e., making the assumption that the effect of the electric field on the particles can be regarded as a small perturbation of their cyclotron motion. This leads to a description using the concept of a drift velocity and therefore, it will be more often applicable to electrons in a plasma rather than to positive ions.

We shall derive a formula for the radius of gyration of a particle in an electric and magnetic field. The force balance in the σ plane gives (fig. 26)

$$m\,\frac{v_\perp^2}{\rho} + \frac{e}{v_\perp}\,|\boldsymbol{E}_\perp \wedge \boldsymbol{v}_\perp| = \frac{e}{c}\,v_\perp B \tag{75}$$

when recalling the assumption made above

$$\rho = \frac{c}{e}\,\frac{\boldsymbol{p}\wedge\boldsymbol{B}}{B^2}\left[1 + c\,\frac{|\boldsymbol{E}_\perp \wedge \boldsymbol{v}_\perp|}{v_\perp{}^2 B}\right]. \tag{76}$$

A differential translation of the centre of gyration is given by

$$\mathrm{d}\xi = \rho' - \rho + \mathrm{d}\boldsymbol{r}.$$

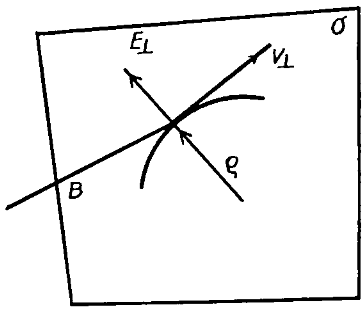


Fig. 26. Motion in crossed electric and magnetic fields.

Let us assume that the magnetic field is uniform. Using a treatment similar to that on page 34 we have

$$\rho' - \rho = \frac{c}{e}\left\{\frac{\mathrm{d}\boldsymbol{p}\wedge\boldsymbol{B}}{B^2} + \frac{\boldsymbol{p}\wedge\boldsymbol{B}}{B^3}\left[\frac{c}{v_\perp{}^2}\,\mathrm{d}\,|\boldsymbol{E}_\perp\wedge\boldsymbol{v}_\perp| \right.\right.$$
$$\left.\left. - \frac{2c}{v_\perp{}^3}\,|\boldsymbol{E}_\perp\wedge\boldsymbol{v}_\perp|\mathrm{d}v_\perp\right]\right\} \tag{77}$$

From

$$\mathrm{d}\boldsymbol{p} = \left(e\boldsymbol{E} + \frac{e}{c}\,\boldsymbol{v}\wedge\boldsymbol{B}\right)\,\mathrm{d}t$$

we get

$$\frac{\mathrm{d}\boldsymbol{p}\wedge\boldsymbol{B}}{B^2} = -\,\frac{e}{c}\,\boldsymbol{v}\,\mathrm{d}t + \frac{e}{c}\,\frac{\boldsymbol{B}(\boldsymbol{v}\cdot\boldsymbol{B})}{B^2}\,\mathrm{d}t + e\,\frac{\boldsymbol{E}\wedge\boldsymbol{B}}{B^2}\,\mathrm{d}t.$$

Substituting these into eq. (77) one gets

$$\boldsymbol{u} = \dot{\xi} = \boldsymbol{v}_\parallel + \frac{c}{eB^2}\left\{e\boldsymbol{E}\wedge\boldsymbol{B} + \frac{mc\,\boldsymbol{v}\wedge\boldsymbol{B}}{Bv_\perp}\left[\frac{1}{v_\perp}\,\frac{\mathrm{d}}{\mathrm{d}t}\,|\boldsymbol{E}_\perp\wedge\boldsymbol{v}_\perp|\right.\right.$$
$$\left.\left. - \frac{2}{v_\perp{}^2}\,|\boldsymbol{E}_\perp\wedge\boldsymbol{v}_\perp|\,\frac{dv_\perp}{\mathrm{d}t}\right]\right\}. \tag{78}$$

Some of the terms in the square brackets are of oscillatory nature, the rest can be neglected in comparison with the $E \wedge B$ term *. Thus

$$u_c = c \frac{E \wedge B}{B^2} \tag{79a}$$

$$u_{\|} = v_{\|}. \tag{79b}$$

This formula does not contain the charge e of the particle and therefore, the drift velocity in crossed electric and magnetic fields does not depend on the sign of the charge. In uniform electric fields the eq. (79a) is valid for any ratio of E and B.

The expression (79a) can be generalized for any force field F by substituting

$$E = \frac{F}{e}.$$

The general result of eq. (79a) is that the drift velocity vector is perpendicular to both the force field F and the magnetic field B.

As our analyses of the effect of grad B in the previous section and that of the electric field E in this section were both developed to the first order in the magnitude of these perturbations one can superimpose the expressions (24a) and (79a) and get a general formula for u_c:

$$u_c = \frac{c}{e} \frac{F \wedge B}{B^2} + \frac{B \wedge \operatorname{grad} B}{\frac{e}{mc} B^3} (\tfrac{1}{2} v_c^2 + v_{\|}^2) \tag{80}$$

or, using the expressions for the magnetic moment μ and the total kinetic energy W one has

$$u_c = \frac{c}{e} \frac{1}{B^2} \left\{ F \wedge B + \left(\frac{2W}{B} - \mu \right) B \wedge \operatorname{grad} B \right\}. \tag{80a}$$

The formula (79a) for the $E \wedge B$ drift has been derived on the assumption of small perturbations to the cyclotron motion. It can be shown that for this type of drift such an assumption is not necessary and we shall derive the above mentioned formula directly from full equations of motion. Let us assume that $E = E_y$, $B = B_z$ and that these fields

* The second term, though usualy small compared to $c \frac{E \wedge B}{B^2}$, is of importance in evaluating the transient current eu_1n in a plasma of density n. Thus $u_1 = \frac{mc^2}{eB^2} \dot{E}_\perp$ (see p. 58).

are uniform. The motion of a charged particle is then described by

$$m\ddot{x} = \frac{e}{c}\,\dot{y}\,B \tag{81a}$$

$$m\ddot{y} = eE - \frac{e}{c}\,\dot{x}\,B \tag{81b}$$

Integrating the first and substituting into the second we get

$$\ddot{y} = \frac{e}{m}\,E - \omega_c^{\,2}\,y - \omega_c\,\dot{x}_o$$

whose solution is

$$y = A\,\sin\,(\omega_c\,t + \varphi) - A\,\sin\,\varphi \tag{82}$$

which gives for the $\dot{x}$ and $\dot{y}$

$$\dot{x} = \omega_c\,A\,\sin\,(\omega_c\,t + \varphi) + c\,\frac{E}{B}$$

$$\dot{y} = \omega_c\,A\,\cos\,(\omega_c\,t + \varphi)$$

The speed of the cyclotron motion is evidently

$$\mathrm{v} = (\dot{x}^2 + \dot{y}^2)^{1/2} = \omega_c\,A \tag{83}$$

whereas a drift appears in the x-motion whose speed is

$$u = c\,\frac{E}{B}$$

identical with expression in eq. (79a).

Comparing v and u it is possible to distinguish three types of motion

a) $v \gg u$ — this is the case corresponding to the perturbational analysis worked out on pp. 53 and 54 (fig. 27a).

b) $v = u$ — the trajectory is the classical cycloid (fig. 27b).

c) $v \ll u$ — the trajectory corresponds to an epicycloid, generated by a circular motion of small amplitude superimposed on fast linear translation (fig. 27c).

In conclusion let us mention the case in which $E > B$. The simple formula (79a) gives $u > c$. This paradox is easily resolved writing the original equations of motion (eqs. (81a, b)) in a relativistic form. It can be shown then that $u < c$, consequently at no time can the Lorentz force

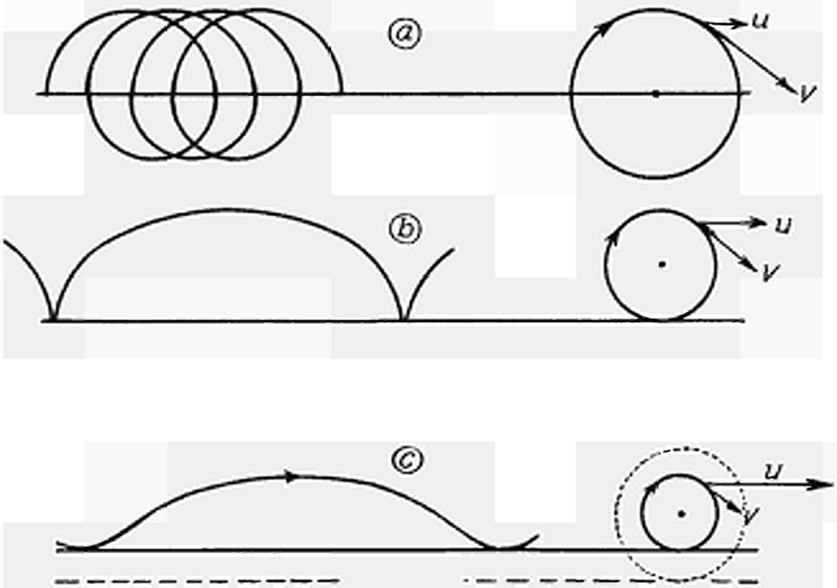


Fig. 27. Motion of a charged particle in crossed electric and magnetic fields.

$\frac{e}{c}\dot{x}B$ ($< eB$) balance the electric force eE and the orbit is an open one, i.e., the particle deviates steadily from the x-axis.

2.3.1. ELECTRIC VORTEX FIELD AND MAGNETIC LENS FIELD

As another application of eq. (79a) let us consider a magnetic field changing in time. Associated with such a magnetic field will be an electric vortex field given by the first of Maxwell's equations

$$\operatorname{curl} \boldsymbol{E} = -\frac{1}{c}\dot{\boldsymbol{B}}.$$

If the magnetic field is homogeneous within a radius $R > r$ one finds

$$E = E_\varphi = \frac{\dot{B}_z}{2c}\, r. \tag{84}$$

Let us assume that the change of B is slow compared to the cyclotron motion, i.e., that

$$\dot{B}\tau \ll B$$

where $\tau = \dfrac{2\pi}{\omega_c}$, and that $\rho \ll r$.

The motion in crossed fields $E_\varphi B_z$ consists then of two component motions (fig. 28).

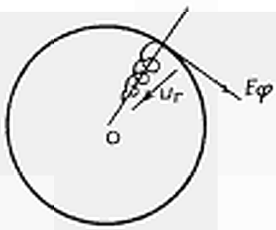


Fig. 28. Drift in a rising cylindrical magnetic field.

(1) A radial drift with the velocity u_r obtained by substituting eq. (84) into eq. (79a)

$$u_r = \dot{r} = -\frac{r}{2}\frac{\dot{B}}{B} \text{ (cm/sec)} \tag{85}$$

giving after integration

$$\frac{r^2}{r_0^2} = \frac{B_0}{B}. \tag{85a}$$

(2) A cyclotron spiral motion in which the particle gains energy from the field $\boldsymbol{E}$. The energy gain per revolution is

$$\begin{aligned}\Delta W &= e \oint_l \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} \\ &= e \int_\Sigma \operatorname{curl} \boldsymbol{E}\, \mathrm{d}\Sigma \\ &= \frac{e}{c}\pi\rho^2\dot{B}\end{aligned}$$

$$\Delta W = \frac{2\pi mc}{e}\left(\tfrac{1}{2}mv_\perp^{2}\right)\frac{\dot{B}}{B^2}. \tag{86}$$

The power absorbed by the particle from the E field is, therefore,

$$\dot{W} = \frac{\Delta W}{\tau}. \tag{86a}$$

Thus $\dot{W} = W_{\perp}(\dot{B}/B)$ where $W_{\perp} = \frac{1}{2}mv_{\perp}^2$. (87)

Integrating we have

$$W_{\perp} = W_{0\perp} \frac{B}{B_0}. \tag{87a}$$

Substituting for B/B_0 from eq. (85a) we have

$$\frac{W}{W_0} = \frac{r_0^2}{r^2}. \tag{88}$$

This ratio of initial energy to final energy is the same as would be achieved by an adiabatic compression of a volume πr_0^2 to a final volume πr^2 containing particles possessing only two degrees of freedom, i.e. r and φ.

Our analysis has been based so far on the assumption of the invariance of the magnetic moment. In case that the magnetic field rises very rapidly this assumption may not be applicable and full equations of motion must be used (ref. 9).

2.4. Motion in Crossed R.F. Electric Field and a Magnetostatic Field

We shall study the motion of a charged particle in an alternating electric and a magnetostatic field. Let the angular frequency of the r.f. field be ω, and the cyclotron frequency ω_c. Four different cases will be studied according to the value of ω/ω_c.

(1) $\omega_c \ll \omega$

In this case the r.f. field appears to be almost a static one and the drift velocity u_c derived in section 2.3. may be used to describe the motion. Thus

$$u = c\,\frac{E_0}{B} \sin\,(\omega t + \phi) \tag{89}$$

and the drift velocity oscillates * with the angular frequency ω (fig. 29).

* Apart from this motion there exists a second order effect. The time variable E field causes a drift in the direction of E (see footnote p. 54). This $u||E$ generates polaribation currents in a plasma in a magnetic field.

(2) $\omega \gg \omega_c$

The magnetic field moves the line of motion slowly away from the direction of E (fig. 30). The angle of deflection after the first half cycle is

$$\psi_1 = \pi \frac{\omega_c}{\omega} . \tag{90}$$

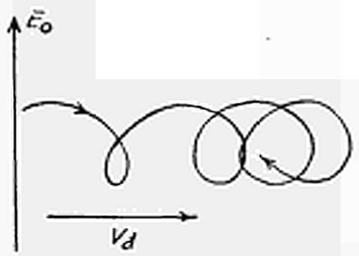


Fig. 29. Motion in crossed alternating electric field and a magnetostatic field for $\omega_c \gg \omega$.

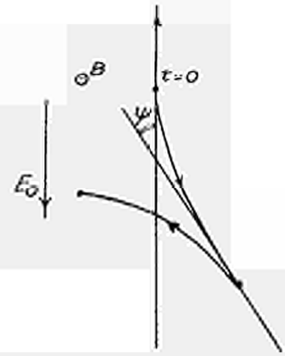


Fig. 30. Motion in an alternating electric field and a magnetostatic field for $\omega \gg \omega_c$.

When $\psi \approx 1/2\pi$ the particle ceases to interact with the field E. The required number of cycles is approximately

$$n \cong \frac{\frac{1}{2}\pi}{\psi_1} = \frac{1}{2} \frac{\omega}{\omega_c} . \tag{90a}$$

(3) $\omega = \omega_c$

This is the case of cyclotron resonance, which is an important type of motion and therefore, it will be described in some detail.

Let us first consider a charged particle in a homogeneous magnetic field $\boldsymbol{B}$ and also in a homogeneous alternating electric field (fig. 31)

$$E = E_0 \cos \omega t$$

where

$$E_0 \perp B \text{ and } \omega = \frac{e}{cm} B.$$

We shall investigate the dynamics of this charged particle, starting with the corresponding equations of motion:

$$\ddot{x} = \frac{e}{m} E_0 \cos \omega t + \frac{e}{cm} B\dot{y} \tag{91a}$$

$$\ddot{y} = -\frac{e}{m} B\dot{x}. \tag{91b}$$

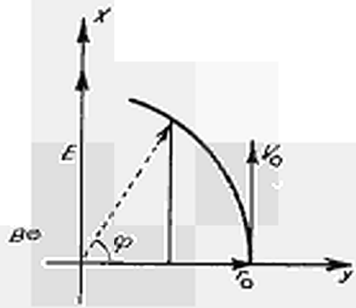


Fig. 31. Diagram of fields and particle position and speed.

Let the initial conditions be $x_0 = 0$ and $\dot{x}_0 = v_0$

$$y_0 = r_0 \text{ and } \dot{y}_0 = 0$$

where $r_0 = v_0/\omega$.

Integrating eqs. (91a) and (91b) and using the initial conditions we have

$$\frac{1}{\omega} \frac{e}{m} E_0 \sin \omega t + \omega y = \dot{x} \tag{92a}$$

$$-\omega x = \dot{y}. \tag{92b}$$

We eliminate x from (92a) and (92b), and get

$$\ddot{y} + \omega^2 y = -\frac{e}{m} E_0 \sin \omega t, \tag{93}$$

the solution of which is

$$y = \frac{v_0}{\omega} \cos(\omega t + \phi_0) - \frac{1}{2} \frac{e}{m} \frac{E_0}{\omega^2} (\sin \omega t - \omega t \cos \omega t). \tag{94a}$$

Similarly

$$x = \frac{v_0}{\omega} \sin(\omega t + \phi_0) + \frac{1}{2} \frac{e}{m} \frac{E_0}{\omega} t \sin \omega t. \tag{94b}$$

The kinetic energy W of the particle is

$$W = \tfrac{1}{2}m\,(\dot{x}^2 + \dot{y}^2). \tag{95}$$

Differentiating (94a) and (94b) and substituting into (95) we have, for $\phi_0 = 0$:

$$W = \tfrac{1}{2}mv_0^2 + \tfrac{1}{2}E_0ev_0t + \frac{1}{8}\,\frac{e^2}{m}E_0^2t^2 + \frac{1}{8}\,\frac{e^2}{m}\,\frac{E_0^2}{\omega^2}\sin^2\omega t$$
$$+ \tfrac{1}{4}v_0\frac{eE_0}{\omega}\sin 2\omega t + \frac{1}{8}\,\frac{e^2}{m}\,\frac{E_0^2}{\omega}\,t\sin 2\omega t. \tag{96}$$

The first term represents the initial kinetic energy, the second increases linearly with time t, the third increases as the square of t, the rest are oscillatory terms of little importance.

Let us now find the time t_1 for which the 3rd term in eq. (96) is equal to the 2nd term. This is

$$t_1 = \frac{4mv_0}{eE_0}.$$

In other words, it is the time at which the impulse eE_0t_1 given to the particle is 4 times the initial momentum mv_0. The time t_1 is short and one may consider that the 3rd term determines the absorption of r.f. energy by the particle.

In eq. (96) we have considered only the particle for which $\phi_0 = 0$. The effect of $\phi_0 \neq 0$ may be predicted from the expressions for x and y (eqs. (94a) and (94b). As t increases the 1st term in those expressions, which alone incorporates ϕ_0, becomes negligible compared with the others. This means that the charges for which $\phi_0 \neq 0$ are eventually pulled into phase with the applied electric field and their energy W increases approximately according to eq. (96).

(4) $\omega \approx \omega_c$

In this case the particle is accelerated up to a maximum kinetic energy, after which it begins to lose energy to the r.f. field. This process of energy exchange is periodic, the frequency being the beat frequency

$$f = \frac{\omega - \omega_c}{2\pi}. \tag{97}$$

The maximum momentum p_{max} gained from the r.f. field is

$$p_{max} = e\bar{E}\,\frac{1}{2f} \tag{98}$$

where E is the average field acting on the particle during the time $1/(2f)$; one has

$$\bar{E} \approx \tfrac{1}{2} E_0$$

and

$$p_{\max} = \tfrac{1}{2}\pi \; eE_0 \; \frac{1}{\omega - \omega_c}. \tag{98a}$$

The maximum energy is

$$W_{\max} = \frac{(p_{\max})^2}{2m} = \frac{\pi^2}{8} \; m \left(c \, \frac{E_0}{B} \right)^2 \left(1 - \frac{\omega}{\omega_c} \right)^{-2} \tag{99}$$

and the maximum radius of gyration is

$$\rho_{\max} = \frac{p_{\max}}{\dfrac{e}{c} B} = \tfrac{1}{2}\pi \, \frac{eE_0}{m\omega_c^{\,2}} \left(1 - \frac{\omega}{\omega_c} \right)^{-1}. \tag{100}$$

*

2.5. The Movement of a Charged Particle in the Field of an Electromagnetic Wave

Let us consider a plane electromagnetic wave in vacuum. The field components are

$$E_x = E \sin\,(\omega t - kz) \tag{101a}$$

$$B_y = E \sin\,(\omega t - kz). \tag{101b}$$

The equations of motion of a particle in this field can be written as

$$\ddot{x} = \frac{eE}{m} \left(1 - \frac{\dot{z}}{c} \right) \sin\,(\omega t - kz) \tag{102a}$$

$$\ddot{z} = \frac{eE}{m} \, \frac{x}{c} \sin\,(\omega t - kz). \tag{102b}$$

In vacuum $k = 2\pi/\lambda = \omega/c$,
and therefore:

$$kz = \frac{\omega}{c} \int_0^t \dot{z} \; dt.$$

If the interaction of the particle and the field over many cycles should not be affeoted by the term kz one must have

$$kz \ll \omega t$$

$$\frac{\omega}{c}\int_0^t \dot{z}\,\mathrm{d}t \ll \omega t$$

i.e.,

$$\frac{\dot{z}}{c} \ll 1. \tag{103}$$

Provided this inequality is satisfied, the set of eqs. (102a, b) reduces to

$$\ddot{x} = ac \sin \omega t \tag{104a}$$

$$\ddot{z} = a\dot{x} \sin \omega t \tag{104b}$$

where

$$a = eE/cm.$$

Integrating eq. (104a) one has

$$\dot{x} = -\frac{ac}{\omega}\cos \omega t + \dot{x}_0$$

and substituting into eq. (104b)

$$\ddot{z} = -\frac{1}{2}\,\frac{a^2c}{\omega}\sin 2\omega t + a\dot{x}_0 \sin \omega t$$

from which

$$z = \frac{1}{8}\,\frac{a^2c}{\omega^3}\sin 2\omega t - \frac{a\dot{x}_0}{\omega^2}\sin \omega t + \dot{z}_0 t. \tag{105}$$

Thus, if $\dot{z}_0 = 0$, there is no mean drift in any direction, the particle moves on a trajectory resembling a figure of eight (fig. 32).

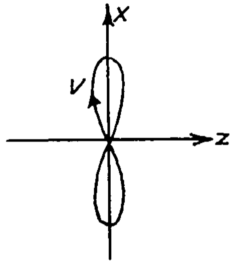


Fig. 32. Motion in the field of an electromagnetic wave.

Let us now suppose that a charged particle is located in a medium which is able to advance the phase of E_x with respect to H_y by $1/2\pi$.* Eqs. (104a, b) are then written as

$$\ddot{x} = ac \sin(\omega t + \tfrac{1}{2}\pi) \tag{106a}$$

$$\ddot{z} = a\dot{x} \sin \omega t. \tag{106b}$$

For $\dot{x}_0 = \dot{z}_0 = 0$ these equations reduce to

$$\ddot{z} = a^2 c\omega^{-1} \sin^2 \omega t.$$

The non-oscillating component of $\dot{z}$ is, therefore,

$$z_0 = \frac{1}{2}\frac{a^2 c}{\omega} t \text{ (cm/sec).} \tag{107}$$

This is the maximum speed a particle with a charge to mass ratio e/m can attain in an electromagnetic wave E_x, H_y.

The average force per particle is then

$$F_r = m\ddot{z} = \frac{1}{2}\frac{e^2 E^2}{mc\omega}. \tag{108}$$

In the case of a single particle a phase shift φ is introduced owing to the influence of the particle on the field-configuration of the wave. The simplified equations of motion are

$$\ddot{x} = a \cdot c \cdot \sin \omega t \tag{109a}$$

$$\ddot{z} = a\dot{x} \sin(\omega t - \varphi) \tag{109b}$$

where $a = \dfrac{eE}{mc}$

the phase shift φ between the E and B vectors represents the reaction of the charge on the radiation field. This is understandable if one imagines the charge to be a perfectly conducting sphere of radius $r = \dfrac{e^2}{mc^2}$ (classical radius of an elementary charged particle). The

* The phase-shift postulated above may be effected by a slab of plasma $\dfrac{c}{\omega_p}$ thick. The number of electrons/cm² in the slab is $N = n\dfrac{c}{\omega_p}$ and the force per cm² of the slab is $F = N \,.\, F_r = 1/2 \dfrac{e^2E^2}{mc\omega_p} \,.\, n \dfrac{c}{\omega_p}$. Since $\omega_p^2 = \dfrac{4\pi e^2 n}{m}$ we get $F = \dfrac{E^2}{8\pi}$ the total radiation pressure.

magnetic lines slip past the sphere along its surface with a speed c; the electric lines are not held up, the conducting sphere represents a short circuit for them. Thus the time τ_B required for the B lines to slip past is $\frac{2r}{c} < \tau_B < \frac{\pi \cdot r}{c}$; the electric lines do it in $\tau_E \simeq \frac{2r}{c}$ (fig. 33). The difference $\tau = \tau_B - \tau_E \cong 0 \cdot 6 \frac{r}{c}$ corresponds to the phase shift $\varphi = \omega\tau$.

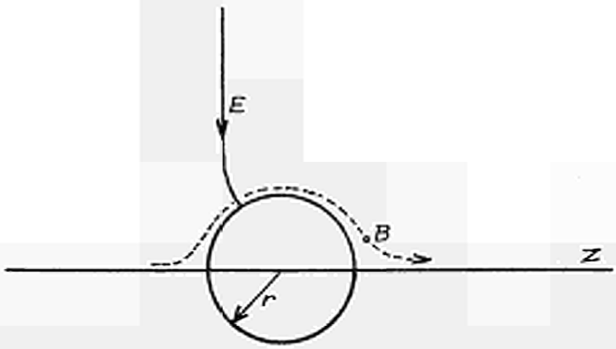


Fig. 33. Electric and magnetic lines slipping past a conducting sphere.

From the two equations we get

$$\ddot{z} = \frac{a^2 c}{\omega} \sin \omega t \cdot \cos (\omega t + \varphi). \tag{110}$$

Integrating we get for the non-oscillating component of the speed

$$\dot{z} = \frac{1}{2} \frac{a^2 c}{\omega} \cdot t \cdot \sin \varphi. \tag{111}$$

from which the steady acceleration is ($\sin \varphi \sim \varphi$)

$$\ddot{\bar{z}} = \frac{1}{2} \frac{a^2 c}{\omega} \varphi. \tag{112}$$

The mean force on the charge is, therefore,

$$\bar{F} = \frac{1}{2} \frac{m\, a^2\, c\, \varphi}{\omega} = \frac{1}{2} mc \frac{e^2 E^2 \tau}{m^2 c^2} \tag{113}$$

let us put * $\tau \sim \frac{2}{3} \frac{r}{c} = \frac{2}{3} \frac{e^2}{mc^3}$ then

* Only to make our heuristic argument more precise numerically.

$$\bar{F} = \frac{1}{3}\left(\frac{e^2}{mc^2}\right)^2 E^2$$

which in terms of Thompsons' cross-section $\left(\sigma_T = \frac{8\pi}{3}\,\frac{e^4}{m^2c^4}\right)$ gives:

$$\bar{F} = \frac{E^2}{8\pi}\cdot\sigma_T\cdot \tag{114}$$

2.6. Radiation from Accelerated Charges

The radiation field of a charged particle experiencing acceleration $\dot{v}$ is (ref. 10)

$$\boldsymbol{E}(t') = \frac{e}{rc^4}\,\frac{[\dot{\boldsymbol{v}} \wedge (\boldsymbol{c} - \boldsymbol{v})] \wedge \boldsymbol{c}}{\left(1 - \frac{\boldsymbol{c}\cdot\boldsymbol{v}}{c^2}\right)^3} \tag{115a}$$

$$\boldsymbol{H}(t') = \frac{1}{c}\,\boldsymbol{c} \wedge \boldsymbol{E} \tag{115b}$$

where $\boldsymbol{c} = \boldsymbol{r}\,c/r$ and v, $\dot{v}$, r and c are taken at a time $t' - r/c$ (see fig. 34).

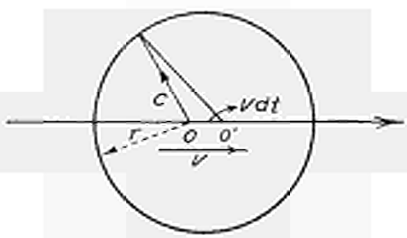


Fig. 34. The position and movement of a point charge in a sphere representing a surface of a related potential.

Let us suppose the charge radiates during a time dt. The time difference dt' between the first and last radiation signal observed at the point where E and H are measured is

$$\mathrm{d}t' = \left(t + \mathrm{d}t + \frac{r}{c} - \frac{\boldsymbol{c}\cdot\boldsymbol{v}}{c^2}\,\mathrm{d}t\right) - \left(t + \frac{r}{c}\right) = \mathrm{d}t\left(1 - \frac{\boldsymbol{c}\cdot\boldsymbol{v}}{c^2}\right)\cdot$$

Thus the energy radiated by the charge during dt in the direction $\boldsymbol{r}$ into a solid angle dΩ is

$$\mathrm{d}^2 w = \frac{c}{4\pi} (\boldsymbol{E} \wedge \boldsymbol{H}) \cdot \left(1 - \frac{\boldsymbol{c} \cdot \boldsymbol{v}}{c^2}\right) r^2 \, \mathrm{d}\Omega \, \mathrm{d}t. \tag{116}$$

Using eqs. (115a, b) and (116) one obtains for a power radiated into unit solid angle

$$\frac{\mathrm{d}w}{\mathrm{d}\Omega} = \frac{e^2}{4\pi c^3} \left\{ \frac{\dot{\boldsymbol{v}}^2}{\left(1 - \frac{\boldsymbol{c} \cdot \boldsymbol{v}}{c^2}\right)^3} + \frac{2\left(\frac{\dot{\boldsymbol{v}} \cdot \boldsymbol{v}}{c}\right)\left(\frac{\dot{\boldsymbol{v}} \cdot \boldsymbol{c}}{c}\right)}{\left(1 - \frac{\boldsymbol{c} \cdot \boldsymbol{v}}{c^2}\right)^4} - \frac{\left(1 - \frac{v^2}{c^2}\right)\left(\frac{\dot{\boldsymbol{v}} \cdot \boldsymbol{c}}{c}\right)^2}{\left(1 - \frac{\boldsymbol{c} \cdot \boldsymbol{v}}{c^2}\right)^5} \right\}. \tag{117}$$

For $v/c \ll 1$ this expression reduces to the well known Hertzian formula

$$\frac{\mathrm{d}w}{\mathrm{d}\Omega} = \frac{e^2}{4\pi c^5} (\dot{\boldsymbol{v}} \wedge \boldsymbol{c})^2. \tag{117a}$$

Let us now integrate the equation (117) with respect to Ω over a large sphere. This gives the total radiation output from the moving charge. The integral is

$$w = \frac{2e^2}{3c^3} \{ \dot{v}_{\parallel}{}^2 \gamma^6 + \dot{v}_{\perp}{}^2 \gamma^4 \} \tag{118}$$

where $\dot{v}$ has components in the direction parallel and perpendicular to v. Transforming $\dot{v}_{\parallel}$ and $\dot{v}_{\perp}$ to the rest system of the particle, i.e., into $\dot{v}_0$, one obtains

$$w = \frac{2e^2}{3c^3} \dot{v}_0{}^2. \tag{118a}$$

Thus it follows that the power radiated by accelerated charges is an invariant of the Lorentz transformation.

From these equations it would appear that charges can radiate only if accelerated. However, eqs. (115) to (117) were derived only for $v < c$. If a charge propagates in or near a medium in which the speed of propagation of radiation is smaller than the speed of the charge these formulae are not applicable and it will be shown that the charge can radiate even if $\dot{v} = 0$.

We shall treat three radiation processes which are of interest in plasma physics. These are: the bremsstrahlung, the cyclotron radiation, the Čerenkov radiation.

For the first two $\dot{v} \neq 0$ whereas in the third case $\dot{v} = 0$.

2.6.1. THE BREMSSTRAHLUNG

As the name indicates, this radiation is emitted by decelerated particles. However, it is now applied to the more general case of particles whose velocity vector changes as a result of Coulomb interaction. Thus, for instance, an electron passing through the electric field of an atomic nucleus will radiate according to formula (118). The validity of this formula is, however, restricted by quantum mechanical laws, and the change of the velocity over a distance equal to the de Broglie wave length of the particle should be very small as compared to the absolute value of the velocity. This can be expressed as

$$\frac{\dot{v}}{v^3} \ll \frac{m}{h}. \tag{119}$$

We shall calculate the radiation emitted by an electron during an elastic collision with an atomic nucleus using the classical formula (118a). During a collision with a nucleus (charge Ze) a non-relativistic electron (mass m, charge e) is subjected to an acceleration (fig. 35).

$$\dot{v} = \frac{Ze^2}{mr^2}. \tag{120}$$

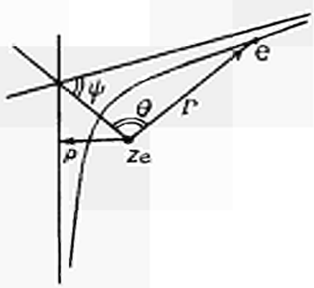


Fig. 35. Geometry of electron-positive ion scattering in which bremsstrahlung is emitted.

Using eq. (118a) the total power radiated away during a collision is (ref. 11)

$$\Delta W = \frac{2}{3} \frac{e^2}{c^3} \int_{-\infty}^{\infty} (\dot{v})^2 \, \mathrm{d}t. \tag{121}$$

One may integrate with respect to the azimuth θ rather than t using the law of constant areas. This is written as

$$r^2 \frac{\mathrm{d}\theta}{\mathrm{d}t} = pv_0 \tag{122}$$

and ΔW becomes

$$\Delta W = \frac{2}{3} \frac{Z^2 e^6}{m^2 c^3} \frac{1}{pv_0} \int_{-(\pi-\psi)}^{\pi-\psi} \frac{1}{r^2} \, \mathrm{d}\theta. \tag{121a}$$

As (see eq. (6))

$$r = \frac{p \tan \psi}{1 + \kappa \cos \theta}$$

one obtains, after integration

$$\Delta W = \frac{2}{3} \frac{Z^2 e^6}{m^2 c^3} \frac{1}{p^3 v_0} \left\{ (\pi - \psi) \left(1 + \frac{3}{\tan^2 \psi} \right) + \frac{3}{\tan \psi} \right\}. \tag{123}$$

The value of the angle ψ is generally larger * than $1/4\pi$, e.g., for electrons whose energy is approximately 100 eV an angle $\psi = 1/4\pi$ is the smallest permissible. For larger electron energies the de Broglie collision parameter

$$p_1 = \frac{h}{mv}$$

is always larger than the parameter p_0 corresponding to a deflection of $2\psi = 90°$, and therefore, the smallest permissible ψ is $\psi_{\min} > 1/4\pi$ (ref. 12).

It thus appears that the value Λ of the expression in the curly brackets in eq. (123) is approximately

$$\Lambda \approx \Lambda(\psi = \tfrac{1}{2}\pi) = \tfrac{1}{2}\pi.$$

Then

$$\Delta W = \frac{\pi}{3} \frac{Z^2 e^6}{m^2 c^3} \frac{1}{p^3 v_0}. \tag{124}$$

* Small angle scattering being the most frequent event.

In a plasma whose electron density is n_e the number of e—p collisions per second with the collision parameter p in the interval $(p, p + dp)$ is $2\pi p\, dp v_0 n_e$. Thus the total radiation power emitted by an electron is

$$W = \int_{p_{min}}^{p_{max}} \Delta W 2\pi p v_0 n_e \, dp$$

where p_{min} follows from eq. (17) chapter 1: $p_{min} \approx h/(2\pi m v_0)$. The value of the integral is insensitive to the choice of p_{max} which could be taken to be infinite and therefore,

$$W = \frac{4\pi^3}{3} \frac{Z^2 e^6}{mc^3 h} n_e v_0.$$

But v_0 can be expressed in terms of the temperature of the electron gas. Assuming that v_0 is equal to the mean random speed $\sqrt{2kT_e/m}$ one has

$$W = 1.71 \times 10^{-27} Z^2 n_e \sqrt{T_e}. \tag{124a}$$

This result agrees reasonably well with a more rigorous wave mechanical calculation (ref. 13). A similar calculation can be performed for an e—e collision, in which the radiation has a predominantly quadrupole character and is, therefore, generally less intense than the dipole radiation from e—p collisions, treated above.

The wave length of radiation emitted corresponds at every instant to the radius of curvature of the trajectory. Thus the shortest wave length wille correspond to $2\pi r_{min}$. For a strong collision (eq. (7)) $r_{min} \sim \dfrac{p_0}{2 \cdot 41}$ and consequently

$$\lambda_{min} \approx \frac{2\pi}{2 \cdot 41} \cdot p_0 \frac{c}{v_{max}}.$$

An approximate value is, therefore,

$$\lambda_{min} \approx 2{\cdot}6\, p_0 \frac{c}{v_0}. \tag{125a}$$

If the trajectory extends from ∞ to $-\infty$ is is evident that $\lambda_{max} \rightarrow \infty$. However, in plasma the upper bound for collision distance is usually the Debye length d and consequently

$$\lambda_{max} \approx 2\pi d \frac{c}{v_0} = \frac{c}{v_0} \sqrt{\frac{\pi k T}{n e^2}}. \tag{125b}$$

The approximate form of the bremsstrahlung spectrum emitted from plasma is shown in fig. 36.

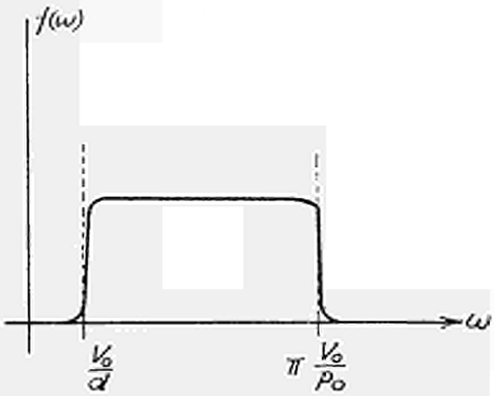


Fig. 36. The Bremsstrahlung spectrum in plasma.

2.6.2. CYCLOTRON (BETATRON, SYNCHROTRON) RADIATION

This is a type of radiation emitted by charges moving in circular orbits. Such a trajectory is traced out by a charged particle in a magnetic field. Clearly

$$v = \dot{v}_{\perp} = \omega_c v.$$

Substituting this into eq. (118a) one has

$$w = \frac{2}{3}\frac{e^2}{c^3}\omega_c^2 v^2 \gamma^4 \tag{126}$$

where $$\gamma = (1 - \beta^2)^{-1/2},\ \beta = \frac{v}{c}.$$

Writing

$$\omega_c = \frac{eB}{m_0 c \gamma}$$

there is *

$$w = \frac{2}{3}\frac{e^4}{m_0^2 c^3} v^2 \gamma^2 B^2. \tag{126a}$$

The time in which a non-relativistic particle will lose practically all its kinetic energy W_k is, therefore,

$$\tau = \frac{W_k}{W} = \frac{3}{4}\frac{m_0^3 c^5}{e^4 B^2} \tag{127}$$

* This formula can also be written as $w = c\sigma_T \left(\frac{\omega_c}{\omega_p}\right)^2 mv^2 . n.$

whereas for a strongly relativistic particle one has

$$\tau = \frac{3}{2} \frac{m_0{}^3 c^5}{e^4 B^2} \cdot \frac{1}{\gamma}. \tag{127a}$$

The angular distribution of this radiation can be expressed as a function of the angle ε which the vector r makes with the plane σ and of the angle δ which the orthogonal projection of r on σ makes with the velocity-vector of the rotating particle (fig. 37). Substituting these into eq. (117) one has

$$\frac{\mathrm{d}w}{\mathrm{d}\Omega} = \frac{e^2 v^2}{4\pi c^3} \left\{ \frac{1}{\left(1 - \frac{v}{c}\cos\varepsilon\cos\delta\right)^3} - \frac{\left(1 - \frac{v^2}{c^2}\right)\cos^2\varepsilon\sin^2\delta}{\left(1 - \frac{v}{c}\cos\varepsilon\cos\delta\right)^5} \right\}. \tag{128}$$

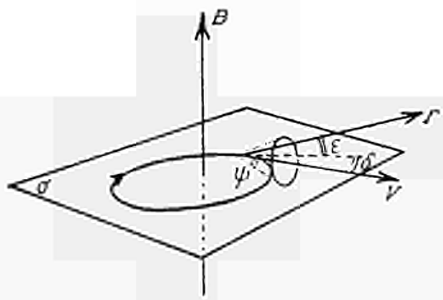


Fig. 37. Cyclotron radiation emitted from circulating relativistic charges.

This shows that the maximum of the radiation power occurs for $\varepsilon = \delta = 0$, i.e., in the direction of the velocity-vector v.

Integrating eq. (128) with respect to ε and δ it can be shown that half of the radiated power is confined within a cone whose axis is $\boldsymbol{v}$ and whose apex angle ψ is approximately

$$\sqrt{\varepsilon\delta} = \frac{m_0 c^2}{W} \approx \frac{1}{\gamma} \tag{129}$$

where W is the kinetic energy of the radiating particle.

The frequency spectrum of cyclotron radiation has been discussed in a number of publications (refs. 14, 15). A full analysis of this problem is beyond the scope of this book and we shall give the main results and a heuristic proof.

A rotating electron radiates a line spectrum of frequencies

$$f_l = \frac{\omega_c}{2\pi} l, \qquad l = 1, 2\ldots\ldots \tag{130}$$

For a non-relativistic particle most of the energy is radiated on the cyclotron frequency, i.e., $l = 1$ only. However, when

$$W > m_0c^2$$

the main portion of the radiated energy appears in higher harmonics of $\omega_c/2\pi$. The harmonic which carries most of this energy has an ordinal number

$$l_{\max} = \left(\frac{W}{m_0c^2}\right)^3. \tag{131}$$

Denoting the output in each harmonic by W_l one has

$$W_l \approx \tfrac{1}{2}e^2 \frac{\omega_c}{r_0} l^{1/3} \tag{132}$$

up to $l = l_{\max}$ after which W_l decreases sharply (fig. 38).

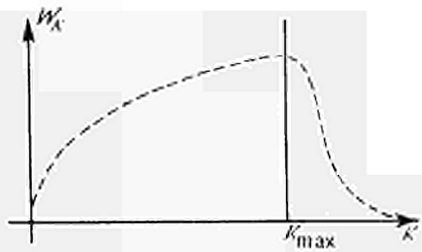


Fig. 38. Spectral distribution of cyclotron radiation from relativistic electrons.

This result can be supported by the following simple argument (ref. 16).

An electron moving in a straight line represents a δ-function of current. This may be Fourier-analysed into a spectrum of harmonic currents with different frequencies each moving with velocity βc. Associated with each current component is an electromagnetic "slow wave" with amplitude falling off in a direction perpendicular to the track. The radial propagation constant k_r is given by

$$k_r^2 = k_0^2 - k_z^2 \text{ where } k_0 = 2\pi/\lambda. \tag{133}$$

As $k_z = k_0/\beta$ it follows that

$$k_r = jk_0 \left(\frac{1}{\beta^2} - 1\right)^{1/2} = jk_0/\beta\gamma. \tag{133a}$$

The radial "fall off" distance is $\Delta \equiv 1/k_r = \beta\gamma/k_0$. (134)

Consider now the electron to be moving in a circle of radius R, then the phase velocity of that part of the "slow wave" which extends to a radius R/β is equal to c. Hence if the distance Δ' between these two circles $\Delta' \ll \Delta$, the wave under consideration will radiate, if the distances $\Delta' \gg \Delta$ it will not contribute appreciably to the radiation from this rotating charge. Equating the distances Δ and Δ' one obtains

$$\beta\gamma\lambda = R(1 - 1/\beta).$$

Now for $\gamma^{-1} \ll 1, \beta = 1 - \frac{1}{2}\gamma^{-2}$ whence

$$\lambda = \pi R/\gamma^3. \tag{135}$$

Radiation of shorter wavelength will not appear in the spectrum. Eq. (135) is equivalent to eq. (131)

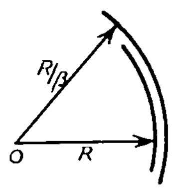


Fig. 39. The cylinder of radius R/β outside of which the near-field becomes a radiation field.

The subject of emission of cyclotron radiation from plasma cannot be treated in this chapter; a greater acquaintance with wave-propagation in plasma is necessary before one can calculate how much of the cyclotron radiation from single particles is reabsorbed and how much can escape from plasma. However, a useful approximation can be derived by assuming that only the energy radiated in harmonics whose frequency is higher than $\frac{1}{2\pi}\,\omega_p$ is lost from the plasma, i.e., considering only $l > \frac{\omega_p}{\omega_c}$. This is obviously not true for infinitely thick plasma which would radiate as a black body, i.e., whose radiation output per unit volume tends to zero. Our approximation gives, in this sense, an upper limit on energy lost per unit volume of plasma.

The power radiated in the l'th cyclotron harmonic is (ref. 14)

$$W_l = \frac{e^2\omega_c}{r} l \left[2\beta^2 J'_{2l}(2l\beta) - (1-\beta^2) \int_0^{2l\beta} J_{2l}(x)dx\right]. \quad (136)$$

Let us restrict ourselves to a non-relativistic plasma, i.e., $\beta \ll 1$.
Since the first maxima of the Bessel functions $J_{2l}(x)$ and $J'_{2l}(x)$ occur at $x \sim 2l$ it follows that $x = 2l\beta$ is a relatively small argument of these functions and one can use the approximation

$$J_p(x) \approx \frac{(1/2\,x)^p}{p!}\left[1 - \frac{x^2}{4(p+1)}\right]. \quad (137)$$

Then

$$W_l \approx \frac{e^2\omega_c}{r} 2l \frac{(2l)^{2l}\beta^{2l+1}(2l+2)}{(2)^{2l+1}(2l)!(2l+1)}\left(1 - \alpha'\beta^2\frac{l}{2}\right) \quad (138)$$

where $\alpha' = \dfrac{2l(l+3)}{(l+1)(2l+3)}$.

Using the expansion $p! = p \cdot \Gamma(p) \approx \sqrt{2\pi}\,\varepsilon^{-p}p^{p+1/2}$ we have for $l \gg 1$

$$W_l \approx \frac{e^2\omega_c}{r}\frac{1}{\varepsilon\sqrt{\pi}}\left(\frac{\varepsilon\beta}{2}\right)^{2l+1}\sqrt{l}\cdot\left(1-\beta^2\frac{l}{2}\right) \quad (139)$$

where $\varepsilon = 2\cdot 72$.

Let us now calculate the power radiated by this electron in all harmonics between $l_1 = \dfrac{\omega_p}{\omega_c}$ and $l_2 = 2/\beta^2$. This is approximately

$$W \cong \frac{e^2\omega_c\beta}{2\sqrt{\pi}\,r}\int_{l_1}^{l_2} l\, a^l\left(\frac{1}{\sqrt{l}} - \frac{\beta^2}{2}\sqrt{l}\right) dl \quad (140)$$

where $a = \left(\dfrac{\varepsilon\beta}{2}\right)^2 \ll 1$. The function of $\sqrt{l}$ in the brackets is varying slowly and provides mainly an expression for a cut-off at l_2. We shall, therefore, use an approximation

$$W \cong \frac{e^2\omega_c\beta\sqrt{l_2}}{2r\sqrt{\pi}}\int_{l_1}^{\infty} a^l\, dl \quad (141)$$

as an upper limit.

Thus

$$W \cong \frac{e^2 \omega_c^2}{c\sqrt{2\pi}} \beta^{-1} \frac{a^{l_1}}{ln\left(\frac{1}{a}\right)}. \tag{142}$$

The ratio of this output to that emitted by the particle in absence of plasma (eq. (126)) is

$$p = \frac{W}{w} = \frac{3\,\varepsilon^3}{32\sqrt{\pi/2}} \frac{a^{l_1 - 3/2}}{ln\left(\frac{1}{a}\right)} \tag{143}$$

where l_1 is determined only by the characteristics of plasma, whereas a depends only on the speed of the particle. Let us take as an example $n = 10^{14}$ el/cm^3, $B = 10^4$ (Gauss), $\beta = 0.1$ then $l_1 \sim 3$ and $p \cong 4 \cdot 10^{-2}$. Assuming a maxwellian distribution of velocities in a plasma whose temperature is T, the power output of incoherent cyclotron radiation will be

$$W_c = 2n\left(\frac{1}{\sqrt{\pi}\,\beta_0}\right)^3 \int_0^\infty W(\beta_\perp) \int_0^\infty \exp\left(-\frac{\beta_\perp^2 + \beta_\parallel^2}{\beta_0^2}\right) 2\pi\beta_\perp\, d\beta_\parallel\, d\beta_\perp \tag{144}$$

where $\beta_0 = \frac{1}{c}\sqrt{\frac{2kT}{m}}$ and $\beta_\perp$ corresponds to the velocity component perpendicular to B. Integrating first over $\beta_\parallel$, putting $\frac{\beta_\perp}{\beta_0} = \sqrt{x}$ and assuming $ln\,\frac{1}{a} \sim ln\,\frac{1}{a_0}$

$$W_c = \frac{e^2 \omega_c^2\, a_0^{l_1 - 1/2}}{c\sqrt{\pi}\; ln\left(\frac{1}{a_0}\right)} n \cdot \left(l_1 - \frac{1}{2}\right)! \qquad \text{(ergs/sec, cm}^3\text{)}\,^* \tag{145}$$

The ratio of this output to that which would be produced were the plasma completely transparent is

$$p' = \frac{W_c}{W_p} \tag{146}$$

* This expression can be also written in terms of T as follows

$$W_c \cong \frac{4e^4\sqrt{\pi}}{cm\; ln\, 1/a_0}\left(\frac{\varepsilon^2 kT}{2mc^2}\right)^{l - 1/2} \frac{l!}{l^{5/2}} \cdot n^2 \tag{145a}$$

where $l = \frac{\omega_p}{\omega_r}$.

where

$$W_p = \frac{2}{3} \frac{e^2}{c\,\beta_0^2} \omega_c^2\, n \int_0^\infty 2\,\beta_\perp^3 \exp\left(-\frac{\beta_\perp^2}{\beta_0^2}\right) \cdot \mathrm{d}\beta_\perp$$

$$= \frac{2}{3} \frac{e^2 \omega^2_c}{c} \beta_0^2 \cdot n \tag{147}$$

and therefore,

$$p' = \frac{3\,\varepsilon^2}{8\sqrt{\pi}} \frac{a_0^{l_1 - 3/2} \left(l_1 - \frac{1}{2}\right)!}{ln\left(\frac{1}{a_0}\right)} \tag{148}$$

Taking values of the last example in which we put $\beta = \beta_0$ ($T \sim 5 \cdot 10^7 \cdot K$) we get $p' \sim 7 \cdot 10^{-2}$.

So far, we have treated the emission of cyclotron radiation as incoherent. If coherence exists, e.g., the electrons are bunched, the mean number of electrons in a bunch being N, the total radiation output will be N times larger than the incoherent one.

2.6.3. ČERENKOV RADIATION

Čerenkov radiation in the broadest sense is electromagnetic radiation from charges in rectilinear and almost uniform motion. The earliest experimental results (in 1934) have been correctly interpreted by S.I. Wawilow, by I.M. Frank and by I.E. Tamm.

It may be shown that such radiation cannot be emitted by electric charges in free space. This is the result of both the energy conservation and the momentum conservation conditions. The relativistic energy W of a particle of rest mass m_0 is given by

$$W^2 = p^2c^2 + m_0^2c^4. \tag{149}$$

Any differential decrease in energy is accompanied by a decrease in the relativistic momentum p according to

$$\mathrm{d}W = \frac{pc^2}{W}\,\mathrm{d}p = \frac{pc^2}{(p^2c^2 + m_0^2c^4)^{1/2}}\,\mathrm{d}p. \tag{150}$$

It is evident that, when $m_0 > 0$,

$$\mathrm{d}W < c\;\mathrm{d}p. \tag{151}$$

If, however, $m_0 = 0$, as is the case with photons, then for plane waves $W' = p'c$ (the prime denotes photons) and

$$\mathrm{d}W' = c\;\mathrm{d}p'. \tag{152}$$

For any type of waves other than plane waves

$$dW' > c \; dp'. \tag{152a}$$

Thus a spontaneous emission of radiation such as is depicted in fig. 40a is forbidden, since it would leave the particle m with larger momentum p than is consistent with eq. (150) (still assuming a rectilinear trajectory). Thus if $dW = dW'$ it follows from eqs. (151) and (152) that $dp > dp'$.

If, however, this excess of momentum ($dp - dp'$) can be transferred, say, to another particle M, the emission of a photon with energy dW' and momentum dp' becomes possible. Such a process is illustrated in fig. 40b. It is evident that the radiation process represented by fig. 40b varies in intensity as m passes M. In order to obtain a constant drain on the excess momentum ($dp - dp'$) it is necessary to provide a system of charges Q, uniform in the direction x (direction of the rectilinear trajectory). With such a system the radiation output W is constant in time.

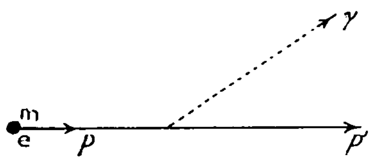


Fig. 40a. Forbidden emission.

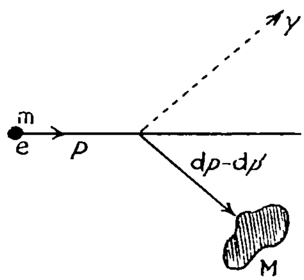


Fig. 40b. Momentum absorber.

It is advisable here to abandon the photon description of the radiation process and use the travelling wave picture instead. Assume that the radiation output from the charge e is constant in time, due to the presence of a suitable momentum absorbing structure S, extended in the x-direction (fig. 41). In terms of travelling waves, such a constant

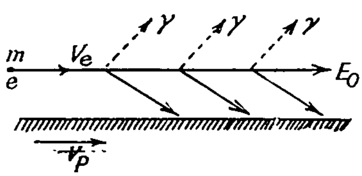


Fig. 41. Distributed momentum-absorption permitting the emission of Čerenkov radiation.

flow of radiation is the result of a constant interaction between an electromagnetic wave $E\omega(x, t)$ and the charge e. The force corresponding to this interaction is

$$F_\omega = eE_0 \exp\ (j\omega t - k_\omega x). \tag{153}$$

But F_ω is constant only if

$$\frac{k_\omega}{\omega} = \frac{1}{v_e} \tag{154}$$

i.e., if the phase velocity of the wave v_p is equal to the velocity v_e of the charge e.

This is obviously impossible in free space, where

$$v_e < c \tag{155}$$

whereas

$$\frac{k_\omega}{\omega} = v_{p\omega} = c. \tag{156}$$

Here eqs. (155) and (156) are equivalent to eqs. (151) and (152).

However, the presence of the system S has a marked effect on the phase velocity $v_{p\omega}$ of the various travelling waves (compatible with the boundary condition on S) and for some frequencies ω

$$v_{p\omega} < c$$

and, for these, the condition of constant interaction

$$v_{p\omega} = v_e$$

can be fulfilled.

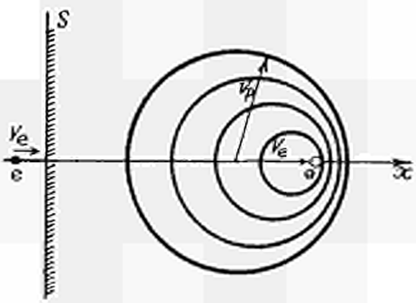


Fig. 42.

From the point of view of the wave concept, the structure S (the momentum drain of the photon picture) is a slow-wave structure.

Such a slow-wave structure (periodic or uniform in the x-direction) may take the form of one of the following well known devices: (1) metal helix; (2) loaded wave guide; (3) dielectric guide; (4) medium whose refractive index $n = (\varepsilon\mu)\ 1/2 > 1$; (5) plasma wave guide.

These structures slow down electromagnetic waves of different frequencies to a different extent owing either to the periodicity of the slow-wave structure or to the absorption.

Infinite dielectric medium

The study of Čerenkov radiation in a refractive medium leads to a description using simple geometric concepts, similar to those found in the study of supersonic phenomena. Imagine the following experiment (fig. 43).

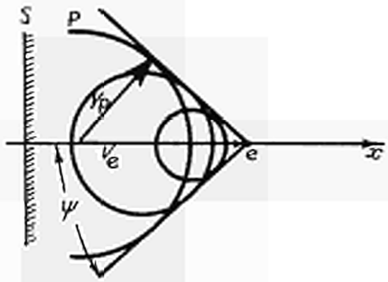


Fig. 43.

A point charge e is shot through a thin metal screen S into a medium whose refractive index is larger than unity

$$n = (\varepsilon\mu)^{1/2} > 1. \tag{157}$$

If the charge moves with a velocity v_e

$$v_e < \frac{c}{n}$$

then the picture of the wave surfaces will be as shown in fig. 42.

However, if the velocity v_e of the charge is larger than the phase velocity v_p of light in the medium i.e.,

$$v_e > \frac{c}{n}$$

the system of the sphrerical surfaces forms a conical envelope C shown in fig. 43. The conical surface is known as the Čerenkov cone.

The angle ψ is given by

$$\sin \psi = \frac{v_p}{v_e} = \frac{c}{v_e} \frac{1}{(\varepsilon\mu)^{1/2}} . \tag{158}$$

It is possible to gain insight into the nature of this type of Čerenkov radiation by noticing that a detector placed near the trajectory will record a sudden electric impulse $E(t)$ (fig. 44). Since the impulse $E(t)$ has an infinitely steep front, it follows that its Fourier spectrum will contain finite components whose frequency $\omega = \infty$.

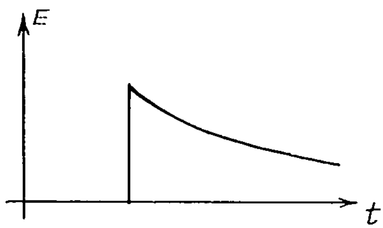


Fig. 44. Field pulse generated by the passage of the surface of the Čerenkov cone.

A more rigorous analysis of the process gives the following formula for the energy frequency spectrum (ref. 17)

$$w(\omega) = \alpha\omega \left(1 - \frac{1}{\varepsilon\mu\beta^2}\right) \tag{159}$$

where $\alpha = v_e e^2/c^2$ and $\beta = v_e/c$.

The formula suggests infinite output at infinitely large frequencies (this type of divergence has been known in the pre-quantum physics as the ultraviolet catastrophe). This is not true because no dielectric or plasma is ideally uniform and thus there are always the effects of periodicity setting in at a wave length λ_0 comparable with intermolecular spacing X_0. It will be appreciated that for instance, for gases, the wave length

$$\lambda_0 = \varepsilon\mu X_0$$

is of the order of $\lambda_0 = (10^{-6})$ cm and thus the maximum w_{max} lies in the visible part of the spectrum.

In order to derive the spectral distribution for a plasma let us rewrite eq. (159) as follows

$$w(\omega) = \alpha\omega \left(1 - \frac{v_s^2}{v^2}\right) \tag{160}$$

where v_s is the speed of propagation of signals of frequency ω in the medium considered. This speed is practically the same as the group velocity v_g, for which we have

$$v_g = \frac{d\omega}{dk} \quad \text{where} \quad k = \frac{2\pi}{\lambda}.$$

For a plasma without a magnetic field (see later p. 150), the dispersion-relation for transversal waves is *

$$\omega^2 - \omega_p^2 = c^2 k^2 \tag{161}$$

from which the group velocity is

$$v_g = c^2 \frac{k}{\omega} = c \sqrt{1 - \frac{\omega_p^2}{\omega^2}}. \tag{162}$$

Substituting for v_g in eq. (160) and integrating over ω we get the total power radiated by the charge

$$\begin{aligned} w &= \frac{e^2}{c^2} v \int_{\omega_1}^{\omega_2} \left[\omega \left(1 - \frac{c^2}{v^2}\right) + \frac{1}{\omega} \frac{\omega_p^2 c^2}{v^2} \right] d\omega \\ &= \frac{e^2}{c^2} v \left[\frac{1}{2} \omega^2 \left(1 - \frac{c^2}{v^2}\right) + \frac{c^2}{v^2} \omega_p^2 \ln \omega \right]_{\omega_1}^{\omega_2}. \end{aligned} \tag{163}$$

The limits of integration are chosen as follows. The lower limit $\omega_1 = \omega_p$ since for $w < w_p$ plasma does not transmit tranversal waves. The upper limit w_2 must be chosen so as not to make the integrand become negative. This means that

$$\omega_2^2 = \frac{\omega_p^2}{1 - v^2/c^2}. \tag{164}$$

Eq. (163) becomes

$$w = \frac{1}{2} \frac{e^2}{e^2} v \omega_p^2 \left[\frac{c^2}{v^2} \ln \left(\frac{1}{1 - \frac{v^2}{c^2}} \right) - 1 \right]. \tag{165}$$

Let us consider only non-relativistic particles. Then $v \ll c$ and expanding the logarithm we get

$$w = \frac{\pi e^4}{mc^4} n v^3. \tag{166}$$

* A more complicated result is obtained if longitudinal modes are also considered.

In case of a relativistic particle in a plasma $v \sim c$ and eq. (163) gives

$$w \sim \frac{1}{2} \frac{e^2 \omega_p^2}{c} \ln \frac{\omega_2}{\omega_1}$$

in this case ω_2 is given by the characteristic distance of the medium. In case of plasma $\omega_2 = \dfrac{v}{d} \sim \omega_p \dfrac{c}{v_t}$ and consequently

$$w \sim \frac{e^2 \omega_p^2}{c} \ln \frac{c}{v_t}. \tag{163a}$$

Using a more rigorous treatment, Kolomenski and Kihara (ref. 18) obtain, instead of our eq. (165),

$$w = \tfrac{1}{2} e^2 \frac{\omega_p^2}{v} \ln \frac{v^2}{v_t^2}. \tag{167}$$

In a plasma having a temperature T the electron velocity-distribution is

$$F(v) = \frac{4 \beta_0^{-1}}{\sqrt{\pi}} \quad c \cdot n \, (v/c)^2 \beta_0^{-3} \exp \left(- \frac{v^2}{2 v_t^2} \right) \tag{168}$$

where $\beta_0 = \dfrac{1}{c}\sqrt{\dfrac{2kT}{m}} = \dfrac{\sqrt{2}\, v_t}{c}$. Provided the electron radiation remains incoherent and the $\omega(k)$ is not substantially affected by the finite temperature effects we have for the intensity of Čerenkov radiation *

$$W \simeq \int_1^{\infty} F(v)\, w(v)\, dv.$$

Using eq. (166)

$$W \approx 0.37 \cdot 10^{-34} \qquad n^2 T_e^{3/2}. \tag{169}$$

It is instructive to plot the radiation power corresponding to the three types of radiation considered so far, i.e. the bremsstrahlung the cyclotron and the Čerenkov radiation. Let us, for this purpose, rewrite the equations (124a), (145a), (169) in the following manner (for $Z = 1$)

$$R_b\, 10^{27} \cdot n_e^{-2}\, W_b \simeq 1.7 \cdot T_e^{1/2} \tag{170a}$$

$$R_c\, 10^{27} \cdot n_e^{-2}\, W_c \simeq 2 \cdot 10^6 \, (0.64 \cdot 10^{-9}\, T_e)^{l-1/2} \frac{l!}{l^{5/2}} \tag{170b}$$

$$R_{ce}\, 10^{27}\, n_e^{-2}\, W_{ce} \simeq 0.37 \cdot 10^{-7}\, T_e^{3/2}. \tag{170c}$$

* The lower limit of the integral is somewhat arbitrary, however, it is plausible to exclude those electrons that move slower than the majority of the distribution.

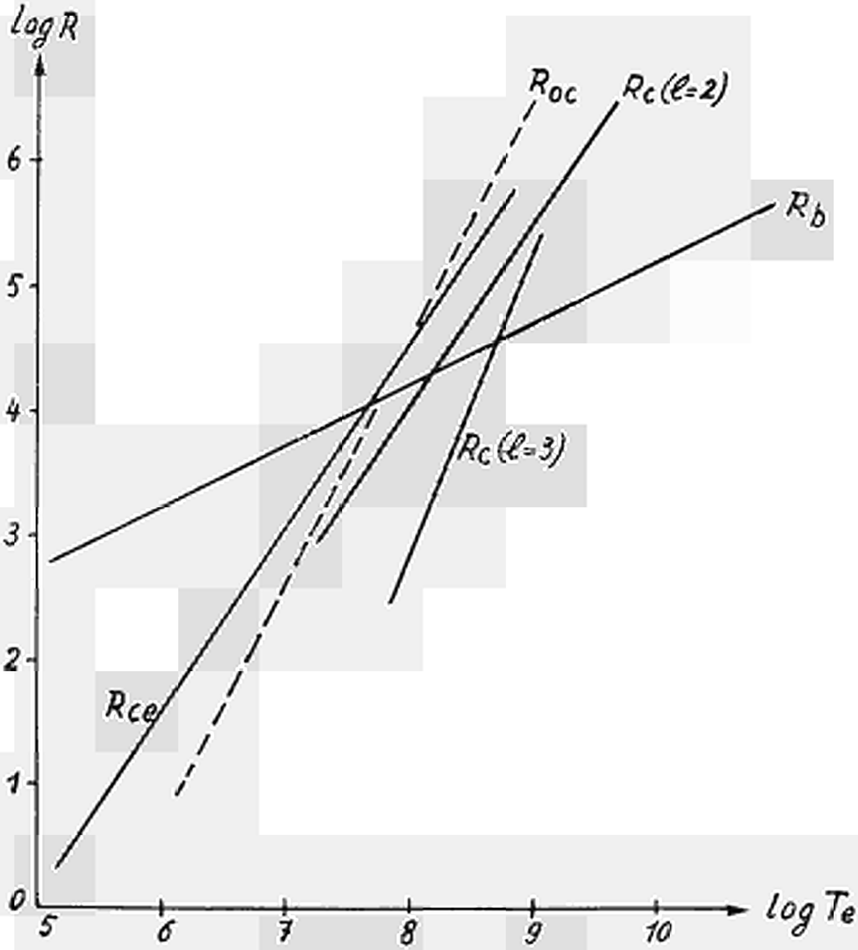


Fig. 45. The three types of radiation.

The corresponding curves are plotted in fig. 45. For the unscreened cyclotron radiation we have used the formula (147) written $\left(\text{for } \frac{B^2}{8\pi} \sim 2nkT\right)$ as

$$R_{oc}\, 10^{27} \cdot n_e^{-2}\, W_p \simeq 3.75 \cdot 10^{-12} \cdot T_e^2.$$

REFERENCES TO CHAPTER 2

1) B. F. Logan *et al.*, *J. assoc. Comput. Mach.* **2**, pp. 28-41 (1955).
2) J. G. Linhart and L. Th. M. Ornstein, CERN report 59-2 (Jan. 1959).
D. L. Morse, *Plasma ph.*, **9**, p. 61 (1967).
3) J. Meixner, Kosmische Strahlung und Magnetfeld der Erde; Vorträge über kosmische Strahlung; edited by W. Heisenberg (Berlin 1943).
4) M. Born and V. Fock, *Zeit. f. Ph.*, **51**, p. 165 (1928).
5) A. Garren *et al.*, Proc. of 2nd Geneva int. conf. on peaceful uses of atomic en., Vol. 31, p. 65 (1958).
6) S. Chandrasekkar, Plasmaphysics, p. 61, Univ. Chicago Press 1962.
7) G. Gibson *et al.*, *Nucl. Fus. Suppl.* **1**, p. 151 (1962), *Phys. Fluids*, **6**, p. 116 (1963).
8) L. Spitzer, Proc. of Geneva conf. 1958, vol. 32, p. 181, United Nations N.Y. 1958.
9) B. Lehnert, Dynamics of charged particles, pp. 99-102, North Holland Pub. Co (1964).
10) Page and Adams, Electrodynamics, pp. 326-328 (Van Nostrand, 1940).
11) A. S. Eddington, The internal constitution of stars, p. 223, CUP (1926).
12) L. Landau and E. Lifshitz, The classical theory of fields, p. 203 for the case $v \ll c$ and p. 213 for $v \sim c$; (Addison and Wesley, 1951).
13) W. Heitler, Quantum theory of radiation, p. 172 and pp. 174-176 (Oxford U.P., 1944 edition).
14) G. Schott, Electromagnetic radiation, CUP (1912).
15) H. Y. Tzu, Proc. Roy. Soc. A, **192**, p. 231 (1948).
16) J. D. Lawson, *Amer. J. Phys.*, **33**, p. 1000 (1965).
17) L. I. Schiff, Quantum mechanics, pp. 262-265 (Mc-Graw Hill, 1949).
18) T. Kihara *et al.*, *Nuclear Fusion*, **1**, p. 181 (1961).

List of symbols used in Chapter 2

A	atomic mass number	i, j, k	unit vectors
B	magnetic field strength	J	longitudinal invariant
c	velocity of light	$j =$	$\sqrt{-1}$
d	Debye length	k	wave number or Boltzmann's constant
E	electric field strength		
e	elementary charge	L	length
F	force	l	quantum number
f	frequency	m	mass of particle
g	unit vector	n	field index of betatron or refractive index or particle density
h	Planck's constant		
i	current density		

p	impact parameter, momentum	ε	angle or dielectric constant or base of natural logarithm
P	momentum	η	mirror ratio B_1/B_0
Q	charge	θ	angle
R	radius of curvature	χ	excentricity of a hyperbola
r	position vector	λ	wave length
s	velocity	μ	magnetic moment or magnetic susceptibility
t	time		
T	temperature	ρ	radius of gyration
u	drift velocity	σ	plane perpendicular to B or cross-section
V	potential		
v	velocity	Σ	surface
v_p	phase velocity	τ	time interval
W	kinetic energy	Φ	magnetic flux
w	power	φ ϕ ψ	angle
x, y, z	coordinates	ξ	position vector of centre of gyration
X_0	intermolecular spacing		
Z	atomic charge	Ω	solid angle
$\beta =$	v/c normalized speed	ω	angular frequency
$\gamma =$	$(1 - v^2/c^2)^{-1/2}$	ω_c	cyclotron frequency
δ	angle	ω_p	plasma frequency

CHAPTER 3

FLUID DESCRIPTION OF PLASMA

Introduction

The behaviour of plasmas is often more appropriately described by means of a fluid model, rather than by the trajectories of individual particles. The fluid is made up from two electrically charged components, one being the electron gas the other the gas of positive ions.

In this chapter we shall derive the equations of motion of these two fluids and transform them into the form used in plasma analysis.

Let us consider a flow in which adjacent particles have vanishingly small velocity relative to each other, the major portion of their total velocity being, therefore, in the direction of the flow. In such a case the flow is called a single stream flow and can be treated appropriately in ordinary configuration space. This description is still adequate for a multi-stream flow provided the component streams are not in any way coupled. If a multistream exists, in which the component streams are coupled, e.g., by collisions between particles composing these streams, a phase-space representation is appropriate (fig. 46).

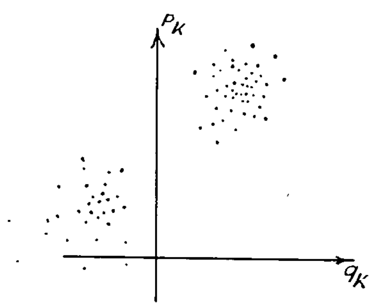


Fig. 46.

The phase-space may be constructed out of canonical variables $q_i p_i$ corresponding to all the degrees of freedom; however, it is often convenient to use a simpler system of orthogonal coordinates q_i

and velocity components v_i. The movement in the velocity subspace is restricted to within a sphere of radius c. Particles which are accelerated to relativistic energies will, therefore, occupy a thin shell near the surface of the c-sphere (fig. 47 also ref. 1).

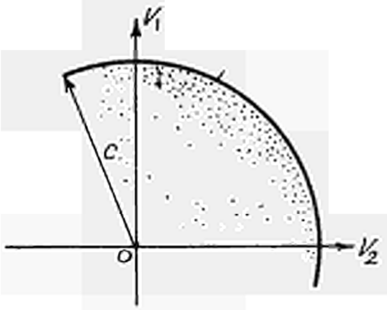


Fig. 47.

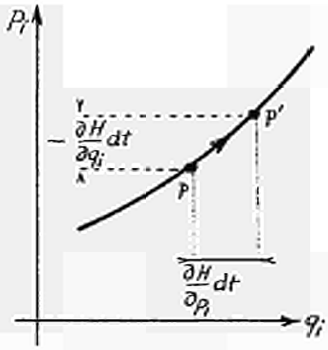


Fig. 48.

The movement of single particles in a phase space is given by the Hamiltonian equations

$$\frac{dq_i}{dt} = \frac{\partial H(q_i p_i t)}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H(q_i p_i t)}{\partial q_i} \qquad (1a, b)$$

where

$$H = c\left[m_0^2c^2 + \left(p - \frac{e}{c}A\right)^2\right]^{1/2} + e\phi. \qquad (2)$$

These equations can be often used for the graphical plotting of a trajectory by a step by step method in a number of p_iq_i graphs. During a time dt, the representative point P (q_ip_i) moves on a trajectory for which $H =$ const., i.e. on a surface of constant energy in the p_iq_i space (fig. 48). For time dependent problems this surface changes its form in time.

We shall apply the phase space description to a system of many particles. Although a Hamiltonian description for any number of interacting particles can be formulated (ref. 2), it is too complicated for problems in plasma physics. One adopts, therefore, a statistical model in which the *particles are considered to form a fluid moving in the p_iq_i space.*

The first general observation one can make about a system of n non-interacting particles located in a small volume $\Delta = \delta q_1 \delta p_1 \ldots \delta q_3 \delta p_3$ of the $p_i q_i$ space is known as Liouville's theorem.

Let the surface of the volume be marked out by the coordinates $p_i q_i$ of a few particles. We then assume that the interaction between the particles themselves can be ignored. The trajectory of each particle in the pq space obeys the Hamiltonian equations (1a, b). The rate at which the volume Δ changes, as the particles move, is given by

$$\frac{\mathrm{d}\Delta}{\mathrm{d}t} = \left(\frac{d(\delta q_1)}{\mathrm{d}t} \delta p_1 + \frac{d(\delta p_1)}{\mathrm{d}t} \delta q_1 \right)$$

$$\delta q_2 \delta p_2 \delta q_3 \delta p_3 + \ldots \text{ permutations} \quad (3)$$

where (fig. 49)

$$\delta q_i = q_i^1 - q_i^2, \; \delta p_i = p_i^1 - p_i^2$$

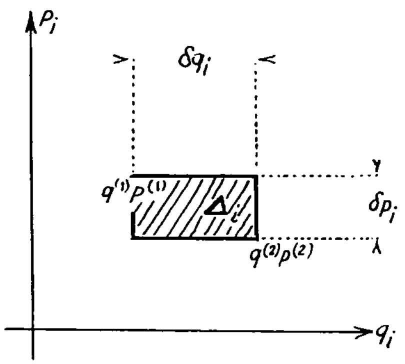


Fig. 49.

and, therefore, using eqs. (1a, b) one obtains *

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta q_i) = \frac{\mathrm{d}q_i^1}{\mathrm{d}t} - \frac{\mathrm{d}q_i^2}{\mathrm{d}t} = \frac{\partial H^1}{\partial p_i} - \frac{\partial H^2}{\partial p_i} = \frac{\partial}{\partial p_i}(\delta H)$$

$$= \delta \left(\frac{\partial H}{\partial p_i} \right)_{p_i = \text{const.}} = \frac{\partial^2 H}{\partial p_i \partial q_i} \partial q_i. \quad (4a)$$

* For interacting particles $\frac{\partial}{\partial p_i}(\delta H) \neq \delta \left(\frac{\partial H}{\partial p_i} \right)$ as H^1 is a function of not only p^1,q^1 but also of p^2,q^2.

Similarly,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta p_i) = \frac{\mathrm{d}p_i^1}{\mathrm{d}t} - \frac{\mathrm{d}p_i^2}{\mathrm{d}t} = -\frac{\partial H^1}{\partial q_i} + \frac{\partial H^2}{\partial q_i}$$

$$= -\delta\left(\frac{\partial H}{\partial q_i}\right)_{q_i=\mathrm{const.}} = -\frac{\partial^2 H}{\partial q_i \partial p_i}\delta p_i. \quad (4b)$$

Substituting eqs. (4a, b) into eq. (3) we get

$$\frac{\mathrm{d}\Delta}{\mathrm{d}t} = \left(\frac{\partial^2 H}{\partial p_1 \partial q_1} - \frac{\partial^2 H}{\partial q_1 \partial p_1}\right)\Delta + \ldots +$$

$$\left(\frac{\partial^2 H}{\partial p_3 \partial q_3} - \frac{\partial^2 H}{\partial q_3 \partial p_3}\right)\Delta \equiv 0. \quad (5)$$

This is the mathematical proof of Liouville's theorem. The theorem asserts that

The phase volume occupied by a system of non-interacting particles is an invariant of their motion.

The result of eq. (5) is often interpreted as the proof of the *invariance of particle density in the phase space.* However, this formulation of the theorem assumes that there are enough particles in the volume $\delta q_i \delta p_i$ to enable us to use the concept of density.

Let us assume that one deals with such a large number of particles that the concept of density is applicable and let us define

$$f(q_i p_i t)\delta q_1 \delta p_1 \ldots \delta q_k \delta p_k$$

as a number of particles in the volume element $\delta q_1 \ldots \delta p_k$. Then the second version of the Liouville's theorem asserts that

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 0 \quad (6)$$

which can be expanded as

$$\frac{\partial f}{\partial t} + \mathrm{Div}\,(Vf) = 0 \quad (6a)$$

or

$$\frac{\partial f}{\partial t} + f\,\mathrm{Div}\,V + V \cdot \mathrm{Grad}\,f = 0 \quad (6b)$$

where V is the velocity of particles in the phase space

$$V = V(\dot{q}_i, \dot{p}_i)$$

and the operators Div and Grad are a divergence and a gradient in the phase space. From Hamilton's equations it follows (similar to eq. (5)) that *

$$\operatorname{Div} V = \frac{\partial}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial}{\partial p_i}\frac{dp_i}{dt} = \frac{\partial^2 H}{\partial q_i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial q_i} = 0. \tag{7}$$

Therefore,

$$\frac{\partial f}{\partial t} + \left[\frac{dq_i}{dt}\frac{\partial f}{\partial q_i} + \frac{dp_i}{dt}\frac{\partial f}{\partial p_i}\right] = 0 \tag{8}$$

or

$$\frac{\partial f}{\partial t} + \left[\frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p_i}\right] = 0. \tag{8a}$$

This equation can be expressed in a more concise form by using the Poisson bracket notation

$$\frac{\partial f}{\partial t} + \{f, H\} = 0. \tag{8b}$$

This is a differential equation of flow in the phase space of a system of particles whose individual motion can be derived from a Hamiltonian H.

3.1. Stationary Distributions

Let us now briefly mention the properties of a stationary density distribution in velocity space i.e., a distribution in which the motion of individual particles does not affect the statistical value of the particle density f (ref. 3, 4). Thus

$$\frac{\partial f}{\partial t} = 0 \tag{9}$$

and consequently

$$\frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p_i} = 0. \tag{10}$$

It can be shown that if initially the density $f(p_i q_i)$ is only a function of the Hamiltonian H of the particles located at $p_i q_i$, then the distribution of f in the p_i space will remain stationary.

* Using the summation convention, i.e., $a_i b_i = \sum_{i=1}^{i=n} a_i b_i$.

In order to prove this, put

$$f = f[H(p_i q_i)]$$

then

$$\frac{\partial f}{\partial q_i} = \frac{\partial f}{\partial H}\frac{\partial H}{\partial q_i}, \quad \frac{\partial f}{\partial p_i} = \frac{\partial f}{\partial H}\frac{\partial H}{\partial p_i}.$$

Putting these expressions into the Poisson's bracket (eq. (8a)) we get

$$\{f, H\} = 0$$

and therefore $\partial f/\partial t = 0$ and the f distribution is a stationary one. Systems in which the particle density in phase-space is a function of the Hamiltonian only, are called "ergodic". Thus we have a theorem: *ergodic distributions are stationary.*

3.2. The Collisionless Boltzmann Equation

Let us transform Liouville's equation (8) into an equation in the $q_i v_i$ space. This is often more useful than the $q_i p_i$ formulation. The equation for $f(q_i v_i)$ in the $q_i v_i$ phase space is then called the collisionless Boltzmann equation (ref. 5).

Let us define the appropriate canonical variables for a relativistic particle in an electromagnetic field. These are

$$q_i, p_i = m_0 v_i \gamma + \frac{e}{c} A_i \tag{11}$$

with

$$\gamma = \frac{1}{\left(1 - \dfrac{v^2}{c^2}\right)^{1/2}}.$$

The corresponding relativistic Hamiltonian is

$$H = c\left\{ m_0^2 c^2 + \left(p - \frac{e}{e} A\right)^2 \right\}^{1/2} + e\phi.$$

Using Hamilton's equations the terms dq_i/dt and dp_i/dt become

$$\frac{dq_i}{dt} \equiv \frac{\partial H}{\partial p_i} = v_i \tag{12}$$

$$\frac{dp_i}{dt} \equiv -\frac{\partial H}{\partial q_i} = +\frac{e}{cm_0\gamma}\left(p_k - \frac{e}{c} A_k\right)\frac{\partial A_k}{\partial q_i} - e\frac{\partial \phi}{\partial q_i}$$

$$= \frac{e}{c} v_k \frac{\partial A_k}{\partial q_i} - e\frac{\partial \phi}{\partial q_i}. \tag{13}$$

With the help of these expressions, eq. (8) can be written

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial q_i} + \left(\frac{e}{c} v_k \frac{\partial A_k}{\partial q_i} - e \frac{\partial \phi}{\partial q_i}\right) \frac{\partial f}{\partial p_i} = 0. \tag{14}$$

This equation can be simplified by interpreting $v_k(\partial A_k/\partial q_i)$ in terms of the force of magnetic field B = curl A on a particle with a velocity vector $\boldsymbol{v}$. This force is F_{M} where

$$\frac{c}{e} \boldsymbol{F}_{\mathrm{M}} = \boldsymbol{v} \wedge \boldsymbol{B} = \boldsymbol{v} \wedge \operatorname{curl} \boldsymbol{A}. \tag{15}$$

Substituting this expression into the term $v_k\, \partial A_k/\partial q_i)$ one obtains

$$v_k \frac{\partial A_k}{\partial q_i} = \boldsymbol{v} \cdot \operatorname{grad} A_i + [\boldsymbol{v}\ (\nabla \wedge \boldsymbol{A})]_i. \tag{16}$$

Let us now transform the expressions $\partial f/\partial p_i$, $\partial f/\partial q_i$ and $\partial f/\partial t$. As p_i is a function of v_i and of q_i (through the A_i's) one has to consider the following transformation of independent variables.

$$\begin{aligned} p_i &= m_0 \gamma v_i + \frac{e}{c} A_i, \\ q_i &= q'_i, \\ t &= t'. \end{aligned} \tag{17}$$

It follows that

$$\frac{\partial f[v_k(p_i), q'_k, t']}{\partial p_i} = \frac{\partial f}{\partial v_k} \frac{\partial v_k}{\partial p_i} + \frac{\partial f}{\partial q'_k} \frac{\partial q'_k}{\partial p_i} + \frac{\partial f}{\partial t'} \frac{\partial t'}{\partial p_i}. \tag{18}$$

The coefficients $\partial v_k/\partial p_i$ and $\partial q'_k/\partial p_i$ can be calculated using the Jacobian of the transformation (17), and are *

$$\frac{\partial v_k}{\partial p_i} = \frac{1}{m_0 \gamma} \left(\delta_{ik} - \frac{v_i v_k}{c^2}\right), \qquad \frac{\partial q'_k}{\partial p_i} = 0, \qquad \frac{\partial t'}{\partial p_i} = 0. \tag{18a}$$

Similarly

$$\frac{\partial f[v_k(p), q'_k, t']}{\partial q_i} = \frac{\partial f}{\partial v_k} \frac{\partial v_k}{\partial q_i} + \frac{\partial f}{\partial q'_k} \frac{\partial q'_k}{\partial q_i} + \frac{\partial f}{\partial t'} \frac{\partial t'}{\partial q_i}. \tag{19}$$

Using the same Jacobian one finds

$$\frac{\partial v_k}{\partial q_i} = \frac{-1}{m_0 \gamma} \frac{e}{c} \frac{\partial A_k}{\partial q_i} \left(\delta_{ik} - \frac{v_i v_k}{c^2}\right), \qquad \frac{\partial q'_k}{\partial q_i} = \delta_{ik}, \qquad \frac{\partial t'}{\partial q_i} = 0. \tag{19a}$$

* δ_{ik} is the Kronecker delta.

The remaining differential operator that needs to be transformed is

$$\frac{\partial f}{\partial t} = \frac{df}{\partial v_k} \frac{\partial v_k}{\partial t} + \frac{\partial f}{\partial q'_k} \frac{\partial q'_k}{\partial t} + \frac{\partial f}{\partial t'} \frac{\partial t'}{\partial t}. \tag{20}$$

Applying the same method as above, we get

$$\frac{\partial v_k}{\partial t} = \frac{-1}{m_0 \gamma} \frac{e}{c} \frac{\partial A_i}{\partial t} \left(\delta_{ik} - \frac{v_i v_k}{c^2} \right), \quad \frac{\partial q'_k}{\partial t} = 0, \quad \frac{\partial t'}{\partial t} = 1. \tag{20a}$$

Using eqs. (16), (18a), (19a) and (20a) one can now write down the transformed eq. (14) as

$$\frac{\partial f}{\partial t'} + v_k \frac{\partial f}{\partial q'_k} + \frac{F_i}{m_0 \gamma} \left(\delta_{ik} - \frac{v_i v_k}{c^2} \right) \frac{\partial f}{\partial v_k} = 0 \tag{21}$$

where

$$F_i = -e \frac{\partial \phi}{\partial q_i} - \frac{e}{c} \frac{\partial A_i}{\partial t} + \frac{e}{c} (\boldsymbol{v} \wedge \boldsymbol{B})_i.$$

Let us develop this equation more explicitly for the cases which are of interest in plasma physics.

3.2.1. NON-RELATIVISTIC ENSEMBLE

The first case is that of a non-relativistic gas, i.e., an ensemble of particles which are confined in the velocity space within a sphere (fig. 50) centred on origin and whose radius is

$$v < 0.1\ c.$$

In this case the terms $v_i v_k / c^2$ in eq. (21) represent at most 1 % correction to the diagonal terms δ_{ik}. This formulation is appropriate

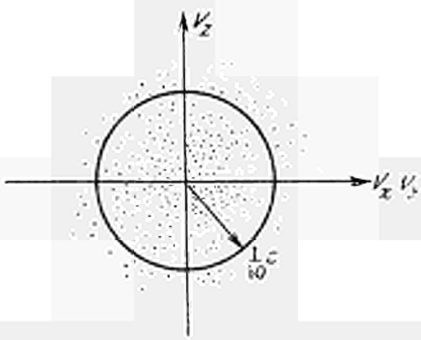


Fig. 50. Non-relativistic gas in the velocity space.

for a plasma near thermal equilibrium in which the distribution is Maxwellian with a temperature

$$T < 6.10^7 \ (^\circ\mathrm{K})$$

in which the mean random speed of the electrons is only

$$\bar{v}_e = \sqrt{\frac{2kT_e}{m_e}} = 0.1\ c.$$

In this case eq. (21) simplifies to (dropping the primes)

$$\frac{\partial f}{\partial t} + v_k \frac{\partial f}{\partial q_k} + \frac{F_k}{m} \frac{\partial f}{\partial v_k} = 0. \tag{22}$$

It is clear that the description of plasma processes in terms of the density f in phase space is appropriate only if there are enough particles per unit volume to give f a meaning. This is possible if both in the velocity space and in the configuration space the fluctuations in f are much smaller than f itself.

This renders the Boltzmann equation inapplicable to regions of velocity space for which the random speed is many times the mean termal speed. A similar limitation applies to the configuration space where processes occurring in too small a volume cannot be described by eq. (21).

The equations (21) and (22) have been derived form the Liouville's theorem and therefore, only considering a system of non-interacting particles. For this reason the potentials ϕ and A must correspond to fields generated by sources outside the plasma. The treatment is applicable, therefore, only to plasmas whose density and pressure are low enough not to perturb noticeably these fields. However, this is too strong a restriction. The Liouville's theorem requires only that there should be no interaction between particles within the small volume $\Delta = (\delta x\ \delta y\ \delta z)\ (\delta p_x\ \delta p_y\ \delta p_z)$ where Δ is very much smaller than the phase space effectively occupied by the plasma. Interaction between particles whose separation L in configuration space is much greater than $L_0 = (\delta x\ \delta y\ \delta z)^{1/3}$ will render equation (21) inapplicable only after a certain relatively long time. In such a case the contributions to ϕ and A from particles located relatively far from Δ can be included in the collisionless Boltzmann equation. The far-field extension of the B-equation is known as the Vlasov equation in which only near interactions are not taken into account. Let us define as a near interaction one in which the distance between two interacting particles is $L \leq L_0$. Since $L_0 \gg l = n^{-1/3}$ it follows that $L > l$. It is evident that the Vlasov approximation may become rigorously valid if such near interactions could be completely neglected. Such a situation may be realised by dividing all electrons and ions into k parts where $k \rightarrow \infty$, mass and electric charge density remaining invariant. In such a case the new interparticle distance is

$$l' = \frac{l}{k^{1/3}}$$

and it is obviously possible to define Δ and therefore L_0 as small as one wishes. Moreover, within the new Δ' the force F' between two charges becomes

$$F' = \left(\frac{e'}{l'}\right)^2 = \left(\frac{e/k}{l/k^{1/3}}\right)^2 = F \cdot k^{-4/3} \qquad (23)$$

and therefore, $F' \rightarrow 0$ even faster than $\dfrac{1}{k}$.

It is interesting to find out how the Debye distance d is influenced by such a subdivision. Recalling (eq. (4a), chapter 1) that

$$d = \sqrt{\frac{\kappa T}{4\pi e^2 n}} = \sqrt{\frac{2/3\, m\bar{u}^2}{4\pi e^2 n}} \qquad (24)$$

where $\bar{u}$ is the mean thermal speed of electrons and considering that $m' = \dfrac{m}{k}$, $n' = kn$ and $e'^2 = e^2k^{-2}$ it follows that d' = d. The Debye distance remains constant after suddivision to any k.

The subdivision in which $k \rightarrow \infty$ corresponds to a model of a homogeneous fluid and let us repeat that the Vlasov equation describes perfectly such a fluid, even to the level of collective interactions *.

Real plasmas, of course, depart in their behaviour from such a model. Let us mention, e.g., the phenomenon of oscillation of a Debye sphere or a Debye layer. We have explained on p. 13 that generating such oscillations by a stochastic process is improbable if the number of electrons N in a Debye sphere is very large. In the homogeneous fluid model the number $N' = N.k \rightarrow \infty$ and therefore, the spontaneous generation of plasma oscillations becomes impossible. Consequently in problems in which one wishes to find out the levels of various fluctuations and oscillating modes in plasma the Vlasov equation is not applicable.

In order to modify the Vlasov equation so as to include at least some of the near-interactions in a real plasma one must add a new $\partial f/\partial t$ term corresponding to collisions. This is done for distant collisions in

* Compare with the Hartree self-consistent model of atom in wave-mechanics.

chapter 8 (the minimum distance in such collisions being much larger than both δ and λ_B).

*

3.2.2. RELATIVISTIC ENSEMBLE

The second situation of interest in plasma physics is the ensemble of relativistic particles whose velocity scatter is small compared with light velocity. In particular, we may assume that there exists a system of reference in which the ensemble can be described as a non-relativistic gas, the velocity v_0 of this frame of reference being close to the velocity of light (fig. 51). In this case

$$v_z \gg v_x, v_y.$$

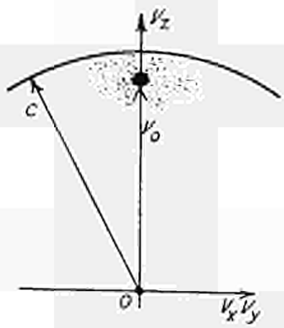


Fig. 51.

Neglecting terms of second order in v_x, v_y, eq. (21) becomes

$$\frac{\partial f}{\partial t} + v_k \frac{\partial f}{\partial q_k} + m_0^{-1}\gamma^{-1}\left[F_x - F_z \frac{v_x v_z}{c^2}\right]\frac{\partial f}{\partial v_x} + m_0^{-1}\gamma^{-1}\left[F_y - F_z \frac{v_y v_z}{c^2}\right]\frac{\partial f}{\partial v_y}$$
$$+ m_0^{-1}\gamma^{-1}\left[F_z\left(1 - \frac{v_z^2}{c^2}\right) - F_x \frac{v_x v_z}{c^2} - F_y \frac{v_y v_z}{c^2}\right]\frac{\partial f}{\partial v_z} = 0. \quad (25)$$

A purely one-dimensional form can be developed by putting $F_x = F_y = 0$, $\partial/\partial x = \partial/\partial y = 0$, $v_x = v_y = 0$. One then obtains

$$\frac{\partial g}{\partial t} + v_z \frac{\partial g}{\partial z} - 2m_0^{-1}\gamma^{-1}F_z \frac{v_z}{c^2}\, g + m_0^{-1}\gamma^{-3}F_z \frac{\partial g}{\partial v_z} = 0 \quad (26)$$

where g is normalized as

$$g = 2\pi \int_0^\infty v_\perp f \, dv_\perp, \quad v_\perp = \sqrt{v_x^2 + v_y^2}. \tag{27}$$

3.3. Integrals of Boltzmann's Equations over the Velocity Space

One is usually interested in the macroscopic properties of a plasma, i.e. in averaged quantities. Let us consider some particle property, for instance, the velocity of the ions and electrons or their energy. Call this particle quality Q, and *let it depend on* v *of the particles only.*

In order to obtain equations concerning Q we multiply the Boltzmann's equation by Q and then integrate over the velocity space π. In order to obtain an average *per particle* the integrated equation will be divided by the number $n(r, t)$ of particles per unit volume; where

$$n(r, t) = \int_\pi f(v, r, t) \, d\pi. \tag{28}$$

Thus the formula defining the average quantity $\bar{Q}$ (depending on r and t only) is:

$$\bar{Q}(r,t) = \frac{\int Q(v) f(v, r, t) \, d\pi}{\int_\pi f(v, r, t) \, d\pi}. \tag{29}$$

3.3.1. NON-RELATIVISTIC CASE

We consider the non-relativistic Boltzmann equation first. Multiplying eq. (22) by $Q(v)$ and integrating over π one obtains the transport equation for the particular average quantity $\bar{Q}$.

In the process of averaging one encounters the following three integrals:

$$\int_\pi Q \frac{\partial f}{\partial t} \, d\pi = \frac{\partial}{\partial t} \int_\pi Q f \, d\pi = \frac{\partial}{\partial t} (n\bar{Q}) \tag{30a}$$

$$\int_\pi Q v_i \frac{\partial f}{\partial q_i} \, d\pi = \frac{\partial}{\partial q_i} n(\overline{v_i Q}) \tag{30b}$$

$$\int_\pi Q \frac{dv_i}{dt} \frac{\partial f}{\partial v_i} \, d\pi = \iint_{\pi_x \pi_y} \left\{ \left| fQ \frac{dv_z}{dt} \right|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f \frac{\partial}{\partial v_z} Q \frac{dv_z}{dt} \, dv_z \right\} dv_x \, dv_y$$

$$+ \ldots \text{ the same terms with } v_x \text{ and } v_y. \tag{30c}$$

The last integral is developed by integration by parts. The first term of the integrand in eq. (30c), containing $|fQ(\mathrm{d}v_i/\mathrm{d}t)|_{-\infty}^{\infty}$ is zero in all physical problems as f has usually the Maxwellian form whereas the particle property Q can increase only as some finite power of v. Each term of the remaining integrals in (30c) has a form

$$\int_\pi f \frac{\partial}{\partial v_i}\left(Q \frac{\mathrm{d}v_i}{\mathrm{d}t}\right) \mathrm{d}\pi.$$

The integral (30c) can thus be written:

$$\int_\pi Q \frac{\mathrm{d}v_i}{\mathrm{d}t} \frac{\partial f}{\partial v_i} \mathrm{d}\pi = -n \overline{\left[\frac{\partial}{\partial v_i}\left(Q \frac{\mathrm{d}v_i}{\mathrm{d}t}\right)\right]}.$$

The transport equation for the quantity Q follows now by writing the previous results as

$$\frac{\partial}{\partial t} (n\bar{Q}) + \frac{\partial}{\partial q_i} [n(\overline{v_i Q})] - n \overline{\left[\frac{\partial}{\partial v_i}\left(Q \frac{\mathrm{d}v_i}{\mathrm{d}t}\right)\right]} = 0. \tag{31}$$

In order not to complicate our analysis unnecessarily we shall consider only such forces for which

$$\frac{\partial}{\partial v_i}\left(\frac{\mathrm{d}v_i}{\mathrm{d}t}\right) = 0.$$

If Q denotes the identity of a particle; then $Q = 1$ and eq. (31) becomes

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial q_i} (n\bar{v}_i) = 0.$$

This could be transformed into

$$\frac{\partial n}{\partial t} + n \frac{\partial \bar{v}_i}{\partial q_i} + \bar{v}_i \frac{\partial n}{\partial q_i} = 0$$

or

$$\frac{\partial n}{\partial t} + \operatorname{div} (n\bar{v}) = 0 \tag{32}$$

which is the situation of continuity of a flow of particles.

We shall now describe the transport of momentum. This is obtained * by letting $Q = v$.

* In the non-relativistic approximation.

Putting this in eq. (31) we obtain

$$\frac{\partial}{\partial t}(n\bar{v}) + \frac{\partial}{\partial q_i}[n\overline{vv_i}] - n\left[\overline{\frac{\partial v}{\partial v_i}\left(\frac{dv_i}{dt}\right)}\right] = 0. \tag{33}$$

The first term in eq. (33) may be expanded as

$$\bar{v}\frac{\partial n}{\partial t} + n\frac{\partial \bar{v}}{\partial t}. \tag{34}$$

The last term is

$$-n\frac{d\bar{v}}{dt}. \tag{35}$$

The second term is a little more difficult. It can be written as:

$$\begin{aligned}\frac{\partial}{\partial x}\int_\pi vv_x f\,d\pi &+ \frac{\partial}{\partial y}\int_\pi vv_y f\,d\pi + \frac{\partial}{\partial z}\int_\pi vv_z f\,d\pi \\ &= i\left\{\frac{\partial}{\partial x}\int_\pi v_x^2 f\,d\pi + \frac{\partial}{\partial y}\int_\pi v_x v_y f\,d\pi + \frac{\partial}{\partial z}\int_\pi v_x v_z f\,d\pi\right\} \\ &+ j\ \ldots\ldots\ldots\ldots\ldots\ldots + \\ &+ k\ \ldots\ldots\ldots\ldots\ldots\ldots\end{aligned} \tag{36}$$

where ($v = vv_x + vv_y + vv_z$).

It is now convenient to define a random velocity related to the average velocity $\bar{v}$ and the velocity v of a particle by

$$u = v - \bar{v}. \tag{37}$$

The definitions of v implies

$$\bar{u} = \frac{1}{n}\int_\pi uf\,d\pi = 0. \tag{38}$$

Let us substitute

$$\overline{v_x} + u_x \text{ for } v_x,$$

etc., into eq. (36). All products in which one component of the average velocity $\bar{v}$ multiplies any component of u drop out because v is not effected by integrations over π and because of eq. (38). There will remain only terms such as $u_x u_y f$ and $v_x v_y f$ integrated over all the velocity space. Thus eq. (36) becomes

$$i\left\{\frac{\partial}{\partial x}n(\bar{v}_x^2+\overline{u_x^2})+\frac{\partial}{\partial y}n(\bar{v}_x\bar{v}_y+\overline{u_xu_y})+\frac{\partial}{\partial z}n(\bar{v}_x\bar{v}_z+\overline{u_xu_z})\right\}$$

$$+\,j\left\{\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\quad\frac{\partial}{\partial y}n(\bar{v}_y^2+\overline{u_y^2})+\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\right\}$$

$$+\,k\ \cdot \qquad (36a)$$

This array can be expressed using a tensor notation. It is composed of two tensors: the $\nabla\psi_v$ tensor which can be written in the ordinary language of tensor algebra as $\nabla(n\bar{v}\bar{v})$ and the $\nabla\psi_u$ tensor, written as $\nabla(n\overline{uu})$ (notice the different way of averaging).

Thus

$$\nabla(\psi_v+\psi_u)=\nabla(n\bar{v}\bar{v})+\nabla(n\overline{uu}). \qquad (36b)$$

In applying the tensor notation to the expansion of eq. (36a) let us first examine the dyadic corresponding to $\bar{v}$, i.e., $\nabla(n\bar{v}\bar{v})$. This can be expanded as:

$$\nabla(n\bar{v}\bar{v})=n\bar{v}\nabla\bar{v}+\bar{v}\nabla\cdot(n\bar{v}). \qquad (39)$$

Thus, the term (36) may be introduced in (33) written as

$$n\bar{v}\nabla\bar{v}+\bar{v}\nabla\cdot(n\bar{v})+\nabla(n\overline{uu}). \qquad (36c)$$

Let us remember that the term $\nabla(n\overline{uu})$ is a complex of integrals and can be evaluated only if the velocity distribution is known, which means that one would first have to know the solution of the Boltzmann equation. However, one is often justified in using a guess at the distribution in the velocity space, compute the $\nabla(\boldsymbol{nuu})$ and obtain a solution for n and $\bar{Q}$'s.

Recalling the continuity equation and using eqs. (36c) and (34) one obtains for eq. (33):

$$n\left(\frac{\partial\bar{v}}{\partial t}+\bar{v}\nabla\cdot\bar{v}\right)=n\frac{F}{m}-\nabla(n\overline{uu}) \qquad (40)$$

where $F=m(d\bar{v}/dt)$.

This is the equation for transfer of momentum in a non-relativistic plasma.

Let us consider an ensemble of particles in which $\boldsymbol{u}$ is isotropic, i.e., a distribution which is spherically symmetrical in the velocity space. An example of this is a Maxwellian distribution.

In that case

$$\nabla(n\overline{uu})=\nabla(n\overline{u^2}). \qquad (41)$$

This expression can be associated with the definition of pressure, as pressure can be defined as the transport of momentum across a surface of unit area in the reference system in which $\bar{v} = 0$.

Thus

$$\nabla(n\overline{u^2}) = \frac{\text{grad } p}{m}. \tag{42}$$

If the u vector defines an ellipsoid in the velocity space (fig. 52) one has

$$\begin{aligned}\nabla(n\overline{uu}) &= \nabla[n(u_x^2 + u_y^2 + u_z^2)] \\ &= \frac{1}{m}(\text{grad } p_x + \text{grad } p_y + \text{grad } p_z)\end{aligned}$$

which is a case often encountered in non-isotropic distribution.

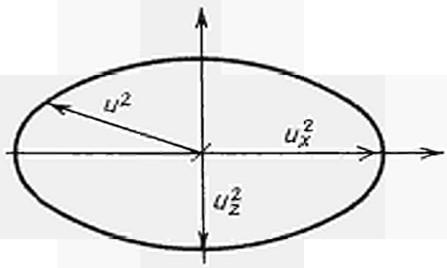


Fig. 52.

For the case of isotropic velocity distribution eq. (40) can be written as

$$n\left(\frac{\partial \bar{v}}{\partial t} + \bar{v}\nabla \cdot \bar{v}\right) = \frac{n}{m}F - \frac{\text{grad } p}{m}. \tag{43}$$

This equation must be applied to each of the two components of the plasma because plasma is a mixture of two gases: the electron gas and the positive ion gas. We will use the subscript e for the electrons and i for the positive ions, v for the velocity of the electrons and w for that of the ions.

If the force is of electromagnetic origin one has

$$F = eE + e\frac{\bar{v}}{c} \wedge B. \tag{44}$$

One can also include other non-electromagnetic forces, for instance a gravitational force, which, however, is of interest only in astrophysics.

Let us now consider the transport of energy. This corresponds to the second moment of $f(t)$ in the velocity space Taking $Q = v^2 = \Sigma_a(\bar{v}_a + u_a)^2$ and substituting in eqs. (30a,b,c) we get for the three terms:

$$\text{1 st.:} \quad \frac{\partial}{\partial t}\,[n\bar{v}^2 + \sum_a \int f u_a{}^2 \, d\pi] = \frac{\partial}{\partial t}\,(\bar{v}^2 + \sum_a u_a{}^2) \cdot n$$

$$\text{2nd:} \quad \frac{\partial}{\partial t} \sum_a \int f v_i \,(\bar{v}_a{}^2 + u_a{}^2 + 2\,\bar{v}_a u_a)\, d\pi =$$

$$= \frac{\partial}{\partial q_i}\,[\bar{v}_a{}^2 \bar{v}_i + \sum_a (\bar{v}_i \overline{u_a{}^2} + 2\,\bar{v}_a\,\overline{u_a u_i} + \overline{u_i u_a{}^2})] \cdot n$$

$$\text{3rd:} \quad -\sum_a \int f \frac{F_i}{m} \frac{\partial}{\partial v_i} v_a{}^2 \, d\pi = -\frac{2\,\bar{F}_i}{m}\,\bar{v}_i\, n$$

where $\bar{v}^2 = \sum_a \bar{v}_a{}^2$ and $\dfrac{\partial F_i}{\partial v_i} = 0.$

Apart from the expression $nm(\overline{u_a u_i})$, which is the pressure-tensor p_{ai} already discussed on pp. 101-102, a new quantity appears, i.e., the vector $Q_i = \frac{1}{2} mn \sum_a \overline{u_i u_a{}^2}$. The physical meaning of this vector is related to the heat-flux. Let us recall that $\sum_a \frac{1}{2} nm\bar{u}_a{}^2$ is the density of internal kinetic energy of our fluid, which in thermal equilibrium can be described by $3/2nkT$. Evidently Q_i expresses the heat transport in the *i-th* direction, due to $\bar{u}_i$. Using this nomenclature and calling $3p = \sum_a nm\bar{u}_a{}^2$, the second moment of the Boltzmann-Vlasov equation can be written as follows

$$\left(\frac{\partial}{\partial t} + \bar{v}_i \frac{\partial}{\partial q_i} + \frac{\partial \bar{v}_i}{\partial q_i}\right)\left(\frac{1}{2}\rho\bar{v}^2 + \frac{3}{2}p\right) + \frac{\partial}{\partial q_i}(\rho \bar{v}_a p_{ai}) + $$
$$+ \frac{\partial Q_i}{\partial q_i} - \rho \bar{v}_i \frac{\bar{F}_i}{m} = 0. \qquad (45)$$

In deriving the successive moments of the B-V equation each new and higher moment required the definition of a new quantity. Thus in writing the equation continuity of flow [(eq. (32)), the zeroth moment of $f(v)$] it was necessary to introduce the expression for mass-velocity vector $\bar{v}_i$, in deriving the equation for the momentum transfer (i.e., 1st moment of the B-V equation) a definition of the pressure — tensor p_{ai} was needed and the equation for energy — transfer (2nd moment) required the introduction of the heat-flux vector Q_i. It can be shown that this necessity persists for higher moments of the B-V equations and consequently the equation system of the moments is an open system and its solution requires the knowledge of one of the moments of $f(v)$ in the velocity-space. In many cases of physical interest we may assume the $f(v)$ or its moments approximately known, in which case the system of moments of the B-V equation can be truncated at a moment of a certain order.

3.3.2. RELATIVISTIC CASE

Let us try to integrate the relativistic Boltzmann equation (21). We shall assume that in the frame S' moving with a speed v the f-distribution is Maxwellian with a mean thermal speed $s' = \sqrt{\overline{u'^2}}$ (fig. 53).

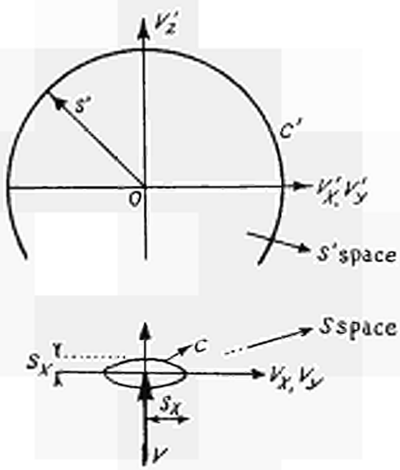


Fig. 53.

In the laboratory frame S the circle C' of S' (corresponding to s') is mapped on a curve C. The transformation that maps s' velocity space on the s velocity space is the theorem for the relativistic addition of velocities. In the s' velocity space

$$\frac{\partial f}{\partial v'_x} = \frac{\partial f}{\partial v'_y} = \frac{\partial f}{\partial v'_z} \sim \frac{f'_0}{\bar{u}'} \sim \frac{n'}{4/3\pi(\bar{u}')^4}. \tag{46}$$

The ratio of densities n and n' is simply

$$\frac{n}{n'} = \bar{\gamma} = \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{1/2}}$$

which is the mean γ for the particles in the S-space.

Also

$$\frac{\bar{u}_x}{\overline{u'_x}} = \frac{\bar{u}_y}{\overline{u'_y}} = \bar{\gamma}^{-1}, \qquad \frac{\bar{u}_z}{\overline{u'_z}} = \bar{\gamma}^{-2}. \tag{47}$$

Let us multiply eq. (25) by v and assume that the u_x, u_y, u_z are small in comparison with C. Then

$$\frac{\partial}{\partial t}(\bar{v}n) + \nabla(n\overline{uu}) + \nabla(n\overline{vv}) + \int_\pi F_i \frac{\partial f}{\partial v_i} v m_0^{-1}\gamma^{-1}\,d\pi$$

$$+ \int_\pi v\left(F_i \frac{\partial f}{\partial v_z} - F_z \frac{\partial f}{\partial v_i}\right)\frac{v_i v_z}{m_0 c^2 \gamma}\,d\pi - \int_\pi F_z \frac{v_z^2}{m_0 c^2 \gamma} v \frac{\partial f}{\partial v_z}\,d\pi = 0. \tag{48}$$

The integration of this equation is too complicated to be performed here. Instead we shall use a simpler argument to obtain a fluid equation whose form is essentially the same as that of the integrated eq. (48).

In the absence of random speeds, i.e., in the limit of $u_x = u_y = u_z = 0$, the equation of the flow of momentum is the well known hydrodynamic equation

$$\frac{\partial v}{\partial t} + \bar{v}\,\mathbf{grad}\cdot\bar{v} = \frac{F_\perp}{m_0\bar{\gamma}} + \frac{F_\parallel}{m_0\bar{\gamma}^3}. \tag{49}$$

It is possible to define additional transversal and longitudinal forces representing pressures in a gas in which $\bar{u} \neq 0$. From eq. (47) it follows that

$$\mathrm{grad}_\perp p = \frac{\partial n m_\perp \overline{u_\perp}^2}{\partial q_\perp} = m_0 \frac{\partial n \overline{u'^2}\bar{\gamma}^{-1}}{\partial q_\perp} \tag{50}$$

$$\mathrm{grad}\, p_\parallel = \frac{\partial n m_\parallel \overline{u_\parallel^2}}{\partial q_\parallel} = m_0 \frac{\partial n \overline{u'^2}\bar{\gamma}^{-1}}{\partial q_\parallel}. \tag{51}$$

Writing

$$m_0\overline{u'^2} = kT' \tag{52}$$

and adding the pressure terms (50) and (51) to eq. (49) we get

$$\frac{\partial \bar{v}}{\partial t} + \bar{v}\,\mathbf{grad}\cdot\bar{v} = \frac{F_\perp}{m_0\bar{\gamma}} + \frac{F_\parallel}{m_0\bar{\gamma}^3} - \frac{1}{m_0 n\bar{\gamma}}\frac{\partial(nkT'\bar{\gamma}^{-1})}{\partial q_\perp}$$

$$- \frac{1}{m_0 n\bar{\gamma}^3}\frac{\partial(nkT'\bar{\gamma}^{-1})}{\partial q_\parallel}. \tag{53}$$

which is the equation for momentum transport in a relativistic gas.

3.4. Fluid Models

The moments of the B-V equation can be written in several different ways, according to the physical situation studied.

As plasma is a medium composed of ions and electrons we shall need two systems of equations, one for the ion gas and one for the electron gas *. Let us denote the quantities related to electrons by a subscript e, those related to ions by a subscript i, except the masses and flow velocities which will be labeled m, M and and v, w respectively.

In absence of dissipative processes (collisions, turbulence) and provided energy transport equations are either unimportant or can be approximated by simple equations of state, it is possible to describe a plasma by continuity of flow and momentum-transport equations. These are

$$\frac{\partial n_e}{\partial t} + \nabla (n_e \boldsymbol{v}) = 0, \qquad \frac{\partial n_i}{\partial t} + \nabla (n_i \boldsymbol{w}) = 0 \tag{54a,b}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\nabla \cdot \boldsymbol{v} = -\frac{e}{m}\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \wedge \boldsymbol{B}\right) - \frac{\nabla (n_e \overline{u_{ej} u_{ek}})}{n_e} \tag{54c}$$

$$\frac{\partial \boldsymbol{w}}{\partial t} + \boldsymbol{w}\nabla \cdot \boldsymbol{w} = \frac{e}{M}\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{w} \wedge \boldsymbol{B}\right) - \frac{\nabla (n_i \overline{u_{ij} u_{ik}})}{n_i} \tag{54d}$$

together with Maxwell's equations

$$\nabla \cdot \boldsymbol{E} = 4\pi e(n_i - n_e), \qquad \nabla \cdot \boldsymbol{B} = 0 \tag{55a,b}$$

$$\nabla \wedge \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}, \qquad \nabla \wedge \boldsymbol{B} = 4\pi e(n_i \boldsymbol{w} - n_e \boldsymbol{v}) + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}. \tag{55c,d}$$

These equations represent the simplest non-relativistic *two-fluid* model of a plasma. This model is useful with plasmas in which the system of electron and ion gases is far from thermodynamic equilibrium, such as e.g., in two-stream plasmas.

If either of the plasma components possesses a relativistic flow speed (this is usually the electron gas) one has to use one relativistic equation of momentum transfer, such as eq. (53) and one non-relativistic equation. Such a formulation is always possible as one can choose a system of reference in which one of the components is at rest. Thus assuming the ions to be non-relativistic

$$m\gamma^3 \left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)_{\|} = -eE_{\|} - \frac{e}{c}(\boldsymbol{v} \wedge \boldsymbol{B})_{\|} - \frac{\partial}{\partial q_{\|}}\frac{(n_e \overline{u_{\|}^2 \gamma})}{n_e} \tag{56a}$$

* We shall consider only once ionized atoms.

$$m\gamma\left(\frac{dv}{dt}\right)_\perp = -eE_\perp - \frac{e}{c}(v \wedge B)_\perp - \frac{\partial}{\partial q_\perp}\frac{(n_e\overline{u_\perp{}^2\gamma})}{n_e} \quad (56b)$$

$$\frac{\partial w}{\partial t} + w \,\mathbf{grad} \cdot w = \frac{e}{M}\left(E + \frac{1}{c} w \wedge B\right) - \frac{\nabla(n_p\overline{u_p u_p})}{n_p}. \quad (57)$$

This describes the relativistic two-fluid model (the equations of continuity being the same as in the non-relativistic case).

In many cases of interest the flow velocity for both components is smaller than the mean thermal speed of the ion gas. In that case both components occupy the region of the velocity space near the origin (fig. 50). If the extension of the system one studies is very much larger than the Debye wave length one can assume a nearly neutral plasma, i.e.. a plasma in which the densities of the two components are almost equal everywhere. In such a case, the movement of both components is coupled to such an extent that it can be represented by equation for a single fluid. One may define the following quantities associated with such a fluid.

Plasma density	ρ	$= n(m + M)$	(58a)
Plasma momentum	ρV	$= n(mv + Mw)$	(58b)
Current density	j	$= -ne(v - w)$	(58c)
Charge density	σ	$= (n_+ - n_-)e.$	(58d)

The equations of continuity can be written as a law of conservation of mass and an equation for electric current and charge. Thus adding and subtracting eqs. (54a,b) we get

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho V) = 0, \qquad \frac{\partial \sigma}{\partial t} + \nabla \cdot j = 0. \quad (59a,b)$$

Let us now form the sum of the momentum-transport equations, i.e., $nm \times$ eq. (54c) + $nM \times$ eq. (54d). We obtain

$$\rho\left(\frac{\partial V}{\partial t} + \frac{m}{M} v \,\mathbf{grad} \cdot v + w \,\mathbf{grad} \cdot w\right) =$$
$$= \sigma E + \frac{1}{c} j \wedge B - \nabla[n(m\overline{u_e u_e} + M\overline{u_p u_p})]. \quad (60)$$

Substracting eq. (54c) from eq. (54d) both multiplied by ne, we have

$$\frac{\partial j}{\partial t} en(v \,\mathbf{grad} \cdot v - w \,\mathbf{grad} \cdot w) = e^2 n\left(\frac{1}{m} + \frac{1}{M}\right) E$$
$$+ e(v - w)\frac{\partial n}{\partial t} + e^2 n \frac{Mv + mw}{Mmc} \wedge B + e\nabla[n(\overline{u_e u_e} - \overline{u_p u_p})]. \quad (61)$$

Our assumptions are equivalent to considering a plasma which is not far from thermal equilibrium, i.e., in which $mu_e^2 \sim Mu_p^2$ and $v^2 \ll u_e^2$, $w^2 \ll u_p^2$, $u_j u_k \ll u_k^2$.

With this eqs. (60) and (61) can be simplified to

$$\rho \frac{\partial V}{\partial t} = \frac{1}{c} j \wedge B + \sigma E - \nabla p \tag{62}$$

$$\frac{\partial j}{\partial t} - j \frac{1}{n} \frac{\partial n}{\partial t} = \frac{e^2 n}{m} \left(E + \frac{1}{c} V \wedge B \right) - \frac{e}{mc} j \wedge B + \frac{e}{m} \nabla p_e. \tag{63}$$

As the flow of the electric current is given mainly by the motion of electrons having a velocity v, whereas the variation of plasma density is related to the ion motion (velocity w) we shall neglect the term $\frac{1}{n} \frac{\partial n}{\partial t}$ in comparison with $\frac{1}{j} \frac{\partial j}{\partial t}$. In cases where $p_e \sim p_i \sim \frac{1}{2} p$ we may write the last equation as

$$\frac{\partial j}{\partial t} = \frac{e^2 n}{m} \left(E + \frac{1}{c} V \wedge B \right) - \frac{e}{mc} j \wedge B + \frac{1}{2} \frac{e}{m} \nabla p. \tag{64}$$

The eq. (62) has the character of an equation of motion, whereas eq. (64) is known as the generalized Ohm's law and the terms on its right-hand side can be interpreted as electromotive force, electromotive force induced by moving through a magnetic field, the Hall electromotive force and the thermo-electric force. These are the equations of the *single-fluid model.* The surface representing $\bar{u}_i$ in velocity space often possesses central symmetry, in which case

$$\sum_i \sum_k \overline{u_i u_k} = \sum_i \sum_k \delta_{ik} \overline{u_i u_k}$$

and the terms

$$m \cdot \nabla n \overline{u_i u_k} = \frac{\partial p_{ki}}{\partial q_k} \tag{65}$$

where p_{ki} is the component of the pressure-tensor.

In many cases the energy-transport and dissipative processes may alter the pressure-tensor, in which case it is necessary to consider the energy-transport equation for the single fluid model. This is obtained by summing equations for energy transport in ion and electron gases (eq. (45)) and adding to this sum eq. (62) multiplied by V. The result is

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \nabla \cdot V \right) (3/2 p + 1/2 \rho V^2) + \nabla \cdot (Q + p_{ik} V_k) = E (j + \sigma V). \tag{66}$$

All this is not applicable to systems in which $\rho V^2 \gg p$. In these, although p may be sometimes neglected, the $\rho V \nabla \cdot V$ may often be important (e.g., centrifugal forces).

The single fluid model can be still further simplified for situations in which the inertial term may be neglected, where pressure is isotropic and where the moving particles of one fluid experience friction against the other fluid. This friction force $\boldsymbol{F}_0$ will be assumed proportional to the difference between the flow velocities of the two components. Thus

$$\begin{aligned} \boldsymbol{F}_0 &= \nu\, n \frac{Mm}{M+m} (\boldsymbol{w} - \boldsymbol{v}) \\ &= \nu \frac{m}{e} \boldsymbol{j}. \end{aligned} \tag{67}$$

Eqs. (62) and (63) become

$$\frac{1}{c} \boldsymbol{j} \wedge \boldsymbol{B} = \operatorname{grad} p \tag{68}$$

$$\nu \boldsymbol{j} = \frac{e^2 n}{m} \left(\boldsymbol{E} + \frac{1}{c} \boldsymbol{V} \wedge \boldsymbol{B} \right) - \frac{e}{mc} \boldsymbol{j} \wedge \boldsymbol{B} + \frac{e}{m} \operatorname{grad} p_e. \tag{69}$$

Substituting eq. (68) into eq. (69) one obtains

$$\nu \boldsymbol{j} = \frac{e^2 n}{m} \left(\boldsymbol{E} + \frac{1}{c} \boldsymbol{V} \wedge \boldsymbol{B} \right) - \frac{e}{m} \operatorname{grad} p_p. \tag{69a}$$

These equations are often used in analysing equilibrium configurations in a magnetic field.

In the hydrodynamical treatment of conducting fluids in a magnetic field one is often justified in putting

$$\nu = 0, \quad \operatorname{grad} p \ll \text{forces of electromagnetic origin}$$

In that case eq. (69a) reduces to

$$\boldsymbol{E} = -\frac{1}{c} \boldsymbol{V} \wedge \boldsymbol{B}. \tag{70}$$

At this point it is convenient to use Maxwell's first equation which reads:

$$\operatorname{curl} \boldsymbol{E} = -\frac{1}{c} \frac{\partial}{\partial t} \boldsymbol{B}.$$

Combining this and eq. (70) one obtains

$$\frac{\partial \boldsymbol{B}}{\partial t} = \operatorname{curl} (\boldsymbol{V} \wedge \boldsymbol{B}). \tag{71}$$

The eqs. (70) and (71) are known as the *hydromagnetic equations*, and the corresponding system of approximations as the *hydromagnetic approximation.*

More complicated fluid models than those discussed sofar can be constructed out of eqs. (56a,b), (57). One of these is a three-fluid model, the fluids being a relativistic electron gas and two components of a plasma whose flow velocity are lower than the mean thermal speed. The appropriate equations are then eq. (56) and eqs. (54) and (55).

REFERENCES TO CHAPTER 3

1) J. L. Synge, The relativistic gas (North-Holland Publ. Co. 1957).
2) C. G. Darwin, *Phil. Mag. 39*, pp. 537-559 (1920).
3) H. Margenau and G. Murphy, The mathematics of physics and chemistry, pp. 428-432 (Van Nostrand, 1943).
4) R. Tolman, The principles of statistical mechanics, pp. 43-70.
5) I. Abonyi, *Beiträge aus der Plasma Physik, 5*, p. 1 (1965).

List of symbols used in Chapter 3

A	vector potential	Q	property of particle
B	magnetic field strength	Q_i	heat-flux vector
c	velocity of light	r	radial coordinate
d	Debye distance	t	time
e	charge of electron	u	velocity
E	electric field strength	T	temperature
f, g	density in velocity space	v, w	velocity
F	force	V	velocity in phase space
$H\ (q_i\ p_i\ t)$	Hamiltonian	W	energy
i, j, k	unit vectors	x, y, z	coordinates
j	current density	$\gamma =$	$(1 - v^2/c^2)^{-1/2}$
k, κ	Boltzmann's constant	δ_{ik}	Kronecker delta
L	length	Φ	potential
m, M	mass of particle	ν_i	coefficient of friction
n	particle density	ρ	plasma density
p	momentum or pressure	λ_D	Debye wave length
p_{ai}	pressure tensor	π	volume in velocity space
q	coordinate	σ	charge density

CHAPTER 4

EQUILIBRIUM CONFIGURATIONS
(Plasmastatics)

Introduction

In this chapter we shall discuss the equilibrium of some typical plasma configurations. As in equilibrium the values for plasma density, velocity and temperature remain constant the time-variation term in eq. (3.62) drop out and the forces acting on plasma are considered. to be constant in time *. We shall use mostly the fluid models for the description of these equilibrium configurations; however, occasionnally it will be instructive to interpret the results in terms of the motion of individual particles.

The simplest example of plasma equilibrium is an isothermal plasma of infinite extension, with no macroscopic internal fields. Such a situation is often approximated in the positive column of large electrical discharges.

As soon as one considers a bounded plasma, pressures arising from gradients in plasma density and temperature appear which cause a flow to the boundaries. In order to reach an equilibrium one must either supply fresh plasma ** at the rate at which it streams away or one must balance the pressures by some system of forces. The latter is achieved on the cosmic scale by gravitational forces, and on the laboratory scale by forces of electromagnetic origin. These forces balancing the plasma pressure are often called the confining forces and if a balance is achieved the plasma is said to be confined.

The confinement by electromagnetic forces can be of four types:

1. purely electrostatic,
2. magnetostatic and electrostatic,
3. magnetostatic,
4. high frequency electromagnetic.

* This does not exclude the steady radiation pressure of alternating electromagnetic fields.

** This is connected with the problems of diffusion and will be studied in chapter 8.

Examples in the first class are rare. One usually requires a plasma in which the electron charge density is only partially neutralised by the charge of positive ions (ref. 1).

The second class contains mostly situations in which the electron gas is immobilized by a magnetic field whereas the positive ion gas is bound by an electric field to this core of the electron gas. Such confinement can be better understood by comparing the radius of gyration ρ of electrons and ions with the dimensions D of the confined plasma. The confinement is characterised by the inequality

$$\rho_e < D < \rho_p. \tag{1}$$

When the confining magnetic field is so strong that

$$\rho_e < \rho_p < D \tag{2}$$

one obtains a situation belonging to the third class. Although the inequality (2) suggests an independent magnetostatic confinement of both plasma components one cannot disregard a confinement mechanism of the second class operating in this case too. This mixture of the second and third class situations occurs when

$$d < \rho_p < D \tag{3}$$

where d is the Debye shielding distance for the positive ions. This particular situation applies to almost all plasma configurations encountered in the research on thermonuclear reactions.

In the fourth class are configurations in which the plasma pressure is balanced by radiation pressure of a high frequency electromagnetic field. As this type of confinement relies on the reflection of h.f. fields by the plasma we shall mention this problem in chapter 5, which is devoted to oscillatory phenomena in plasma.

The examples of plasma equilibria studied in this chapter are drawn mainly from classes 2 and 3 of electromagnetic confinements.

4.1. Plasma in an External Magnetic Field

When only external magnetic fields are used for the confinement of plasma one is usually not interested in confining a plasma one of whose components possesses a large stream velocity.

It thus follows that a plasma in an external magnetic field can be adequately described by means of the single-fluid model. On the other hand, owing to the anisotropic nature of the Lorentz force it can be expected that the velocity distribution of both the electron gas and the positive ion gas will be somewhat anisotropic.

The equations corresponding to this model are eqs. (3.68) and (3.69a) in which $\nu_i = 0$.

Thus

$$c \operatorname{grad} p = j \wedge B \tag{4}$$

$$E + \frac{1}{c} V \wedge B = \frac{1}{en_e} \operatorname{grad} p_p. \tag{5}$$

The field intensities E, B, the current density j, and the charge density q are related by the time independent Maxwell's equations.
Thus

$$\operatorname{curl} E = 0 \tag{6a}$$

$$\operatorname{curl} B = \frac{4\pi}{c} j \tag{6b}$$

$$\operatorname{div} E = 4\pi q = 4\pi e(n_p Z - n_e) \tag{6c}$$

$$\operatorname{div} B = 0. \tag{6d}$$

Let us consider first a plasma with an almost isotropic velocity distribution. Using eqs. (4) and (6b) we get

$$4\pi \operatorname{grad} p = (\operatorname{curl} B) \wedge B. \tag{7}$$

Equation (5) does not contain additional information on the distribution of B and p, this can be determined from the equations (7) and (6d) alone. We shall write these together as they are the basic equations for the calculations of plasma equilibria

$$\nabla B = 0 \tag{8}$$

$$\nabla B^2 - 2\, B\nabla B = -\, 8\pi\nabla p \tag{9}$$

and multiplying eq. (40) by $\wedge\, B$ we get

$$j_\perp = c\, \frac{B \wedge \operatorname{grad} p}{B^2}. \tag{10}$$

At this point one may make a fundamental observation.
It follows from eq. (7) that the pressure-gradient is perpendicular to B and j (fig. 54). This can also be expressed by saying that the magnetic field-lines and stream-lines of electric current are loci of constant plasma pressure.

In many geometries the term $B\, \nabla \cdot B = 0$ and eq. (9) can then be integrated with respect to r. One gets

$$p + \frac{B^2}{8\pi} = \text{const.} \tag{11}$$

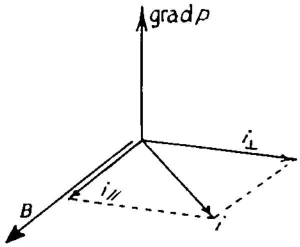


Fig. 54. Current densities in a plasma confined by a magnetic field.

This shows clearly the diamagnetic behaviour of plasma in a magnetic field; as the plasma pressure increases the magnetic field inside plasma decreases (fig. 55). Evidently $B^2/8\pi$ can be regarded as a magnetic pressure and eq. (11) represents the balance of this pressure and the pressure of plasma.

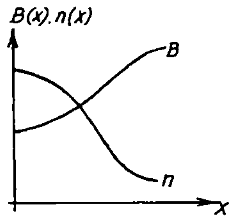


Fig. 55. Magnetic field and particle density in a confined plasma.

It is instructive to interpret the fluid description of this simple plasma - equilibrium in terms of particle orbits. Each particle will exhibit three types of motion

a) a cyclotron motion with a frequency $\omega_c = eB/mc$,
b) a drift motion due to the non-uniformity of the magnetic field,
c) a drift motion due to the presence of the radial electric force

$$E = \frac{1}{ne} \text{ grad } p_p.$$

Only the first two motions give rise to a current distribution *. The corresponding current densities can be evaluated as follows.

* When the grad p_p is large the c) drifts generate electrical currents. See p. 118.

Each gyrating particle possesses a magnetic moment (eq. (2.29)).

$$\mu = \frac{W_\perp}{B}.$$

In a plasma with a Maxwellian velocity distribution one can express

$$W_\perp = kT.$$

The magnetic moment density is then

$$M = -\frac{nkT_s}{B} = -\frac{p_s}{B} \quad \text{where } s = e, p. \tag{12}$$

The current density generated by this magnetic moment distribution is

$$i_1 = c\frac{\partial M}{\partial r}. \tag{13}$$

The drift velocity of the electrons due to inhomogeneous field is

$$v = \frac{c}{e}\frac{\dfrac{\partial B}{\partial r}}{B^2}kT \tag{14}$$

and the corresponding current density

$$i_2 = -cp\frac{\dfrac{\partial B}{\partial r}}{B^2}. \tag{15}$$

The total current density is, therefore, *

$$i = i_1 + i_2 = -\frac{c}{B}\frac{\partial p}{\partial r} \tag{16}$$

and the magnetic field follows from

$$\operatorname{curl} B \equiv -\frac{\partial B}{\partial r} = -4\pi\frac{\dfrac{\partial(p_e + p_p)}{\partial r}}{B}$$

or

$$\frac{\partial p}{\partial r} = \frac{1}{8\pi}\frac{\partial B^2}{\partial r} \tag{17}$$

which is the same as eq. (11).

* The cancellation of i_2 and the term $-c(p/B^2)(\partial B/\partial r)$ of i_1 has been analysed in detail by L. Tonks (*Phys. Rev.*, *97*, p. 1443). The term $-c(p/B^2)(\partial B/\partial r)$ is shown to represent a circulating current due to the gradient in the radius of gyration of the confined particles.

In order to examine to what extent the ions are electrostatically confined, let us make use of the, so far neglected, eq. (5).

Multiplying eq. (5) vectorially by $\boldsymbol{B}$, we have

$$\boldsymbol{V}B^2 - \boldsymbol{B}(\boldsymbol{V}\cdot\boldsymbol{B}) = c\boldsymbol{B}\wedge\left(-\boldsymbol{E} + \frac{1}{en}\,\mathrm{grad}\,p_\mathrm{p}\right)$$

giving
$$\boldsymbol{V}_\perp = c\frac{\boldsymbol{B}}{B^2}\wedge\left(-\boldsymbol{E} + \frac{1}{en}\,\mathrm{grad}\,p_\mathrm{p}\right). \tag{18}$$

For infinitely heavy ions, exercising a pressure p_i, the confinement cannot be other than electrostatic and therefore,

$$\boldsymbol{E} = \frac{1}{en}\nabla p_\mathrm{p} \tag{19}$$

in such a case $V_\perp = 0$. This field is also called the ambipolar field E_a. On the other hand, one can imagine that the magnetic field can be so strong that even ions are confined magnetically and then $\boldsymbol{E} = 0$ and

$$V_\perp = \frac{c}{enB^2}\boldsymbol{B}\wedge\nabla p_\mathrm{p}. \tag{20}$$

This formula can be rewritten using $p_\mathrm{p} = p_\mathrm{e} = nkT$, $T = \mathrm{const.}$, $v_t = \sqrt{\dfrac{2kT}{M}}$ and $\rho = \dfrac{Mv_t}{e/c\,B}$ as follows

$$V_\perp = v_t\frac{\rho}{2}\frac{\dfrac{\partial n}{\partial x}}{n}. \tag{21}$$

This can be interpreted considering two adjacent ion orbits a and b, whose plane is perpendicular to $\boldsymbol{B}$ and to ∇n (fig. 56). The centra of these orbits are 2ρ apart and consequently if the number of particles circulating in the orbit a is proportional to n, the number in orbit b is proportional to $n + 2\rho\dfrac{\partial n}{\partial x}$. Let us observe a strip of the σ plane intersected by these particles. The net flow of ions is proportional to $v_t\,\rho\dfrac{\partial n}{\partial x}$. The same argument gives the same flux for electrons, however, since electrons are much lighter than ions the net mass-flow is essentially given by the flow of ions, hence the expression for $V_\perp \neq 0$, eq. (21).

We see now that the $V_\perp$ exists only because of the gradient in density of the magnetically confined plasma.

Let us discuss now the criterion determining whether the ions are confined electrostatically or magnetically. It is evident that when the

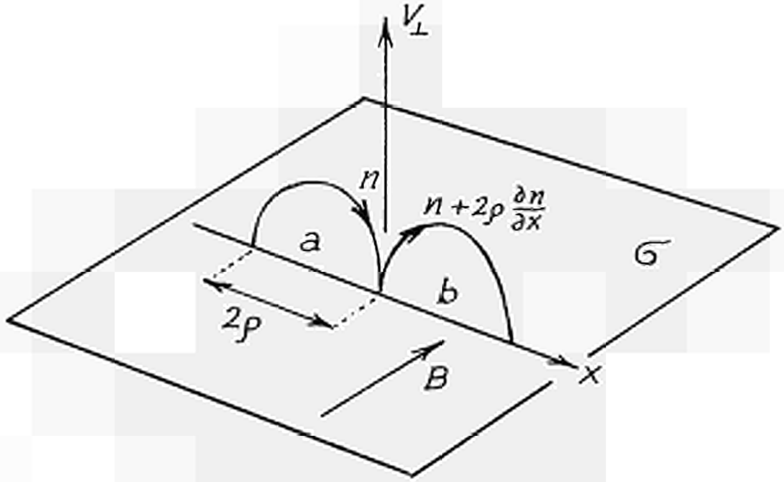


Fig. 56. Flow generated by charged particles with density gradient in a magnetic field.

ambipolar field is so strong that the ion-motion on a Larmor orbit is sensibly perturbed, the ions begin to be confined electrostatically. Thus in order that the ions are confined magnetically, the energy gained and lost from the ambipolar field along the orbit must be much smaller than the mean kinetic energy of the ion in one degree of freedom. This can be expressed as

$$2\rho_+ eE \ll \tfrac{1}{2}kT \tag{22}$$

where from eqs. (18) and (19)

$$E < E_a = \frac{kT}{en}\,\frac{\partial n}{\partial x} = \frac{kT}{e}\,\frac{1}{D} \tag{23}$$

where D is a distance characterizing the $n(x)$ distribution. The criterion (22) becomes even more stringent if we put

$$2\rho_+ eE_a \ll \tfrac{1}{2}kT. \tag{22a}$$

Substituting for the E_a we get a criterion for the magnetic confinement of the ions

$$\rho_+ \ll \tfrac{1}{4}D. \tag{24}$$

Example: Let $T = 10^5(°\mathrm{K})$, $B = 100$ (kG) then for deuterium $\rho_+ \sim 1.5$ cm and $D \gg 6$ cm (e.g., $D = 50$ cm). This shows that ions in experiments on controlled fusion are not, in most cases, confined purely magnetically.

When the criterion is not satisfield the orbit analysis of pp. 114-115 is not valid since the non-uniform distribution of n, B and E no longer permits us to assume that the drifts due to $E = \frac{1}{en}$ grad p_i do not induce electric currents. In order to see this clearly let us suppose that $\rho_+ \gg D$. In such a case a boundary layer of positive charge develops,

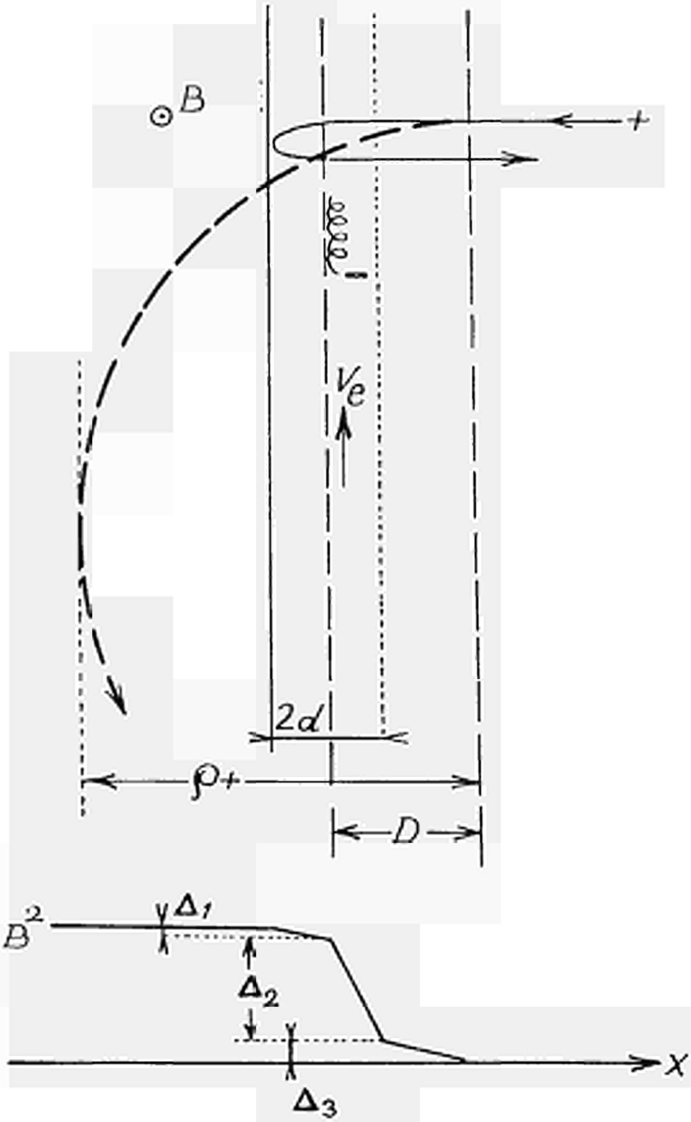


Fig. 57.

which is held to the electron current region (D thick) by the electric field E (fig. 57). This boundary layer and electric field resembles very much the situation of a Debye-layer, whose thickness is d, discussed in chapter 1, on page 8. It is clear that the E field does not cause any drift in the ion gas since its extension is only $D + d \ll \rho_+$ and our formula (2-79a) corresponds to a whole ion orbit and not to only a small section of it. If, on the other hand, $\rho_- < d$ the electrons drift with a speed $v_e = c \dfrac{E_a}{B}$ which corresponds to a Lorentz force

$$F = neDE_a \sim nkT \frac{D}{d} \tag{25}$$

using the equation (25) in which $D \sim d$, F is, therefore, equal to nkT, the pressure of the ion gas.

The criterion for an electrostatic confinement of the ions is, therefore, not simply an inversion of the inequality (22), but

$$\rho_+ \gg d \tag{26}$$

as already mentioned in the introduction to this chapter.

Even this model of the electrostatic confinement within a layer d thick is not realistic, as in such a case the v_e is equal to $c \dfrac{kT}{ed} \dfrac{1}{B_a}$ where $B_a \sim \sqrt{4\pi nkT}$ and therefore, $v_e \sim c$. In order that $v_e \ll v_{te}$ as requires the assumption of small perturbation of the cyclotron motion of the electrons, the electric field must be distributed over a thickness $D' \gg d \dfrac{c}{v_{te}} = \dfrac{c}{\omega_p}$. This thickness is also known as the collisionless skin-depth (see later, p. 153). This consideration yields, therefore, another and more precise criterion for electrostatic confinement of the ions, i.e.,

$$\rho_+ \gg \frac{c}{\omega_p}. \tag{27}$$

Finally a word on the basic equations (8) and (9).

Any magnetic field distribution satisfying eq. (7) determines a certain pressure distribution. However, inversely a pressure distribution could determine a magnetic field whose divergence does not vanish everywhere or the generation of such a field might require an additional current distribution within the plasma.

An example of the latter is a spherically symmetrical pressure distribution where

$$p = \text{const.} \quad \text{for } r = \text{const.}$$

and therefore, according to eq. (43) $B_r = 0$ and

$$4\pi \frac{\partial p}{\partial r} = -\frac{1}{2} \frac{\partial (B_\theta^2 + B_\varphi^2)}{\partial r} = \text{const.} \quad \text{for} \quad r = \text{const.}$$

This requires that $\partial B_\theta/\partial r$ and $\partial B_\varphi/\partial r$ do not vanish simultaneously on the spherical surface r = const. and therefore, the existence of a current distribution i_r in which there are at least 2 poles at which $i_r \to \infty$ (fig. 58). We shall see later that if alternating B fields are admitted a spherically symmetrical distribution of pressure can be generated.

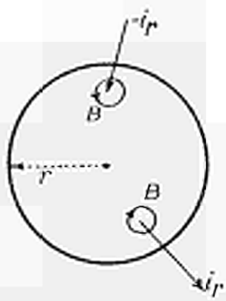


Fig. 58. Supplementary currents needed to realize a given p-distribution.

4.1.1. CYLINDRICAL AND CUSP GEOMETRY

Many complex and interesting plasma configurations can be obtained by the superposition of cylindrical and cusped magnetic fields.

Let us consider first the case of a cylindrical field produced by a current loop and an infinitely long solenoid (fig. 59). The field on the axis is

$$B_A = B_0 + B_1 f(z) \tag{28}$$

where B_0 is the uniform field of the solenoid and $B_1 = \dfrac{2\pi I}{R}$, I being the current in the loop and R the radius of the loop. Let us introduce plasma into a flux tube ϕ on the axis. If $\beta = \dfrac{8\pi p}{B^2} \ll 1$, then evidently the flux tube will change its form only slightly. Let the radius of ϕ in absence of plasma be $r(z) \ll R$ and let us assume that the region in which $\dfrac{\partial p}{\partial r} \neq 0$ is D thick and is relatively thin, i.e., that $D \ll r$.

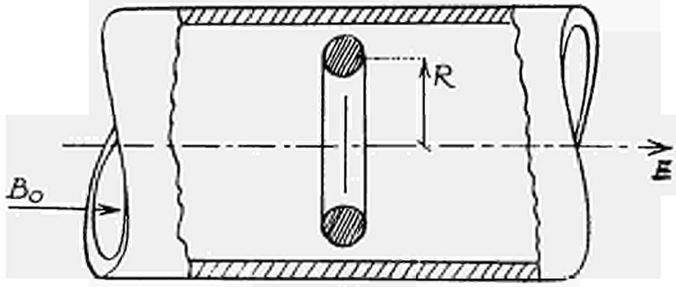


Fig. 59.

The surface current induced on the plasma tube is given by

$$\frac{1}{c} i \cdot B_A = p$$

i.e.,

$$i \simeq c \frac{p}{B}. \tag{29}$$

In equilibrium we know that p is constant along lines of force and consequently $i \propto \frac{1}{B}$.

If $r \ll R$ the field B_p corresponding to this current distribution can be approximated by that of a solenoid whose current per unit length is i. Thus

$$-B_p \simeq \frac{4\pi}{c} i = \frac{\beta}{2} B_A \tag{30}$$

or

$$B_p \sim -\frac{\beta}{2} B_0 f(z). \tag{31}$$

This is the magnetic field due to the diamagnetic behaviour of the plasma. The total field is, therefore,

$$B = B_A + B_p = B_A \left(1 - \frac{\beta}{2}\right). \tag{32}$$

Since the total flux ϕ must remain constant even after the introduction of a diamagnetic medium it follows that the flux tube must expand. Thus

$$\pi B_A r^2 = \pi B_A \left(1 - \frac{\beta}{2}\right) (r + \Delta r)^2$$

from which

$$\Delta r = \frac{r}{4}\beta. \tag{33}$$

The tube expands more the larger is its radius, i.e., the weaker is the vacuum magnetic field B_A.

Let us now consider the other extreme, i.e., $\beta = 1$. This means that a tube of plasma is able to exclude completely the vacuum field B_A. We can imagine, e.g., that the axial region is in a direct contact with a large reservoir of perfectly conducting plasma at a pressure p. Such a plasma is known as a "free-boundary" plasma. The field B must be parallel to such a boundary Σ and since $p =$ const. on Σ it follows that $B =$ const. on Σ and also that $i = 2c\dfrac{p}{B} =$ const. on Σ.

It is clear that if $B_0{}^2 < 8\pi p$ no equilibrium is possible, i.e. the plasma will completely enclose the current-loop I. However, if we assume that the B_0 field is generated by currents in a solenoid whose radius is R_0, then the flux $\pi\ R_0{}^2B_0$ is conserved within this solenoid (the flux πB_0R^2 is conserved within the loop in any case) and a plasma equilibrium becomes possible. The self-field of i is as before

$$-B_p = \frac{4\pi}{c}i = \frac{8\pi p}{B},$$

where $B = \sqrt{8\pi p} = B_p$.

The solution at $|z| \gg R$ is obtained from $B(R_0{}^2 - r_0{}^2) = B_0R_0{}^2$ and $B^2 = 8\pi p$ and it is

$$r_0 = R_0\sqrt{1 - \frac{B_0}{\sqrt{8\pi p}}}. \tag{34}$$

Near the loop, four solutions are possible. In the first and second all or none of the plasma threads the loop (fig. 60a,b), in the third a part of the plasma tube envelopes the loop and a part threads it, in the fourth all the plasma envelopes the loop (fig. 61a,b). The first and second occur when $r_0 \ll R$. The presence of the plasma-conductor in the loop does not modify very much the vacuum field and if

$$B_0 + B_1f > \sqrt{8\pi p}$$

the plasma cannot exist for $|z| < z_1$, as determined from the above relationship. Of particular interest is the configuration corresponding to $z_1 = 0$, i.e., to

$$B_0 + B_1 = \sqrt{8\pi p}$$

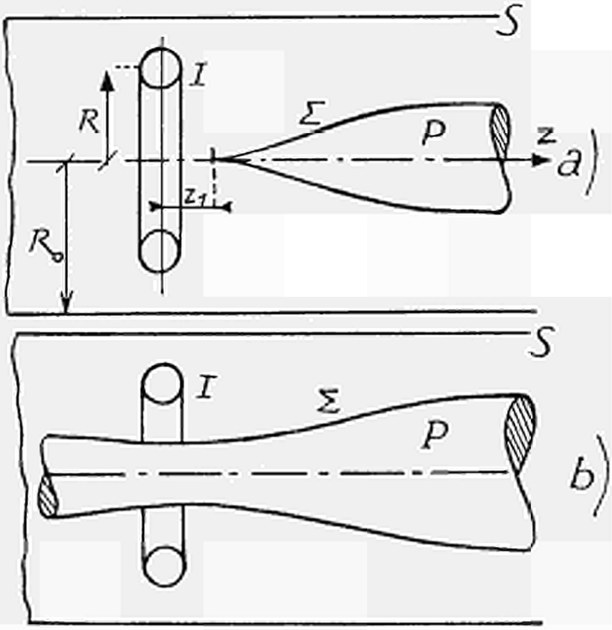


Fig. 60.

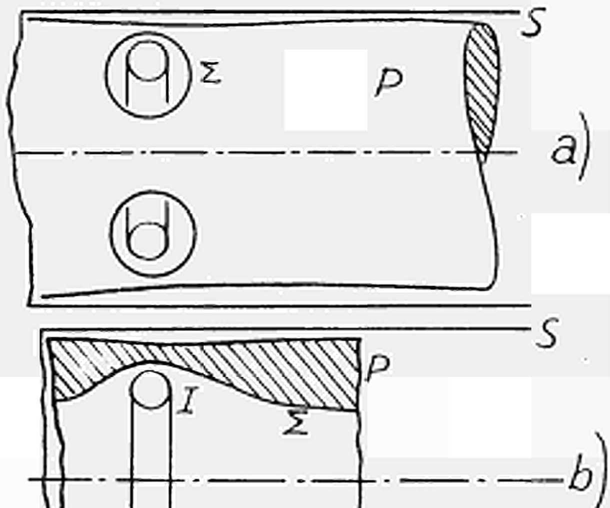


Fig. 61.

or $b = \dfrac{\sqrt{8\pi p}}{B_0}$ where $b = 1 + \dfrac{B_1}{B_0}$ is known as the mirror-ratio.

In this case the field $\boldsymbol{B}_1$ is just sufficiently strong to confine the plasma on both sides of $z = 0$ (fig. 62).

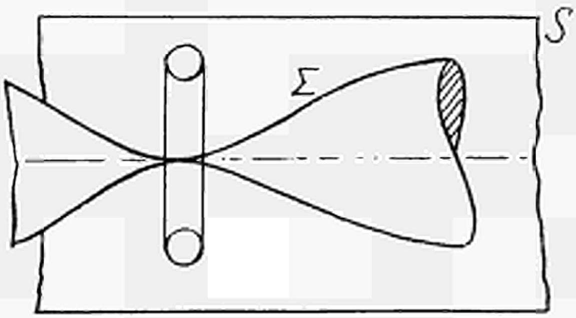


Fig. 62. Case of $z_1 = 0$

The orthogonality of the grad p lines and $\boldsymbol{B}$ lines enables us to use many well known methods of field computation, such as conformal mapping, to solve the problems of plasma equilibrium. For instance, it is possible to transform a simple or even a trivial equilibrium configuration into a more complicated one. As an example let us consider a two-dimensional quadrupole magnetic field configuration, free of plasma. Such a field is produced, e.g., by four parallel line currents, the adjacent ones flowing in opposite directions (fig. 63).

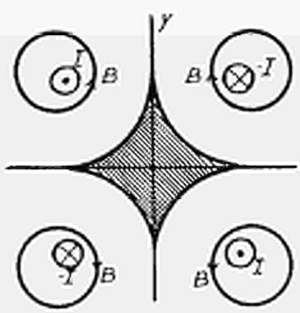


Fig. 63. Quadrupole confinement of a free surface plasma.

The vacuum magnetic field near $x = y = 0$ can be expressed as *

$$B_{x0} = bx$$
$$B_{y0} = -\, by. \tag{35}$$

Let us consider a volume of plasma of such a large conductivity that the above mentioned vacuum field is completely eliminated from its volume. The field lines $\boldsymbol{B}$ must be parallel to the plasma boundary. This gives us the first boundary condition $\boldsymbol{B} \cdot \text{grad } B = 0$ or $A \equiv A_z = \text{const.}$ where $\boldsymbol{B} = \text{curl } A$. Let us integrate eq. (7) across such a free boundary σ. One obtains

$$4\pi p = -\, \tfrac{1}{2} B^2 = \frac{4\pi}{c}\, \boldsymbol{j} \wedge \boldsymbol{B} \tag{36}$$

where p is a constant.

This is equivalent to a second boundary condition.

The other equation determining the shape of the boundary σ is the field equation $\Delta \boldsymbol{B} = 0$.

The surface currents at the boundary σ must be able to produce a field

$$B_x = -\, bx$$
$$B_y = by \tag{37}$$

in order to cancel the vacuum field inside the plasma.

It can be shown that the curve representing this boundary is a hypocycloid

$$x^{2/3} + y^{2/3} = a^{2/3}. \tag{38}$$

The family of boundaries of plasmas confined by the quadrupole field of four parallel conductors can be now obtained by conformal transformation of the hypocycloid boundary. Thus, e.g., an inversion with respect to a circle passing through the four line currents gives (fig. 64).

$$\frac{x^{2/3} + y^{2/3}}{(x^2 + y^2)^{2/3}} = \left(\frac{a}{\alpha}\right)^{2/3}. \tag{38a}$$

The cusp geometry is an example of a system in which the magnetic pressure $\dfrac{B^2}{8\pi}$ increases with the distance from the centre of the system.

* This follows from eqs. (6b and d). Thus if div $\boldsymbol{B} = 0$ and curl $\boldsymbol{B} = 0$ we have curl $\boldsymbol{B} = -\, \Delta \boldsymbol{B} + \text{grad div } \boldsymbol{B} = 0$. Therefore $\Delta \boldsymbol{B} = 0$.

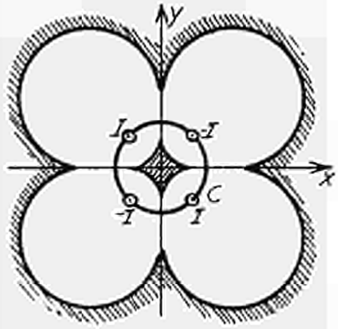


Fig. 64. Two free surface-plasmas related by inversion with respect to the circle C.

This is seen from

$$B^2 = b^2(x^2 + y^2). \tag{39}$$

It is evident that the pressure increases with both x and y.

It is possible to add a constant b_z field to the B_x, B_y field of a cusp. The magnetic pressure distribution has still a minimum on the z-axis, though this minimum is relatively more shallow than that of a purely cuspidal field. Superimposing the B_z field we have eliminated the zero-field on the z-axis. The system is known as the "stuffed cusp" and is a good example of a class of magnetic configurations called the non-zero minimum B configuration (see p. 208).

4.2. Confinement by Magnetic Fields Generated by Currents in the Plasma

4.2.1. NON-RELATIVISTIC STREAMS

Let us treat first the case of a non-relativistic axial electron flow in a cylindrical plasma.

We shall use the two-fluid model of a plasma and restrict ourselves to the case of fully ionized hydrogen of such low density that electron-proton collisions may be neglected.

The vector equations (3.54c) and (3.54d) can be written in cylindrical co-ordinates and for a current flow parallel to the z-axis as follows:

$$\frac{\partial v_z}{\partial t} + \frac{\partial v_z}{\partial r} v_r = -\frac{e}{mc}(cE_0 + cE' + v_r B), \tag{40}$$

$$\frac{\partial v_r}{\partial t} + \frac{\partial v_r}{\partial r} v_r = -\frac{e}{mc}(cE_r - v_z B) - \frac{1}{n_e m}\frac{\partial p_e}{\partial r}, \tag{41}$$

$$\frac{\partial w_z}{\partial t} + \frac{\partial w_z}{\partial r} w_r = \frac{e}{Mc}(cE_0 + cE' + w_r B), \tag{42}$$

$$\frac{\partial w_r}{\partial t} + \frac{\partial w_r}{\partial r} w_r = \frac{e}{Mc}(cE_r - w_z B) - \frac{1}{n_p M}\frac{\partial p_p}{\partial r}, \tag{43}$$

where v_z, v_r and w_z, w_r are the z- and r-components of $\boldsymbol{v}$, $\boldsymbol{w}$. E_0 is the applied electric field, E' is the self-induced electric field generated by $\partial B/\partial t$, E_r is the electric field due to a radial electron-proton space-charge separation and B is the azimuthal magnetic field of the axial electric current.

We now invoke the assumption of charge neutrality of the plasma and let

$$v_r = w_r = w,\ n_e = n_p = n.$$

This assumption results in neglecting radial oscillations of the electron gas with respect to the protons. This is justifiable as the processes of plasma contraction are slow compared with electron oscillations (chapter 5).

The radial electric field can now be eliminated from eqs. (41) and (43).

Also, owing to $m/M \ll 1$ and $w_z v_z \ll 1$, eq. (42) is of little importance. It thus follows that only two equations need be used to describe the motion of the volume element of plasma column. These are (putting $v_z = v$):

$$\frac{\partial v}{\partial t} + w\frac{\partial v}{\partial r} = -\frac{eE_0}{m} - \frac{eE'}{m} - \frac{e}{m}\frac{w}{c}B, \tag{44}$$

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial r} = \frac{e}{Mc}vB - \frac{1}{nM}\frac{\partial}{\partial r}(p_e + p_p). \tag{45}$$

The fields E' and B can be expressed by means of a vector potential A having an axial component only; hence

$$E' = -\frac{1}{c}\frac{\partial A}{\partial t}, \tag{46a}$$

$$B = -\frac{\partial A}{\partial r}. \tag{46b}$$

From Maxwell's equations one obtains the relation between A and the current density

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right)+\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2}=\frac{4\pi}{c}i. \tag{47}$$

To these equations must be added finally the equation of continuity

$$\frac{\partial n}{\partial t}+\operatorname{div} nw=0. \tag{48}$$

Substituting (46a) and (46b) in eqs. (44) and (45), expanding eq. (48) and putting $i=-nev$ we get the following system of equations:

$$\frac{\partial v}{\partial t}+w\frac{\partial v}{\partial r}=-\frac{e}{m}E_0+\frac{e}{mc}\frac{\partial A}{\partial t}+\frac{e}{mc}w\frac{\partial A}{\partial r}, \tag{44a}$$

$$\frac{\partial w}{\partial t}+w\frac{\partial w}{\partial r}=-\frac{e}{Mc}v\frac{\partial A}{\partial r}-\frac{1}{nM}\frac{\partial}{\partial r}(p_e+p_p), \tag{45a}$$

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right)+\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2}=-\frac{4\pi}{c}nev, \tag{47a}$$

$$\frac{\partial n}{\partial t}+w\frac{\partial n}{\partial r}+\frac{n}{r}\frac{\partial}{\partial r}(rw)=0. \tag{48a}$$

At this point the connexion of the pressures p_e and p_p with n, r and t must be introduced. Assuming a nearly Maxwellian distribution of random velocities in the electron and proton gases with temperatures T_e and T_p,

$$p_e+p_p=nk(T_e+T_p)=nkT. \tag{49}$$

In general, T_e and T_p as well as n may be functions of the radius r. However, in most cases of interest the conduction and convection of heat in the cross-section of the plasma column is sufficiently intense to assume that the plasma is isothermal, i.e., *

$$\frac{\partial T_e}{\partial r}\cong\frac{\partial T_p}{\partial r}\cong 0.$$

* Assuming that all changes of volume are adiabatic it can be shown that T obeys the following differential equation

$$\frac{\partial T}{\partial t}=\frac{4\pi}{3}\frac{T}{N}\int_0^R n\frac{\partial}{\partial r}(rw)\,dr, \tag{50}$$

where N is the number of electrons per cm length of the plasma column.

In principle, using eqs. (44a) to (48a) and (50), one can determine the dependent variables v, w, n, A and T as functions of the independent variables r and t.

Let us find the solution corresponding to a steady state, i.e. a case for which

$$\frac{\partial}{\partial t} = 0, \quad E_0 = 0, \quad w = 0.$$

The only equations that are not identically satisfied are eqs. (45a) and (47a), which become

$$\frac{\partial A}{\partial r} = \frac{kT}{(ev/c)} \frac{-\partial n/\partial r}{n}, \tag{51}$$

$$\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial A}{\partial r} \right) = \frac{4\pi}{c} nev. \tag{52}$$

The solution of these equations corresponding to v = const. is the well known Bennett distribution

$$n = \frac{n_0}{\{1 + r^2/r_0^2\}^2}, \tag{53}$$

where

$$\frac{e^2}{mc^2} \pi r_0^2 n_0 = \frac{kT}{\tfrac{1}{2}mv^2},$$

which can be also written as

$$\frac{e^2}{mc^2} N = \nu = \frac{kT}{\tfrac{1}{2}mv^2}, \tag{54}$$

where N is the electron line density and e^2/mc^2 is the classical electron radius.

The plasma column is evidently a very sharply defined structure. one-half of its mass being concentrated within the cylinder of radius r_0 (and 90 % of its mass located within $r = 3r_0$).

This result and eq. (53) can also be obtained by considering the Boltzmann distributions of electrons and protons in their respective rest systems, due to each other's electromagnetic fields (ref. 2).

Defining as the total current $I = \frac{e}{c} Nv$ we can also write eq. (54) as

$$I^2 = 2\,NK(T_+ + T_-) \tag{55}$$

known as the Bennett's relation.

This can be also derived directly considering a superficial current density $i = \dfrac{I}{2\pi r}$. In order that the plasma pressure $2nkT$ can be balanced by the magnetic pressure $\dfrac{B^2}{8\pi}$, where $B = \dfrac{2I}{r}$ there must be

$$nk(T_+ + T_-) = \frac{1}{2\pi}\,\frac{I^2}{r^2}$$

or

$$2\,Nk(T_+ + T_-) = I^2$$

It can be shown that eq. (55) is quite general and is valid for any distribution of n and v (ref. 3).

It is interesting to follow the trajectories of individual particles in such a stream. The character of the trajectory depends on the ratio of stored magnetic energy W_M to the kinetic energy of the electron flow W_K. When the latter predominates the electron trajectory is sinusoidal, whereas when the magnetic energy is larger than the kinetic energy, the trajectory resembles a trochoid (fig. 65).

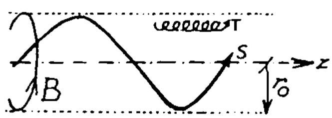


Fig. 65. Motion of electrons for $v < 1$ (S) and for $v > 1$ (T).

This criterion can be derived from the comparison of the radius of gyration ρ_e of the electrons at r_0 and of the radius r_0 of the stream. The condition corresponding to the trochoidal path is

$$\rho_e < r_0$$

which is

$$mv < \frac{e}{c}\,B_0 r_0, \qquad \text{where } B_0 = B(r_0).$$

Using eqs. (51), (53) and (54) this becomes

$$v > 1. \tag{56}$$

The smooth trajectory of fig. 65 corresponds to $v < 1$, whereas the tightly wound trochoid of corresponds to $v > 1$.

This is approximately equivalent to the above mentioned energy-condition as

$$\frac{W_M}{W_K} = \frac{\frac{1}{4}\int\limits_0^R B^2 r\,dr}{\pi m v^2 \int\limits_0^R n r\,dr}$$

$$= \frac{\nu}{\pi}\int\limits_0^R \frac{r^3}{r_0^4}\left(1+\frac{r^2}{r_0^2}\right)^{-2} dr$$

where $R \gg r_0$.

Thus

$$\frac{W_M}{W_K} = \nu\,\frac{1 + 2\ln\dfrac{R}{r_0}}{2\pi}. \tag{57}$$

The term

$$\frac{1 + 2\ln\dfrac{R}{r_0}}{2\pi}$$

in most cases does not differ from unity and therefore, eq. (56) can be also interpreted as an energy criterion.

4.2.2. RELATIVISTIC STREAMS

As a second example of plasma confined by self-fields we shall treat briefly the equilibrium of a relativistic electron-stream whose geometrical form is that of a sheet * (fig. 66). This can be done using the relativistic two-fluid equations (56a,b) and (57) in chapter 3.

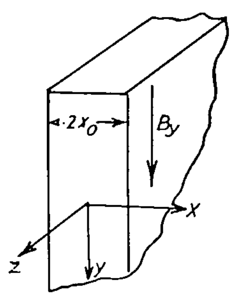


Fig. 66.

* This model resembles a section of the E layer as proposed in the Astron fusion experiment (ref. 4).

Neglecting the scattering and radiation loss and assuming that in a relativistic case $v_{\|}$ = constant, an equilibrium is possible if

$$E_{\|} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t} = 0, \; v_{\perp} = w_{\perp} = w_{\|} = 0, \quad E_{\perp} = E,$$

$$v_{\|} = v = \text{const.}, \quad B_{\perp} = B_y = B.$$

Eqs. (3-56) and (3-57) reduce to

$$eE = e\frac{v}{c}B - \frac{1}{n_e}\frac{\partial p_e}{\partial x} \tag{58}$$

$$eE = \frac{1}{n_p}\frac{\partial p_p}{\partial x} \tag{59}$$

whereas Maxwell's 1st and 3rd equations give

$$\frac{\partial B}{\partial x} = -\frac{4\pi e}{c}n_e v \tag{60}$$

$$\frac{\partial E}{\partial x} = 4\pi e(n_p - n_e). \tag{61}$$

In an isothermal and Maxwellian ensemble of particles one can put

$$\frac{\partial}{\partial x}p_e = kT_e\frac{\partial}{\partial x}n_e \tag{62a}$$

$$\frac{\partial}{\partial x}p_p = kT_p\frac{\partial}{\partial x}n_p \tag{62b}$$

where T_e and T_p are related to the mean thermal velocity in the x-direction.

From eqs. (58, (59), (60) and (62a) one obtains

$$\frac{\partial}{\partial x}\left(\frac{T_e}{T_p}\frac{\partial n_e/\partial x}{n_e} + \frac{\partial n_p/\partial x}{n_p}\right) = -\frac{4\pi e^2}{kT_p c^2}v^2 n_e \tag{63}$$

whereas from eqs. (59), (61) and (62b) one has

$$\frac{\partial}{\partial x}\left(\frac{\partial n_p/\partial x}{n_p}\right) = \frac{4\pi e^2}{kT_p}(n_p - n_e). \tag{64}$$

These simultaneous non-linear differential equations are difficult to solve. A particular solution is obtained if one assumes that (ref. 5)

$$n_p = \eta n_e \tag{65}$$

where η is a constant. In that case eqs. (63) and (64) are compatible if

$$\left(\frac{T_e}{T_p} + 1\right)(1 - \eta) = \beta^2. \tag{66}$$

However, this set of particular solutions does not include some of the more realistic density distributions. We shall treat two cases which are of interest here

a) $T_e \gg T_p \qquad (T_p \approx 0)$

b) $T_p \gg T_e \qquad (T_e \approx 0)$.

a) In this case eq. (63) simplifies to

$$\frac{\partial}{\partial x}\left(\frac{\partial n_e/\partial x}{n_e}\right) = -\frac{4\pi e^2\beta^2}{kT_e} n_e \tag{67}$$

whose solution is (fig. 67):

$$n_e = \frac{n_{0e}}{(\cosh x/x_0)^2} \tag{68}$$

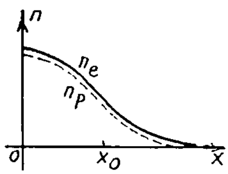


Fig. 67. Density distribution in a sheet beam for $T_e \gg T_p$.

where

$$n_{0e}x_0^2 = \frac{1}{2\pi}\frac{kT_e}{e^2\beta^2} \tag{69}$$

which is the pinch relation for a sheet-beam. It then follows from eq. (64) that

$$n_p = \frac{n_{0p}}{(\cosh x/x_0)^2} \tag{70}$$

and

$$n_{0p} - n_{0e} \to 0 \text{ as } T_p \to 0.$$

This leads us to a model of a quasi-neutral relativistic plasma; we shall call this configuration a mixed beam model.

b) In this case the electromagnetic forces balance the transversal pressure of the positive ions only.

Eq. 63) gives

$$\frac{\partial}{\partial x}\left(\frac{\partial n_p/\partial x}{n_p}\right) = -\frac{4\pi e^2\beta^2}{kT_p}\,n_e \tag{71}$$

whereas eq. (64) is written as

$$\frac{\partial}{\partial x}\left(\frac{\partial n_p/\partial x}{n_p}\right) = \frac{4\pi e^2}{kT_p}\,(n_p - n_e). \tag{72}$$

A solution satisfying both equations is possible only if

$$-\,n_e\beta^2 = n_p - n_e \tag{73}$$

i.e., if

$$n_e = n_p\gamma^2 = \frac{n_{0e}}{(\cosh x/x_0)^2}. \tag{73a}$$

If such a solution were to hold for the whole of the space occupied by the beam, the linear densities of electrons and positives would have to be also in the ratio γ^2, and the solution would be of the type $n_p = \eta n_e$ mentioned above. This corresponds to a non-physical situation, as the stored electric energy tends to infinity.

However, if one postulates an overall neutrality of the stream the electron distribution becomes cut-off at a certain distance x_1. For $x > x_1$ there is only a positive ion atmosphere confined by the electric field of the cold electron core (fig. 68). Eqs. (71) and (72) give for $x < x_1$

$$n_e = n_p\gamma^2 = \frac{n_{0e}}{(\cosh x/x_0)^2}$$

whereas for $x > x_1$ one has to consider only eq. (72) with $n_e = 0$. This gives

$$n_p = \frac{n_1}{\left(1 + \dfrac{x - x_1}{x^2}\right)^2} \tag{74}$$

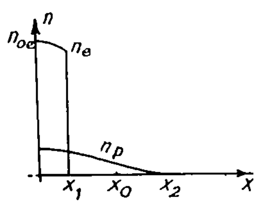


Fig. 68. Density distribution in a sheet beam for $T_e \ll T_p$.

where

$$x_2 = \frac{kT}{4\pi e^2 n_{0e}(1 - \gamma^{-2}) x_0 \tanh \dfrac{x_1}{x_0}}$$

$$n_1 = \gamma^{-2} \frac{n_{0e}}{(\cosh x_1/x_0)^2}\,.$$

Matching the n_p and $\partial n_p/\partial x$ distributions at $x = x_1$ one obtains for $\gamma \gg 1$

$$x_1 = x_0 \gamma^{-1}, \qquad x_2 \cong x_0 \gamma \tag{75a, b}$$

and a pinch condition

$$x_2 \nu = \frac{1}{\pi} \frac{kT_p}{mv^2} \tag{76}$$

where

$$\nu = 2 \frac{e^2}{mc^2} \int_0^{x_1} \frac{n_{0e}\,dx}{(\cosh x/x_0)^2}.$$

This equilibrium configuration will be called the cold electron-core model.

4.3. Plasma Equilibrium in External and Self-Fields

4.3.1. STABILIZED Z-PINCH

A plasma column confined by the B_φ field of a current passing parallel to the axis of the column is, generally, not a stable configuration as will be shown in chapter 5. In order to improve the stability an axial magnetic field B_z can be superimposed. Let us discuss the equilibrium of a plasma in such a combination of self- and external fields.

The nonrelativistic case can be treated using the one-fluid model. The corresponding equations are

$$\frac{\partial p}{\partial r} = \frac{1}{8\pi} \frac{\partial}{\partial r} (B_\varphi^2 + B_z^2) \tag{77}$$

$$\frac{\partial B_\varphi}{\partial \varphi} = \frac{\partial B_z}{\partial z} = 0. \tag{78}$$

It is, therefore, possible to choose such functions $B_\varphi(r)$, $B_z(r)$ to which corresponds a $p(r)$ in equilibrium.

Let us first consider the case of superficial currents. In that case (fig. 69) $B_\varphi^2 + B_z^2 - B_1^2 = 8\pi p$, where B_2 and B_1 are the B_z fields outside and inside the plasma column.

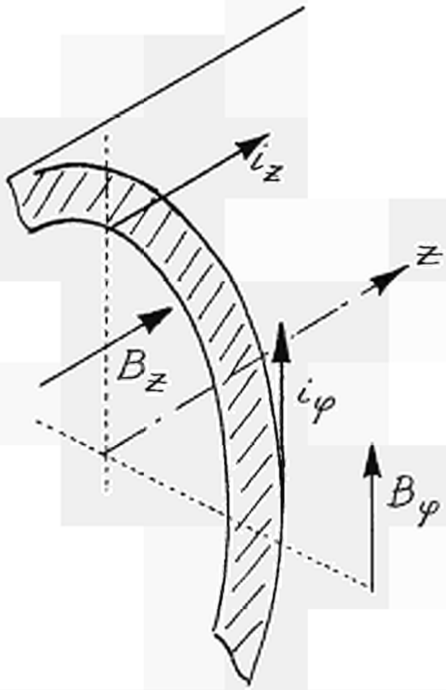


Fig. 69. Stablized Z-pinch with surface currents.

In reality, owing to the finite conductivity of the plasma, the currents i_z and i_φ spread radially and the distribution of B_φ and B_z is not discontinuous.

The situation becomes somewhat complicated if the plasma exhibits different conductivities $\sigma_\parallel$ and $\sigma_\perp$ in the direction parallel and perpendicular to the magnetic fields. Let us assume, as an example, that the plasma pressure is negligible and that current can flow only parallel to the lines of magnetic field $\boldsymbol{B} = \boldsymbol{B}_\varphi + \boldsymbol{B}_z$. Consequently

$$\frac{i_\varphi}{i_z} = \frac{B_\varphi}{B_z} \tag{79}$$

where i_φ and i_z, the current-densities follow from Maxwell's 2nd equation:

$$i_z = \frac{c}{4\pi}\frac{1}{r}\frac{\partial}{\partial r}(B_\varphi r), \quad i_\varphi = -\frac{c}{4\pi}\frac{\partial B_z}{\partial r}. \tag{80a, b}$$

The eq. (79) expresses also the pressure balance and is equivalent to eqs. (77) and (78) when $\frac{\partial p}{\partial r} = 0$ (a special case of force-free fields which will be discussed later on p. 140).

Taking mean values of B_φ and B_z, e.g.,

$$\langle B_\varphi \rangle = \tfrac{1}{2}B_\varphi, \quad \langle B_z \rangle = B_1 + \tfrac{1}{2}\Delta B = B \tag{81a,b}$$

where

$$\Delta B = B_2 - B_1$$

we get

$$\Delta B = \frac{B_\varphi^2}{2B}. \tag{82}$$

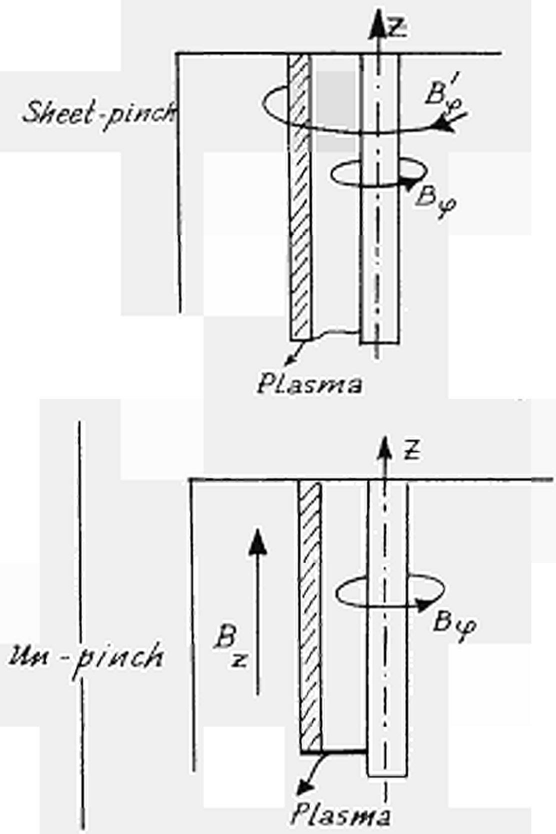


Fig. 70. The hard-core configurations.

This intensification of the B_z field occurs even if the conductivity does not assume such extreme values as considered here and the effect is known as the paramagnetic behaviour of a plasma (ref. 6).

The geometry of stabilized pinches can be somewhat complicated by using a central conductor whose B_φ field is capable to create a cylindrical hole in a plasma. Such pinches are called hard-core or tubular pinches. A separate magnetic field is then required to balance the pressure of the plasma on the outer surface of the plasma tube. This can be achieved by either B_φ or B_z field (fig. 70). In the first case the configuration is known as triax or sheet-pinch, in the second case a nome of un-pinch, has been used (ref. 7, 8). It is evident that in equilibrium

$$\frac{B^2_{\text{out}}}{8\pi} = p = \frac{B^2_{\text{in}}}{8\pi} \tag{83}$$

whatever is the geometry of the inside and outside fields.

4.3.2. TOROIDAL PLASMA LOOP

Let us discuss briefly the equilibrium of a plasma filament in which the plasma pressure is balanced by the constriction force of the magnetic field of a current flowing in the plasma, the position of the filament being determined by the force of an externally generated magnetic field on the plasma current. Such a model is usually called the string model.

The most interesting application of the string model is to the toroidal pinch configuration. Let us consider a current carrying plasma beam forming a circle of radius R and whose cross-section is nearly circular with a radius r_0 (fig. 71).

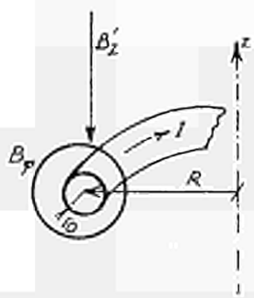


Fig. 71. Field configuration of a current loop in equilibrium.

When

$$\frac{R}{r_0} \gg 1$$

the pressure and the current distribution in the cross-section of the beam are of the Bennett type and are independent of R and the self-magnetic field at $r < 2r_0$ can be represented as a sum of a Bennett field distribution B_φ and an axial component B_z

$$B_z \ll B_\varphi .$$

This axial field gives rise to a loop expanding force

$$F_M = 2\pi B_z IR \tag{84}$$

where I is the total current in the plasma filament and F_M is evaluated over the whole periphery of the loop.

The magnitude of this force can be calculated from the theorem of virtual displacement. Thus if the stored energy of the magnetic field of current I is $W = \frac{1}{2}LI^2$ where L is the corresponding self-inductance which is $L = 4\pi R(\ln (8R/r_0) + \frac{1}{4})$ (henrys), the theorem of virtual displacement asserts that

$$F\delta R = -\delta W$$

or

$$F = -\frac{\partial W}{\partial R}.$$

As only L is explicitely dependent on R one gets

$$F_M = -2\pi I^2 \left(\ln \frac{8R}{r_0} + 1.25 \right) \tag{84a}$$

The force F_M is not the only force tending to expand the plasma filament. One should take into account also the centrifugal force F_c of the electrons and positive ions. For a hydrogen plasma

$$F_c = \left(Nm \frac{v^2}{R} + NM \frac{w^2}{R} \right) 2\pi R \tag{85}$$

where N is the linear density of electrons in the filament, and $m = \gamma m_0$ is the electronic mass. It can be shown that in most cases

$$mv \approx Mw$$

and, therefore,

$$F_c \cong 2\pi N m_0 \gamma v^2. \tag{85a}$$

In order to compare his force with the hoop force F_M let us express the current I in terms of N and v.

Eq. (84a) can then be written as

$$F_{\mathrm{M}} = -2\pi \frac{e^2}{c^2} N^2 v^2 \left(\ln \frac{8R}{r_0} + 1.25 \right).$$

Let us form a ratio

$$\frac{F_{\mathrm{M}}}{F_{\mathrm{c}}} = \frac{\frac{c^2}{e^2} N \left(\ln \frac{8R}{r_0} + 1.25 \right)}{m_0 \gamma}. \tag{86}$$

It is useful, when dealing with plasma filaments carrying electric currents, to define a new line density

$$\nu = \frac{e^2}{m_0 c^2} N.$$

This is the number of electrons per e^2/m_0c^2 length of the filament, where e^2/m_0c^2 is the classical radius of an electron. Using this quantity, it follows from eq. (86) that the force F_{M} due to the self-field of the current loop predominates over the centrifugal force F_{c} of the circulating electrons if

$$\frac{\nu \left(\ln \frac{8R}{r_0} + 1.25 \right)}{\gamma} > 1. \tag{87}$$

In order to obtain an equilibrium it is necessary to balance this expanding force. This can be achieved by using an external axial magnetic field opposite to B_z. This compensating field is, therefore

$$B'_z = \frac{I}{\nu R} \left[\nu \left(\ln \frac{8R}{r_0} + 1.25 \right) + \gamma \right]. \tag{88}$$

The proportionality of B'_z to I suggests that the field B_z' should be produced by induction. Two induction mechanisms can be considered

a) A betatron type of induction, in which a primary current $I_1 \propto I \propto B'_z$.

b) Image currents. In this case the current I induces image currents I' on a suitably shaped conducting surface. Again $I \propto I' \propto B'_z$.

4.3.3. FORCE-FREE MAGNETIC FIELDS

An equilibrium of hot plasma in a magnetic field generally requires a force distribution

$$\boldsymbol{f} = \boldsymbol{i} \wedge \boldsymbol{B}$$

in order to balance a pressure gradient distribution

$$f' = -f = \text{grad}\, p.$$

If in addition to a set of currents $\boldsymbol{i}_\perp$ there exists a set of currents $\boldsymbol{i}_\parallel$

$$\boldsymbol{i}_\parallel = \frac{(\boldsymbol{i} \cdot \boldsymbol{B})\, \boldsymbol{B}}{B^2} \tag{89}$$

the force distribution will remain unchanged.

Let us now consider a cold plasma, or any other conductor in which $p = 0$ everywhere. In such a medium a magnetic field could be generated by a system of $i_\parallel$ currents in a force-free manner. The differential equations which such a field obeys follows from eqs. (7) and (6b), i.e.,

$$\text{curl}\, \boldsymbol{B} \wedge \boldsymbol{B} = 0 \tag{90}$$

$$\text{div}\, \boldsymbol{B} = 0. \tag{91}$$

Let us discuss a distribution having cylindrical symmetry and in which $\partial/\partial\varphi = \partial/\partial z = 0$. In this case only fields whose radial component is zero everywhere satisfy eq. (91), and eq. (90) can be written as

$$r^2 \frac{\mathrm{d}B_z^2}{\mathrm{d}r} + \frac{\mathrm{d}(r^2 B_\varphi^2)}{\mathrm{d}r} = 0. \tag{90a}$$

A trivial solution is, evidently, a purely axial homogeneous field

$$B_z = \text{const.}, \quad B_\varphi = 0.$$

Non-trivial solutions would be those in which the energy density $W = \frac{1}{8\pi}(B_z^2 + B_\varphi^2)$ could be prescribed and the individual B_φ and B_z would follow from eq. (90a). Substituting W into eq. (90a) we get

$$B_\varphi^2 = -\tfrac{1}{2} r \frac{\mathrm{d}W}{\mathrm{d}r} \tag{92}$$

$$B_z^2 = \tfrac{1}{2} r \frac{\mathrm{d}W}{\mathrm{d}r} + W. \tag{93}$$

Although these equations may yield non-trivial solutions, such solutions can be shown to correspond to non-physical configurations. By non-physical one means that either the field B or the total stored energy

$$W_0 = 2\pi \int_0^\infty W r \,\mathrm{d}r$$

or some other component of Maxwell's stress tensor is infinite (ref. 9).

Notwithstanding the non-physical character of force-free fields they are of interest as physically possible magnetic fields may be often approximately force-free (ref. 10).

A typical example of force-free configurations is a thin tubular conductor in which the two component of surface current-density are equal, i.e.,

$$i_z = i_\varphi$$

in which case $B_z = B_\varphi$ taken at the surface of the tube, which implies a direct balance of magnetic pressures. In order that the conductor is not subjected to any pressure, the vector of current density $\boldsymbol{i}$ must be always parallel to $\boldsymbol{B}$; thus on the outer surface $i = i_\varphi$ whereas on the inside surface $i = i_z$ (configuration similar to that of fig. 69).

REFERENCES TO CHAPTER 4

1) W. Paul *et al.*, *Z. für Phys.*, ***152***, pp. 143-182 (1958).
   W. C. Elmore *et al.*, *Ph. of Fluids*, ***2***, p. 239 (1959).
2) W. H. Bennett, *Phys. Rev.*, ***45***, p. 89 (1934).
3) E. Fünfer and G. Lehner, Ergebnisse der exakten Naturwissenschaften, ***34***, p. 26, Springer Verlag 1962.
4) J. G. Linhart and A. Schoch, *Nucl. Instr. and Methods*, ***4***, p. 332 (1959).
5) R. V. Polovin and N. L. Tsintsadze, *Jour. Tech. Fiziky*, ***2***, p. 1354 (1957).
6) I. N. Golovin *et al.*, Proc. Geneva Conf. 1959, vol. ***32***, p. 80, U.N., N.Y.
7) O. A. Anderson *et al.*, Proc. Geneva conf. 1958, vol. ***32***, p. 150, U.N., N.Y.
8) O. A. Anderson *et al.*, *Phys. of Fluids*, ***1***, p. 489 (1958).
9) A. Schlüter, *Z. f. Naturfors.*, ***12a***, pp. 855-859 (1957).

List of symbols used in Chapter 4

A, $\boldsymbol{A}$	vector potential	p	pressure
B, $\boldsymbol{B}$	magnetic field strength	q	charge density
c	velocity of light	r	radial coordinate
d	Debye distance	R	radius
D	length	t	time
e	charge of electron	T	temperature
E, $\boldsymbol{E}$	electric field strength	u, v, w	velocity
f	force density	W	kinetic energy
F	force	x, y, z	cartesian coordinates
i, j	current density	Z	atomic number
I	current	$\beta =$	v/c
k	Boltzmann's constant	γ	$(1 - v^2/c^2)^{-1/2}$
L	self-inductance	μ	magnetic moment
m, M	particle mass	$\nu =$	e^2N/mc^2 normalized line-density
M	magnetic moment density		
n	number density	ρ	radius of gyration
N	linear number density	ω_p	plasma frequency

CHAPTER 5

WAVES AND INSTABILITIES IN PLASMA

Introduction

The total energy content of a plasma configuration is made up of four components.

1. The thermal energy, i.e, the energy of random motion. The density of this energy is

$$W_{\mathrm{T}} = \tfrac{1}{2} n_{\mathrm{e}} m \overline{u_{\mathrm{e}}^2} + \tfrac{1}{2} n_{\mathrm{p}} M \overline{u_{\mathrm{p}}^2}.$$

2. The kinetic energy of ordered motion, whose density is *

$$W_{\mathrm{K}} = \tfrac{1}{2} \sum_i n_{\mathrm{e}i} m v_i^2 + \tfrac{1}{2} \sum_j n_{\mathrm{p}j} M v_j^2$$

where i is the number of separate electron streams and j is the number of positive ion streams.

3. Stored energy of electric fields in plasma. The energy-density is $W_{\mathrm{E}} = E^2/8\pi$.

4. Stored energy of magnetic fields in plasma. The density of this energy is $W_{\mathrm{M}} = B^2/8\pi$.

All these four energy reservoirs are mutually coupled through the motion and distribution of plasma particles and by their electromagnetic fields. Such a coupling can give rise to either an oscillatory or a unidirectional exchange of energy between these reservoirs (fig. 72). The equations of motion and Maxwell's equations suggest many such conversion possibilities.

This chapter is devoted to such conversion processes and is divided into three main sections. In the first we shall study oscillatory processes in a plasma in which the motion of the electron gas plays the major rôle. The second section is devoted to oscillatory processes in which the motion of positive ions is of importance. The third section

* In a multi-stream plasma the distinction between the energy of random motion and the kinetic energy is not always very clear especially in turbulent plasmas.

treats the case where the energy conversion proceeds in one direction only. Such a process is known as plasma instability.

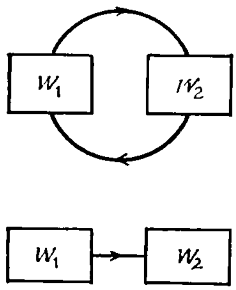


Fig. 72. Energy exchange diagrammes.

5.1. Electron Oscillations in Plasma

High frequency waves in free space can be represented by a wave equation derived from Maxwell's equations

$$\text{curl curl } \boldsymbol{E} + \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0. \tag{1}$$

In such a wave electric energy is transformed into magnetic energy during each half a cycle according to the equation

$$\text{curl } \boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}. \tag{2}$$

In a plasma, the waves induce oscillating currents and eq. (2) must be modified to

$$\text{curl } \boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} + 4\pi \frac{e}{c}(n_p Z \boldsymbol{w} - n_e \boldsymbol{v}) \tag{2a}$$

where $\boldsymbol{w}$ and $\boldsymbol{v}$ follow from the equations of motion of the two-fluid model of plasma.

It can be appreciated that the greatest contribution to the term on the r.h.s. of eq. (2a) comes from the electron motion and that the positives, owing to their large mass, can be considered in most cases as a stationary positive gas.

From the same equation it follows that W_E is transformed into W_M and into the kinetic energy of the electrons W_{Ke}.

The density of the kinetic energy of electrons W_{Ke} becomes comparable to W_{E} and W_{M} when $\dfrac{4\pi}{c}\, i = -\, 4\pi\, \dfrac{e}{c}\, nv$ becomes equal to the displacement flux $(1/c)\ (\partial E/\partial t)$.

For a harmonic oscillation with angular frequency ω the equation of motion for the electron gas becomes

$$m\mathrm{j}\omega \boldsymbol{v} = -\, e\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \wedge \boldsymbol{B}\right). \tag{3}$$

In cases in which $v/c \ll 1$,

$$v \cong j\, \frac{eE}{m\omega} \tag{4}$$

and it follows, that only when

$$4\pi\, \frac{e^2}{mc}\, \frac{nE}{\omega} \ll \frac{\omega}{c}\, E \tag{5}$$

can one consider the effect of a plasma on the electromagnetic wave as a small perturbation. This criterion can be also written as

$$\omega^2 \gg \frac{4\pi e^2 n}{m} = \omega_{\mathrm{p}}{}^2 \tag{5a}$$

where ω_{p} is a characteristic frequency of the electron component of plasma. The value of ω_{p} indicates also the range of the frequency spectrum in which we shall be interested in our analysis of electron oscillations in plasma.

We shall treat first of all an infinite and uniform plasma in which the random velocity of the electrons is zero. However, the presence of a magnetostatic field will be admitted.

The basic equations are:
the wave equation

$$\text{curl curl}\ \boldsymbol{E} + \frac{1}{c^2}\, \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 4\pi\, \frac{e}{c^2}\, \frac{\partial}{\partial t}\, (n_{\mathrm{e}} \boldsymbol{v}) \tag{6}$$

the divergence equation

$$\text{div}\ \boldsymbol{E} = 4\pi e(Z_{\mathrm{p}} n_{\mathrm{p}} - n_{\mathrm{e}}) \tag{7}$$

the equation of motion (eq. (3.57))

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\ \mathbf{grad} \cdot \boldsymbol{v} = -\frac{\mathrm{e}}{m}\left[\boldsymbol{E} + \frac{1}{c}\, \boldsymbol{v} \wedge (\boldsymbol{B} + \boldsymbol{B}_0)\right] \tag{8}$$

and the equation of continuity

$$-\frac{\partial n_e}{\partial t} = \text{div}\ (n_e \boldsymbol{v}). \tag{9}$$

In the presence of a strong magnetostatic field $\boldsymbol{B}_0$ the term $\boldsymbol{v} \wedge \boldsymbol{B}_0$ must be retained. However, as in most cases

$$|B| \leq |E| \text{ and } v \ll c,$$

the term $\boldsymbol{v} \wedge \boldsymbol{B}$ will be neglected.

If the amplitude of oscillations is small, one may consider the time variation to be of the form $e^{j\omega t}$. This covers many cases of practical interests. Eqs. (6)-(9) become

$$\text{curl curl}\ \boldsymbol{E} - \frac{\omega^2}{c^2}\boldsymbol{E} = 4\pi j \frac{e}{c^2}\omega n_0 \boldsymbol{v} \tag{10}$$

$$\text{div}\ \boldsymbol{E} = 4\pi e n_1, \tag{11}$$

$$j\omega \boldsymbol{v} = -\frac{e}{m}\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \wedge \boldsymbol{B}_0\right) \tag{12}$$

(the same as eq. (3))

$$j\omega n_1 = n_0\ \text{div}\ \boldsymbol{v} \tag{13}$$

where we have put

$$n_e = n_0 + n_1 e^{j\omega t} \tag{14}$$

and according to the small amplitude model, $n_1 \ll n_0$.

The high frequency-component of the current density in the rest system of the electron gas is

$$\boldsymbol{i} = -e n_0 \boldsymbol{v}. \tag{15}$$

Eqs. (10)-(13) can now be written in terms of electromagnetic quantities only

$$\text{curl curl}\ \boldsymbol{E} - \frac{\omega^2}{c^2}\boldsymbol{E} = -\frac{4\pi}{c^2} j\omega \boldsymbol{i} \tag{10a}$$

$$-\text{div}\ \boldsymbol{E} = 4\pi \frac{\text{div}\ \boldsymbol{i}}{j\omega} \tag{11a}$$

$$j\omega \boldsymbol{i} = \frac{e^2 n_0}{m}\boldsymbol{E} - \frac{e}{mc}\boldsymbol{i} \wedge \boldsymbol{B}_0. \tag{12a}$$

The form of these equations suggests that the field $\boldsymbol{E}$ is associated with two types of current i_1 and i_2, one which is solenoidal, i.e.,

$$\text{div}\ \boldsymbol{i}_1 = 0,\ \text{curl}\ \boldsymbol{i}_1 \neq 0$$

and the other one which is irrotational, thus

$$\operatorname{div} \boldsymbol{i}_2 \neq 0, \operatorname{curl} \boldsymbol{i}_2 = 0.$$

The first type of current distribution does not exhibit any accumulation of charge and represents therefore, a purely transversal wave, whereas the i_2-distribution belongs to a longitudinal oscillation. These two types of waves may be coupled through the magnetic field $\boldsymbol{B}_0$ and form a hybrid wave.

As eqs. (10a)-(12a) are linear one may apply the superposition principle for their solutions. Thus a complex solution can be constructed out of simple waves having a harmonic variation along a certain direction, called the direction of propagation. If this direction is parallel to the z axis, all the oscillating quantities vary as

$$e^{j(\omega t + kz)}. \tag{16}$$

Using these assumptions we shall investigate the propagation of elementary longitudinal, transversal and hybrid electron waves in an unbounded uniform cold plasma and give two simple examples of electron waves in a bounded and cold plasma.

5.1.1. THE LONGITUDINAL OSCILLATIONS

Eqs. (11a) and (12a) give

$$E_z = -\frac{4\pi}{j\omega} i_z, \quad j\omega i_z = \frac{e^2 n_0}{m} E_z \tag{17a,b}$$

which determine a characteristic frequency

$$\omega = \omega_p \tag{18}$$

for which such oscillations are possible. The wave number k does not enter into the analysis and one may thus conclude that waves or oscillatory disturbances of the longitudinal type of any dimension can exist, always oscillating with the angular frequency ω_p (see hot plasma later in 5.2.1).

5.1.2. THE TRANSVERSAL OSCILLATIONS

The only electric field components are E_x, E_y. Let us analyse two cases: $\boldsymbol{B}_0 \,//\, \boldsymbol{k}$ and $\boldsymbol{B}_0 \perp \boldsymbol{k}$.

a) Eqs. (10a), (12a) corresponding to the case $\boldsymbol{B}_0 \,//\, \boldsymbol{k}$ (fig. 73) have the following form

$$\left(k^2 - \frac{\omega^2}{c^2}\right) E_x = -\frac{4\pi}{c^2} \mathrm{j}\omega i_x \tag{19}$$

$$\left(k^2 - \frac{\omega^2}{c^2}\right) E_y = -\frac{4\pi}{c^2} \mathrm{j}\omega i_y \tag{20}$$

$$\mathrm{j}\omega i_x = \frac{\omega_p^2}{4\pi} E_x - \omega_c i_y \tag{21}$$

$$\mathrm{j}\omega i_y = \frac{\omega_p^2}{4\pi} E_y - \omega_c i_x \tag{22}$$

where

$$\omega_c = \frac{eB_0}{mc}.$$

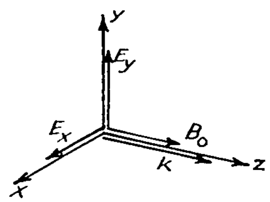


Fig. 73. Vector diagramme for a tranversal wave with $B_0 \,/\!/\, k$.

In order that this system of homogeneous linear equations shall have a solution, its determinant D must be zero. This condition is a relation between k and ω and therefore, a dispersion relation governing the propagation of electromagnetic waves parallel to B_0.

One gets

$$D \equiv \left[\left(k^2 - \frac{\omega^2}{c^2}\right) - \frac{\omega_p^2}{c^2}\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1}\right]^2 - \frac{\omega_p^4 \omega_c^2}{\omega^2 c^4}\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-2} = 0$$

from which

$$k^2_{1,2} = \frac{\omega^2}{c^2}\left[1 - \frac{\omega_p^2}{\omega^2}\left(1 \pm \frac{\omega_c}{\omega}\right)^{-1}\right]. \tag{23}$$

The phase velocity of the wave is

$$v_p = \frac{\omega}{k}$$

for which one obtains

$$v^2_{p1,2} = \frac{c^2}{1 - \frac{\omega_p^2}{\omega^2}\left(1 \pm \frac{\omega_c}{\omega}\right)^{-1}} . \tag{23a}$$

The function $v_p(\omega)$ is plotted in fig. 74 for several values of ω_c/ω_p. Substituting the dispersion relation (23) back into eqs. (19) and (20) one finds that the minus sign in eq. (23) corresponds to a ratio

$$\frac{E_x}{E_y} = + j$$

i.e., to a circularly polarized wave whose electric vector rotates in the same sense as the gyrating electrons, whereas the plus sign corresponds to $E_x/E_y = - j$, i.e., to a circularly polarized wave whose E-vector rotates in the opposite sense.

It is evident, therefore, that a plane polarized wave incident on a slab of plasma in a parallel magnetic field will be split into two circularly polarized waves, the extraordinary and the ordinary wave, each propagating with a different phase velocity. After passing through the slab of thickness z the plane of polarization of the original incident wave will be rotated by an angle

$$\psi = 2\pi\left(\frac{z}{\lambda_1} - \frac{z}{\lambda_2}\right) = z(k_1 - k_2) \tag{24}$$

which is analogous to the Faraday effect in crystals.

The formula (23a) can be applied only to that part of the frequency spectrum for which

$$\omega \gg \omega_{\text{ions}} \tag{25}$$

where

$$\omega^2_{\text{ions}} = \frac{4\pi e^2 Z^2 n_p}{M} .$$

For ω comparable to ω_{ion} the electron wave is coupled to an ion hydromagnetic wave *.

Let us plot $v_p(\omega)$ for several values of ω_c/ω_p. There are two branches to each curve, the v_p tending to infinity for

$$1 \pm \frac{\omega_c}{\omega} = \frac{\omega_p^2}{\omega^2} .$$

The left branches are not complete because of the criterion (25).

* See section 5.2.2.

Of special interest is the case of $B_0 = 0$, i.e. $\omega_c/\omega_p = 0$. Here the dispersion relation simplifies to *

$$v_p^2 = \frac{c^2}{1 - \frac{\omega_p^2}{\omega^2}} \tag{26}$$

which corresponds to the curve 0 in fig. 74.

b) Purely transversal waves for which $\boldsymbol{B}_0 \perp \boldsymbol{k}$ can exist only if $\boldsymbol{E} \parallel \boldsymbol{B}_0$ (fig. 75). In this case $\boldsymbol{i} \parallel \boldsymbol{B}$ and the magnetic field has no

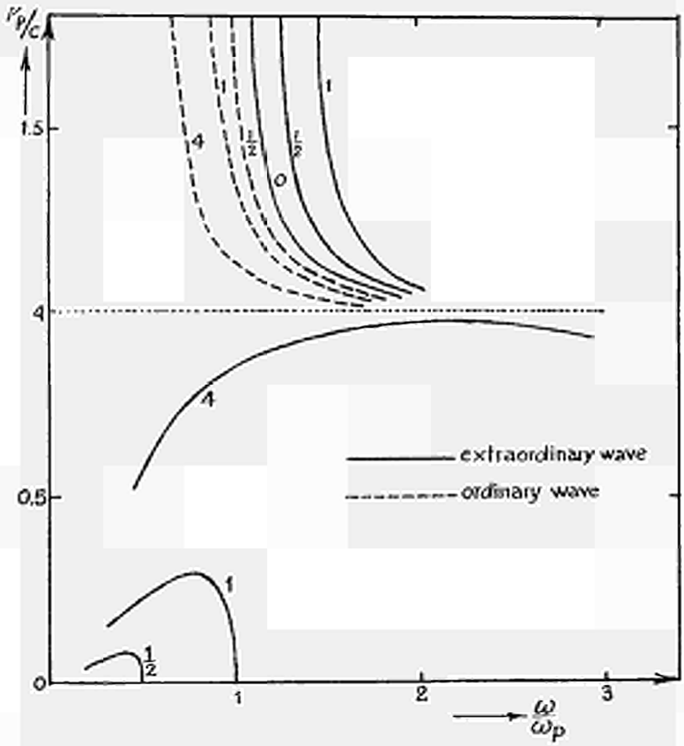


Fig. 74. Graph of a dispersion relationship for a transversal electron wave when $\boldsymbol{B}_0 \parallel \boldsymbol{k}$. The attached numbers represent ω_c/ω_p.

* It is evident that $v_p > c$. The signal velocity (or group velocity) v_g is related to v_p by the relation

$$v_p v_g = c$$

and consequently

$$v_g < c$$

(see also ref. 1).

influence on the propagation of these waves. The wave is the same as the one described above for which $B_0 = 0$ and it conforms to the same dispersion relation as derived in eq. (26).

*

5.1.3. HYBRID TRANSVERSAL AND LONGITUDINAL WAVES

These are waves in which a transversal oscillating current is transformed by the action of a magnetostatic field $\boldsymbol{B}_0$ into a longitudinal current (fig. 76). It thus follows that if the transversal current density is i_y the magnetic field required for such a transformation is B_{0x}

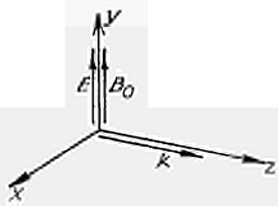


Fig. 75. Vector diagrammes for a transeversal wave with $\boldsymbol{E} \parallel \boldsymbol{B}_0$.

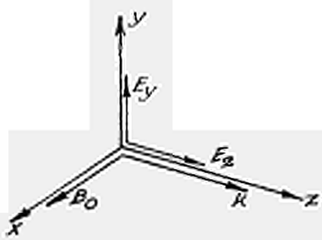


Fig. 76. Vector diagramme for a hybrid wave.

and the longitudinal current density is i_z. The corresponding equations follow from the general eqs. (10a)-(12a) and are

$$\left(k^2 - \frac{\omega^2}{c^2}\right) E_y = -\frac{4\pi}{c^2}\,\mathrm{j}\omega i_y \tag{27}$$

$$-\frac{\omega^2}{c^2}\,E_z = -\frac{4\pi}{c^2}\,\mathrm{j}\omega i_z \tag{28}$$

$$\mathrm{j}\omega i_z = \frac{\omega_p^2}{4\pi}\,E_z - i_y\omega_{cx} \tag{29}$$

$$\mathrm{j}\omega i_y = \frac{\omega_p^2}{4\pi}\,E_y + i_z\omega_{cx} \tag{30}$$

which can be reduced to a characteristic equation (put $\omega_{cx} = \omega_c$)

$$c^2\,\frac{k^2}{\omega^2} = \frac{\left(1 - \dfrac{\omega_p^2}{\omega^2}\right)^2 - \dfrac{\omega_c^2}{\omega^2}}{1 - \dfrac{\omega_p^2}{\omega^2} - \dfrac{\omega_c^2}{\omega^2}}. \tag{31}$$

It is readily seen from eqs. (29) and (30) that for $\omega_c = 0$ the coupling between the transversal and longitudinal waves disappears and one recovers the two dispersion relations (18) and (26).

The phase velocity $v_p = \omega/k$ is plotted for several values of the parameter ω_c/ω_p (fig. 77). It is seen that for $\omega_c > 0$ there is a band of frequencies for which k is purely imaginary and which therefore cannot propagate.

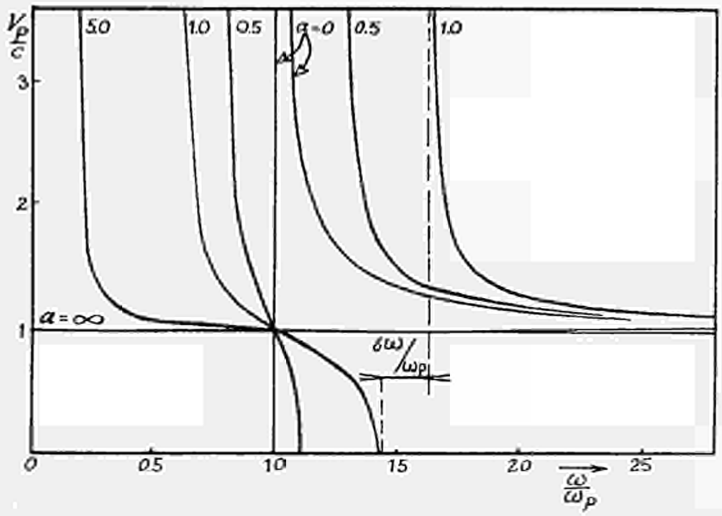


Fig. 77. Graph of a dispersion relationship for hybrid electron waves; $\alpha = \omega_c/\omega_p$.

The width $\delta\omega$ of this stop-band is obtained from eq. (31) by putting k = 0, ∞. For $\omega_c \ll \omega_p$ one obtains

$$\delta\omega \sim \frac{\omega_c}{2}. \tag{32}$$

5.1.4. REFLECTION OF ELECTROMAGNETIC WAVES BY PLASMA

Consider a simple dispersion relationship, such as expressed by eq. (26). It is evident that for

$$\frac{\omega_p^2}{\omega^2} > 1 \tag{33}$$

the velocity v_p becomes an imaginary quantity. Remembering that

$$v_p = \frac{\omega}{k}$$

one may also say that k becomes imaginary, i.e., $k = jK$. This implies that the space variation of the oscillation corresponds to an evanescent wave. Thus

$$E = E_0 e^{-Kz}. \tag{33a}$$

Similar conclusion can be drawn, even when plasma is in a magnetostatic field, (gyrotropic plasma), only the condition (33) involves also ω_c. An example of such a relation is eq. (23) which shows that transversal waves can propagate in a gyrotropic plasma even if their frequency is well below the plasma frequency.

The depth at which the amplitude of a non-propagating wave decreases by a factor e is called the reflection skin-depth δ *. Let us write eq. (26) as

$$-K^2 = \frac{1 - \omega_p^2/\omega^2}{c^2/\omega^2}$$

from which

$$K = \frac{\omega_p}{c}\left(1 - \frac{\omega^2}{\omega_p^2}\right)^{1/2} \tag{34}$$

and from eq. (33a) it follows that

$$\delta = \frac{c}{\omega_p}\,(1 - \omega^2/\omega_p^2)^{-1/2} \tag{35}$$

For $\omega \ll \omega_p$ one has $\delta \simeq 0.56 \times 10^6 n^{-1/2}$ (cm) **. (35a)

This reflection property of a plasma resembles the reflection by dielectrics, where it is found that a wave is totally reflected if its angle of incidence is larger than the Brewster's angle θ_0 (fig. 78) which is given by

$$\tan\theta_0 = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}.$$

In order that the reflection be total or all θ one must have $\theta_0 = 0$ and therefore, $\varepsilon_1 = 0$.

* In contradistinction to absorption skin-depth $\delta' = \sqrt{\rho/\omega}$.

** If electrons filled space completely, i.e., if $n\,\frac{4}{3}\pi r_e^3 = 1$ (where $r_e = \frac{e^2}{mc^2}$ the classical radius of electron), the reflection skin-depth would be

$$\delta \geqq \frac{c}{\sqrt{\frac{4\pi e^2 n}{m}}} = \frac{r_e}{\sqrt{3}} \quad \text{and} \quad \omega_p = \sqrt{3}\,\frac{c}{r_e}. \tag{36}$$

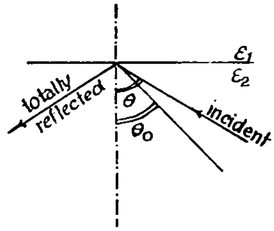


Fig. 78. Reflection at a boundary between two dielectrics.

If now ε_2 corresponds to vacuum and ε_1 to a plasma medium one has

$$\varepsilon_2 = 1,$$

and from Maxwell's second equation and eq. (4)

$$\begin{aligned} \operatorname{curl} \boldsymbol{B} &= \frac{1}{c}\dot{\boldsymbol{E}} + \frac{4\pi \boldsymbol{i}}{c} \\ &= \left(\frac{\mathrm{j}\omega}{c} + \frac{4\pi n e^2}{jc\omega m}\right)\boldsymbol{E} \\ &= \frac{\mathrm{j}\omega}{c}\left(1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2}\right)\boldsymbol{E}. \end{aligned} \tag{37}$$

If plasma is to be compared with a dielectric then its dielectric constant is

$$\varepsilon_1 = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2} \tag{38}$$

and according to Brewster's condition for total reflection it becomes totally reflective for

$$\frac{\omega_{\mathrm{p}}^2}{\omega^2} \geq 1$$

which is exactly the condition (33).

The reflection criterion, i.e., $\omega < \omega_{\mathrm{p}}$ can be also derived using considerations based on microscopic processes of radiation scattering by a single electron. The scattering cross-section is the well-known Thomson cross-section $\sigma_{\mathrm{T}} = \dfrac{8\pi}{3} r_e^2$ (see pp. 66 and 171). A bunch of N electrons scattering coherently will, therefore, present a cross-section $N^2\sigma_{\mathrm{T}}$. If the radius of the bunch is R all the incident radiation will be scattered when $N^2\sigma_{\mathrm{T}} \geq \pi R^2$ from which $R \lesseqgtr 1.35 \dfrac{c}{\omega_{\mathrm{p}}}$. The coefficient 1.35

appears owing to the spherical geometry of the bunch; in a plane case $R = \dfrac{c}{\omega_p} = \delta$.

This effect of total reflection can be used in measurements of plasma density. Plasma slab or cylinder is irradiated by a beam of microwaves (fig. 79). It is necessary that the dimensions of the plasma are much

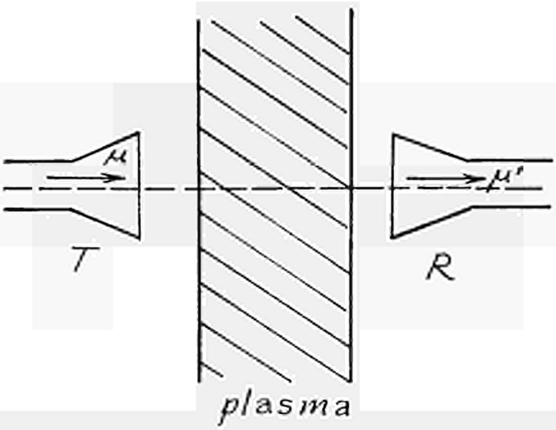


Fig. 79. Principle of density measurements with microwaves.

larger than the wave length of the incident radiation. As long as at least a part of the beam is transmitted, $\omega_p < \omega$ and consequently $n < \dfrac{m}{4\pi e^2} \cdot \omega^2$. When the density of the plasma increases so that $\omega_p > \omega$ the transmission is cut off and the ω determines the plasma density at that moment. Using several beams at different frequencies (e.g. corresponding to wave lengths of 3, 0.8 and 0.4 cm) it is possible to estimate the time dependence of n (fig. 80, ref. 2).

*

5.1.5. ELECTRON WAVES ON A PLASMA CYLINDER

In this section we shall derive first the general wave equation of electron oscillations on a cylinder of plasma, and apply this to a cylindrical plasma in a magnetic field.

Let us formulate eqs. (10a) and (12a) in cylindrical coordinates z, r, θ. Furthermore we shall restrict ourselves to waves of rotational symmetry.

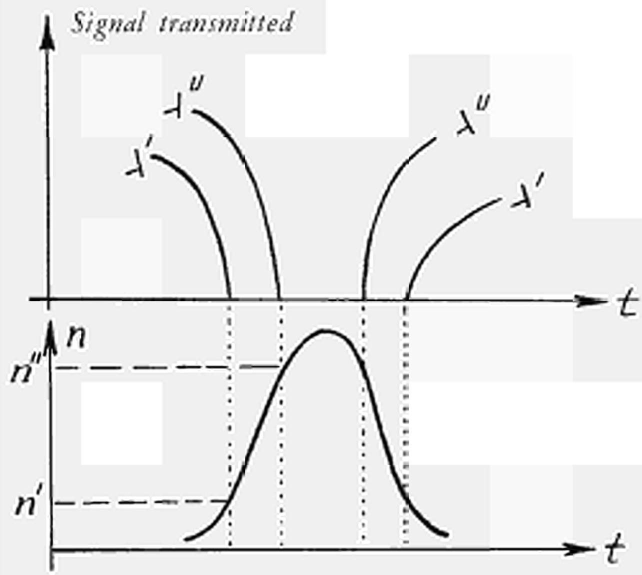


Fig. 80. Cut-off measurements of $n(t)$.

The wave eq. (10a) then becomes

$$\frac{\partial^2 E_z}{\partial z \partial r} - \frac{\partial^2 E_r}{\partial z^2} - \frac{\omega^2}{c^2} E_r = -4\pi \mathrm{j} \frac{\omega}{c^2} i_r \tag{39a}$$

$$\frac{1}{2} \frac{\partial^2 r E_r}{\partial z \partial r} - \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} E_z - \frac{\omega^2}{c^2} E_z = -4\pi \mathrm{j} \frac{\omega}{c^2} i_z \tag{39b}$$

$$-\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial r E_\theta}{\partial r} \quad \frac{\partial^2 E_\theta}{\partial z^2} - \frac{\omega^2}{c^2} E_\theta = -4\pi \mathrm{j} \frac{\omega}{c^2} i_\theta . \tag{39c}$$

The equation (12a) for current density, considering only a B_z field, is

$$\mathrm{j}\omega i_z = \frac{\omega_p^2}{4\pi} E_z \tag{40a}$$

$$\mathrm{j}\omega i_r + i_\theta \omega_{cz} = \frac{\omega_p^2}{4\pi} E_r \tag{40b}$$

$$\mathrm{j}\omega i_\theta - i_r \omega_{cz} = \frac{\omega_p^2}{4\pi} E_\theta . \tag{40c}$$

Let us assume that the field B_z is strong enough to prevent an appreciable h.f. current flow in the radial and azimuthal direction *

* In other words $\omega_c \gg \omega$.

and that the field pattern varies with z as e^{jkz}. Eq. (39a) gives

$$E_r = jk\left(\frac{\omega^2}{c^2} - k^2\right)^{-1} \frac{\partial E_z}{\partial r}. \tag{41}$$

Substituting eqs. (40a) and (41) into eq. (39a) we get

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial E_z}{\partial r} + \left(\frac{\omega^2}{c^2} - k^2\right)\left(1 - \frac{\omega_p^2}{\omega^2}\right) E_z = 0 \tag{42}$$

which is the wave equation describing the propagation of an electron-wave in a plasma-cylinder. Comparing this equation with a wave equation in a dielectric medium one finds that the plasma behaves as an anisotropic dielectric with

$$\varepsilon_r = 1, \quad \varepsilon_z = 1 - \frac{\omega_p^2}{\omega^2}. \tag{43a,b}$$

The solution of eq. (42) is *

$$E_z = aJ_0(|\chi|r) \tag{44}$$

where

$$\chi^2 = \left(\frac{\omega^2}{c^2} - k^2\right)\left(1 - \frac{\omega_p^2}{\omega^2}\right).$$

The corresponding solution for B_θ can be obtained from Maxwell's second equation. Thus

$$\operatorname{curl} E = -j\frac{\omega}{c} B_\theta.$$

Using eqs. (41) and (44) one has

$$B_\theta = j\frac{a|\chi|\dfrac{\omega}{c}}{\dfrac{\omega^2}{c^2} - k^2} J_1(|\chi|r). \tag{45}$$

Assuming that the space outside the plasma cylinder is vacuum, the appropriate solution of the wave equation for $r > r_0$ and for a guided cylindrical wave is

$$E_z = bK_0(|\chi_0|r), \quad \chi_0^2 = \frac{\omega^2}{c^2} - k^2 \tag{46}$$

$$B_\theta = j\frac{b|\chi_0|\dfrac{\omega}{c}}{\dfrac{\omega^2}{c^2} - k^2} K_1(|\chi_0|r). \tag{47}$$

* J_n and K_n being the usual symbols for Bessel functions.

The constants a and b are obtained from the condition that the E_z and B_θ components must be continuous functions of radius. Thus at the boundary of the plasma cylinder

$$aJ_0(|\chi|r_0) - bK_0(|\chi_0|r_0) = 0 \tag{48}$$

$$a\,|\chi|\,J_1(|\chi_0|r_0) - b\,|\chi_0|\,K_1(|\chi_0|r_0) = 0. \tag{49}$$

The secular equation of this system is

$$\frac{J_0(|\chi|r_0)}{J_1(|\chi|r_0} = \frac{|\chi|}{|\chi_0|}\,\frac{K_0(|\chi_0|r_0)}{K_1(|\chi_0|r_0)}. \tag{50}$$

This can be written as

$$\frac{J_0(|\chi|r_0)}{J_1(|\chi|r_0)} = \left(\frac{\omega_p^2}{\omega^2} - 1\right)^{1/2} \frac{K_0(|\chi_0|r_0)}{K_1(|\chi_0|r_0)}. \tag{50a}$$

It is useful to introduce the following system of non-dimensional quantities

$$\xi = \frac{\omega^2}{\omega_p^2}, \qquad \kappa = \frac{k^2c^2}{\omega_p^2}, \qquad \rho = r_0\frac{\omega_p}{c} = 2\sqrt{\nu}.$$

In terms of these, eq. (50a) becomes

$$\frac{J_0\{[4\nu(1-1/\xi)\,(\xi-\kappa)]^{1/2}\}}{J_1\{[4\nu(1-1/\xi)\,(\xi-\kappa)]^{1/2}\}} = \left(\frac{1}{\xi} - 1\right)^{1/2} \frac{K_0\{[4\nu(\kappa-\xi)]^{1/2}\}}{K_1\{[4\nu(\kappa-\xi)]^{1/2}\}} \tag{50b}$$

which is the dispersion equation for the guided waves on a plasma cylinder confined by a B_z field.

Interpretation of the Dispersion Relation

The curve representing the dispersion equation (50b) can be plotted in a ξ, κ coordinate systems (fig. 81). One has

$$\xi = \frac{\omega^2}{\omega_p^2} = \left(\frac{\pi r_0}{\lambda}\right)^2 \nu^{-1}$$

$$\kappa = \frac{k^2c^2}{\omega_p^2} = \left(\frac{\pi r_0}{\lambda_g}\right)^2 \nu^{-1}$$

where λ_g is the guide-wave length, λ and r_0 may be expressed as functions of ξ and κ. Thus

$$\lambda = \lambda_g\sqrt{\frac{\kappa}{\xi}} \text{ and } r_0 = \frac{\lambda_g}{\pi}\sqrt{\kappa\nu}.$$

From, these, and with the help of the $\kappa = \kappa(\xi)$ curves one can plot curves representing

$$\lambda = \lambda(r_0)$$

in which v and λ_g are parameters (fig. 82).

It follows from fig. 81 that waves whose frequency is larger than $\omega_p/2\pi$ cannot be guided, whereas waves whose frequency is much smaller than $\omega_p/2\pi$ are only weakly attached to the plasma cylinder, most of their energy being located outside the plasma, and their phase velocity approaches that of light.

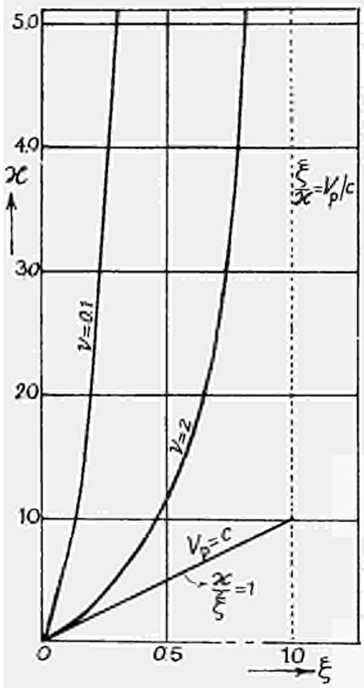


Fig. 81.

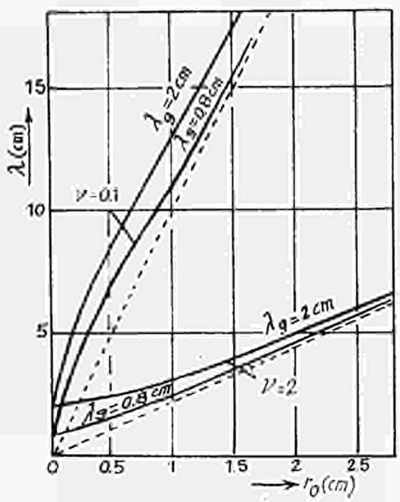


Fig. 82.

Between these two extremes lies a range of frequencies for which waves can be guided by the plasma cylinder, most of the wave energy being stored in the plasma.

In the first experiments on plasma wave guides a cylindrical glass tube containing a positive column of a glow-discharge was used as the central conductor in a coaxial cable structure (ref. 3). Only coaxial-modes of propagation were thus studied. Later, using a glass tube containing a high frequency discharge was used to study the propagation of surface waves (ref. 4). First experiments on propagation in free plasma columns, i.e., not in contact with a glass tube, were done in 1958 (ref. 5), using a plasma produced by a PIG discharge (PIG = Penning ionization gauge).

5.1.6. EFFECTS OF RANDOM VELOCITIES ON WAVES IN PLASMA

Up to now we have assumed that the electrons (and the ions) had no appreciable random velocities and plasma has, therefore, behaved as a perfectly stationary dispersive medium. In order to make our results more realistic it is necessary to consider a space-velocity distribution $f(v_i, x_i)$ of the electrons and also stochastic fluctuations of such a distribution, in the phase-space v_i, x_i.

As soon as one considers a velocity spread, the model of cold plasma, so far used by us, has to be dropped. The fluid equations can still be used, containing now the pressure terms (see later 5.2.1.). The inclusion of these terms allows us to take into account the various sound phenomena and the dispersion relations of plasma oscillations are correspondingly modified. If analysis is to be further refined we have to use the Boltzmann-Vlasov equations rather than those corresponding to the two-fluid model. Most of the dispersion relations derived so far remain practically unchanged, however, a new phenomenon makes its appearance: damping.

The stochastic fluctuations are connected with phenomena similar to those encountered in gases, where small density fluctuations scatter a monochromatic wave. In hot plasmas such scattering of waves can be caused not only by density fluctuations but also by velocity fluctuations — a type of Doppler broadening.

We shall first treat a typical case of damping, known as Landau damping and then we shall develop a simple theory of scattering due to plasma fluctuations.

Landau damping

The existence of damping of longitudinal plasma oscillations has been discovered by Landau (ref. 6).

Starting from Vlasov and Poisson equations for electrons we write

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - \frac{e}{m} \cdot E \frac{\partial f}{\partial v} = 0 \tag{51}$$

$$\frac{\partial E}{\partial x} = - 4\pi en \int_{-\infty}^{+\infty} (f - f_0)\, dv \tag{52}$$

where $f = f(x, v, t)$ is the perturbed distribution function, $f_0 = f_0(v)$ is the unperturbed velocity distribution, $E = E(x, t)$ and $\int_{-\infty}^{+\infty} f_0 \mathrm{d}v = 1$.

If we assume a spatial-temporal dependence $e^{j(kx-\omega t)}$ and keeping only first order terms in E and $f_1 = f - f_0$ we have

$$j \cdot f_1 (-\omega + k \cdot v) = \frac{e \cdot E}{m} \frac{\partial f_0}{\partial v} \tag{53}$$

and the Poisson's equation becomes

$$jkE = -\frac{4\pi e^2 n}{jm} E \int_{-\infty}^{+\infty} \frac{\frac{\partial f_0}{\partial v}}{-\omega + kv} \mathrm{d}v \tag{54}$$

or

$$1 = -\frac{\omega_p^2}{k} \int_{-\infty}^{+\infty} \frac{f_0}{\omega - kv} \mathrm{d}v. \tag{55}$$

Evidently there is a pole at $\omega = kv$ and the integration path has to be deformed around it. If the path of integration passes below the pole and if the distribution $f_0(v)$ is Maxwellian we get

$$\omega = \omega_p \left(1 + \frac{3}{2} k^2 d^2\right) - j \sqrt{\frac{\pi}{8}}\, \omega_p (kd)^{-3} \exp\left[-\frac{1}{2}(kd)^{-2} - \frac{3}{2}\right] \tag{56}$$

valid for $k^2 d^2 \ll 1$.

The first term on the r.h.s. corresponds to a new dispersion relationship, the second to damping.

This result is open to criticism (ref. 8, 7) which touches such points as continuity of f_0, analytic behaviour of f'_0 and the form of the initial perturbation. However, as we have not introduced any new physical mechanisms, it should be possible to interpret the eq. (56) in terms of particle-wave interaction. This is easy for the undamped part and will be done later (p. 173). The damping term will now be derived using the model of particle-wave interaction in a linear accelerator.

Let us follow the movement of particles in the x, p_x space. This can be described by Hamiltonian equations written in the frame of reference moving with the phase-velocity of the wave (ref. 10).

$$\frac{\mathrm{d}p_x}{\mathrm{d}t} = -\frac{\partial H}{\partial x}, \quad \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial H}{\partial p} \tag{57, 58}$$

where

$$H = \sqrt{m_0^2c^4 + p^2c^2} - pv_p - eE\frac{v_p}{\omega}\cos kx; \tag{59}$$

$$\frac{\omega}{k} = v_p \text{ (phase velocity).}$$

For non-relativistic electrons we get $H - m_0c^2 = W_{tot} = \frac{1}{2}mv^2 - e\frac{E}{k}(\cos kx - 1)$ corresponding to closed or open orbits in the v, x plane i.e., to trapped * or free particles (fig. 83). The separatrix between these two classes of orbits corresponds to a particle initially at $kx = -\pi$, $v = 0$ and its W_{TOT} is

$$W_{TOT} = 2\frac{eE}{k}$$

from which

$$v = \left[\frac{2\,eE}{mk}(\cos kx + 1)\right]^{1/2}. \tag{60}$$

The equation of motion is

$$\dot{v} = \ddot{x} = -\frac{e}{m}E\sin(kx). \tag{61}$$

For $kx \ll \frac{\pi}{2}$ we can use the approximation

$$\ddot{x} = \frac{-e}{m}Ekx. \tag{62}$$

This shows that particles moving on closed orbits (inside a "bucket") rotate in the phase space with an angular frequency

$$\omega_0 = \sqrt{\frac{e}{m}E\cdot k}. \tag{63}$$

The maximum speed is $v_{1\max} = \left(\frac{e}{m}Ek\right)^{1/2}\cdot x_0$. From conservation of energy (eq. (59)) follows that the highest speed any particle can possess and still remain trapped in the bucket is (fig. 83)

$$\delta v = 2\sqrt{\frac{eE}{mk}}. \tag{64}$$

The linear approximation ($kx \ll \pi/2$) is certainly not true near the inside boundary of the bucket, however, we shall not make an

* Particles in a "bucket".

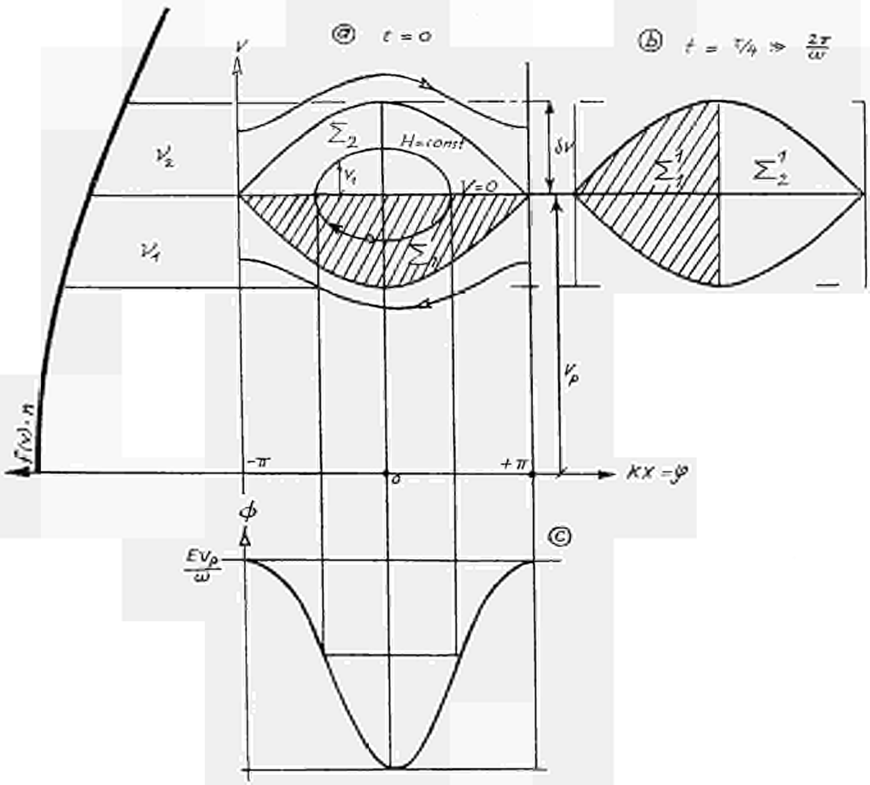


Fig. 83. Damping due to particle-wave interaction in a linear accelerator.

a The bucket in phase space at $t = 0$;
b The same at $t = \tau/4$;
c The potential ϕ in the wave.

unreasonable error if we suppose that *all* the particles in the bucket rotate in it with the angular speed ω_0 which tends to zero as $E \to 0$ and, therefore, for weak fields

$$\omega_0 \ll \omega.$$

After a time $\tau/4 = \dfrac{\pi}{2\omega_0}$ all the ν_1 particles which at $t = 0$ occupied the cross-hatched crescent Σ_1 will be in Σ'_1. Similarly the group ν_2 in Σ_2 will rotate into Σ'_2. In this new configuration the particles will damp or amplify the wave. The work done by these particles on the field of the wave is

$$\Delta W = e\bar{E}v_p\ (\nu_1 - \nu_2) \cdot \Delta t \tag{65}$$

where $\bar{E}$ is the mean electric field *, e.g., $\bar{E} = \left(\frac{2}{\pi}\right) E$. The mean energy over a length λ is

$$W = \frac{E^2}{16\pi} \lambda. \tag{66}$$

Let us write

$$\frac{\Delta W}{\Delta t} = \frac{E^2 \lambda}{16\pi} 32\, e v_p \frac{\nu_1 - \nu_2}{\lambda E}. \tag{67}$$

If, as will be shown later $\frac{\nu_1 - \nu_2}{E}$ is independent of E, we can integrate equation (67) and get

$$W = W_0 \cdot \exp\,(-2\delta \cdot t) \tag{68}$$

where

$$\delta = 16\, e v_p \frac{\nu_1 - \nu_2}{\lambda E} \tag{69}$$

is the damping coefficient.

Let us now evaluate $\nu_1 - \nu_2$ for a Maxwellian distribution. There

$$\nu_1 - \nu_2 = -n \left|\frac{\partial f}{\partial v}\right|_{v=v_p} \overline{\delta v} \cdot \Sigma \cdot \frac{\lambda}{2\pi} \tag{70}$$

where

$$\left|\frac{\partial f}{\partial v}\right|_{v=v_p} = -\frac{2}{\pi^{3/2}} \frac{v_p}{u^3} \exp\left(-\frac{v_p^2}{2u^2}\right) \tag{71}$$

$$\Sigma = k \int_{-\lambda/2}^{\lambda/2} \left[\frac{2eE}{mk}(\cos kx + 1)\right]^{1/2} dx = 4 \cdot \delta v \tag{72}$$

$$u = \omega_p d, \qquad \overline{\delta v} = \frac{\Sigma}{2\pi} = \frac{2}{\pi} \cdot \delta v. \tag{73}$$

Using eqs. (69)-(73) we get for δ

$$\frac{\delta}{\omega_p} = 0.73 \left(\frac{\omega}{\omega_p}\right)^2 (kd)^{-3} \exp\left[-\tfrac{1}{2}\left(\frac{\omega}{\omega_p}\right)^2 (kd)^{-2}\right] \tag{74}$$

where in the exponent $(\omega/\omega_p)^2 = 1 + 3\,(kd)^2$.

Comparing eqs. (74) and (56) we see that, apart form the term $\frac{\omega}{\omega_p}$ which is nearly always near unity, the formula (74) derived on the

* The mean $\bar{E}$ implies averaging over $\frac{\lambda}{2}$.

basis of the physical model of linear accelerators predicts a higher damping than the Landau formula (fig. 84). At at time $t_0 = \frac{1}{\delta}$, the bucket shrings vertically by factor $e^{-1/2}$ (see eqs. (64) (66) (68)) and some of the closed orbits will have been in the meantime transformed

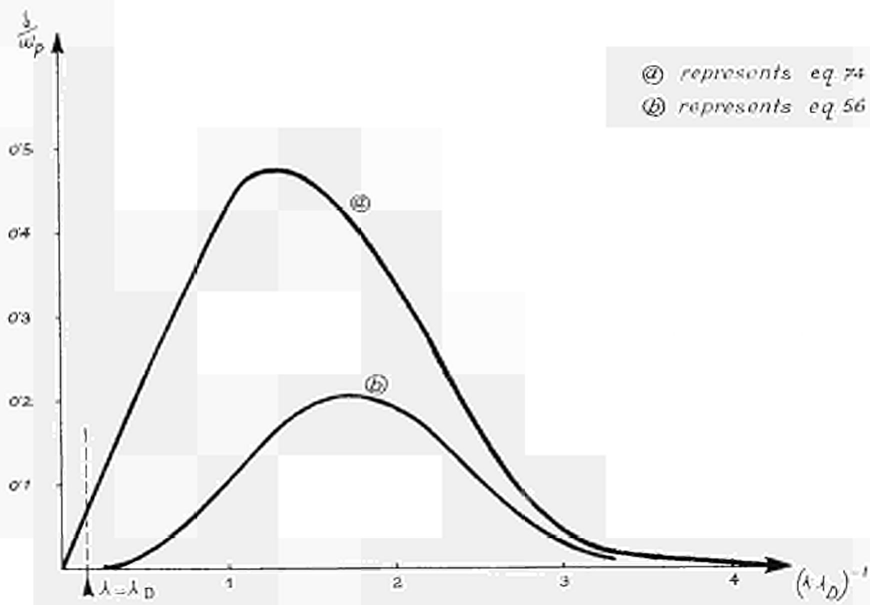


Fig. 84.

into open ones (fig. 85). Since there are more particles lost in this way at the top of the bucket than at its bottom, more particles are accelerated to $v_p + \delta v$ than decelerated to $v_p - \delta v$. This tends to flatten out the $f(v)$ distribution around $v = v_p$. Thus the departure from the equilibrium distribution for oscillating modes (only one being excited) is being eliminated (by damping the mode out) in favour of a departure of the $f(v)$ from the Maxwellian distribution *.

We have assumed that it is permissible to evaluate the interaction of particles and wave using maximum values of E and of $(v_1 - v_2)$. Consequently we must regard our δ as an upper limit to a $\delta(t)$. in fact near $t = 0$ no damping occurs since none of the v_1 particle has

* As the degree of organization of our system remains unchanged, the total entropy is constant.

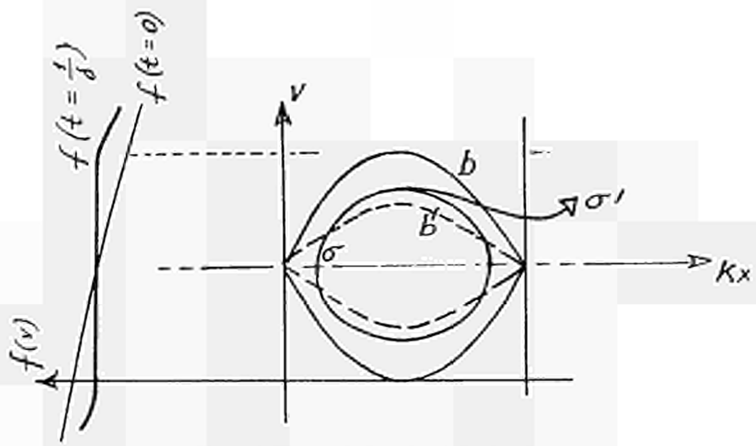


Fig. 85. Vertical shrink of the bucket.

yet traversed the v_0 boundary. This shows how the instantaneous damping coefficient depends on the initial distribution of ν in v and x and that for certain distributions no damping will occur for some period *.

For a well developed Landau damping we require, therefore, that $t_0 \gtrless \tau/4$. This implies (for $\omega \sim \omega_p$) that

$$\frac{\omega_p}{\sqrt{e/m}} \sqrt{\frac{\lambda}{E}} \leqslant 1930 \left(\frac{d}{\lambda}\right)^3 \exp\left[1/2\,(kd)^{-2}\right]. \tag{75}$$

Introducing $eE_m = \frac{\pi}{2}\frac{m}{2}\omega_p^2 d$, the maximum field in spontaneous oscillations in plasma (see p. 9), our condition can be written

$$\frac{E}{E_m} \geqslant 0.65 \cdot 10^{-3} \left(\frac{\lambda}{d}\right)^7 \exp\left(-\frac{1}{8\pi^2}\frac{\lambda^2}{d^2}\right) \tag{76}$$

This shows that Landau damping is operative only when the induced electric field is comparable with the E_m fields.

On the other hand, if $\tau/4 \ll T_0$, the ν_1 and ν_2 will both partially spill out of the bucket and during the second $\tau/4$ more particles will be decelerated than accelerated by the wave. This action causes amplifica-

* Most of the controversies concerning Landau's analysis are caused by the differences in the prescribed initial distributions. In this connection it is interesting to note that the electron distribution in the bucket is much less perturbed by the space charge induced by E than by the rotation corresponding to ω_0. The two perturbations are in a ratio of approx. $\frac{u}{v_p}\frac{d}{\lambda}\sqrt{\frac{\Delta n}{n}}$, where $\Delta n = \frac{E}{2\pi\lambda e}$.

tion of the wave and the total damping will have to be found averaging over τ. This will lead to a damping term much smaller than that of Landau.

In an experiment where a longitudinal wave is excited at one plane in a plasma and propagates through a thickness L of plasma, Landau damping should be observed only if $L \approx v_p \cdot \frac{\tau}{4}$ which can be also written as (see also ref. 10)

$$\frac{L}{\lambda} \approx \frac{1}{4\sqrt{\pi}} \sqrt{\frac{E_{m\lambda}}{E}}$$

where $E_{m\lambda} = 4\pi en \frac{\lambda}{2}$ is the maximum field obtainable in plasma on fully separating negative and positive charges over a distance λ. Example:

$n = 10^{14}$, $\lambda = 1$, $E = 30$ elst · u/cm then $E_{m\lambda} = 3 \cdot 10^5$ elst · u/cm and $L \sim 15$ cm.

Scattering of waves by plasma fluctuations

The fluid description of plasma, accomplished by means of the Boltzmann equation and of the Maxwell's equations, breaks down when a volume is considered whose dimensions are smaller than the Debye wave length d. In such a volume the fluctuations of both, number density n and fluid velocity v, can become comparable with the mean values of density and random velocity, i.e., with n_0 and u, where $u = \sqrt{\frac{2kT}{m}}$. The fluctuations of n and v will, of course, be present even in larger volumes than $\lambda_D{}^3$. In absence of particle interaction, i.e., for a perfect gas, the mean root square of a relative fluctuation in density is given by the normal distribution law (ref. 9)

$$\delta = \frac{\langle \delta n^2 \rangle^{1/2}}{n} = N^{-1/2} = \frac{1}{\sqrt{n\Omega}} \tag{77}$$

where N is the mean number of particles in a volume Ω in which the fluctuation is observed.

Let us consider a plasma in which the ions are infinitely heavier than the electrons. The ion distribution cannot change in response to electron fluctuations and these will be, therefore, limited in amplitude by the electrostatic forces induced by the perturbed charge neutrality. The law governing δ in plasmas can be derived as follows.

Let us consider all the modes of oscillations in a cube whose side is L. Owing to boundary conditions only modes for which $\frac{2L}{\lambda_i} = \frac{Lk_i}{\pi}$ is an integer ($i = x, y, z$) can exist. If two adjacent modes have wave numbers k_i and $k_i + \delta k_i$ then

$$\frac{2L}{\pi}\,[k_i + (k_i + \delta k_i)] = 1 \tag{78}$$

or

$$\delta k_i = \frac{\pi}{2L} \tag{79}$$

and the density of modes in the (k_i, q_i) space is

$$\nu = \frac{L^{-3}}{\delta k_x\, \delta k_y\, \delta k_z} = 8\pi^{-3}. \tag{80}$$

In a space Ω the principal fluctuation will have a wave length $\frac{\lambda_0}{2} \sim \Omega^{1/3}$. Modes whose wave length λ is not much different from λ_0 will contribute to the amplitude of the fluctuation, e.g., if $\frac{3\lambda_0}{4} < \lambda < \frac{5\lambda_0}{4}$ the modes add up in a coherent manner over one period of the fluctuation *.

The corresponding number of modes is then

$$N_\lambda = \left(\frac{2\delta k}{\pi}\right)^3 \tag{81}$$

where
$$|\delta k| = \frac{2\pi}{\lambda_0^2}\cdot|\delta\lambda| = \frac{\pi}{\lambda_0}.$$

In thermodynamic equilibrium, each mode should possess an energy κT. The energy associated with the principal fluctuation in 1 cm^3 is, therefore,

$$W = N \cdot \kappa T \cdot = \frac{8\kappa T}{\lambda_0^3}. \tag{82}$$

If the maximum electric field of the electron fluctuation is E, then the mean electrical energy density is $\frac{E^2}{16\pi}$ and since at one moment all the energy of the fluctuation is in the electric field we get

$$E = 8\left(\frac{\pi\kappa T}{\lambda_0^3}\right)^{1/2}. \tag{83}$$

* A rather arbitrary choice, implying that the numerical coefficient in eq. (87) is correct only to an order of magnitude.

The Poisson's equation gives for a plane perturbation

$$4\pi e \cdot \delta n = kE \tag{84}$$

from which

$$\delta n = \frac{4\sqrt{\pi}}{e} \lambda_0^{-5/2} \sqrt{\kappa T} \tag{85}$$

where $\delta n = \langle \delta n^2 \rangle^{1/2}$. Finally

$$\delta = \frac{\delta n}{n} = 8\pi \frac{\lambda\, d}{\lambda_0} n^{-1/2} \lambda_0^{-3/2} \tag{86}$$

if N is the number of electrons in the space Ω we get

$$\delta = \sqrt{8\pi} \frac{\lambda\, d}{\lambda_0} N^{-1/2}. \tag{87}$$

The factor $\frac{8.9\, d}{\lambda_0}$ corresponds to the departure of plasma behaviour from that of a perfect gas. The latter is recuperated only when $\lambda_0 \cong 8.9\, d$ (more rigorous considerations can be found in ref. 11). For $\lambda < d$ it is doubtful that one can still talk of modes of oscillation and the term $\frac{d}{\lambda_0}$ is no longer applicable and consequently $\delta \sim N^{-1/2}$.

It is to be expected that the propagation of any wave through a plasma will be affected by these fluctuations in a manner similar to that of scattering of light by density fluctuations in gases (ref. 9), p. 214).

The effect of the velocity fluctuation can be understood as a Doppler broadening of the frequency of the original wave. The density fluctuations scatter the wave by creating small changes in the mean dielectric constant of the medium. Both of these mechanisms cause the phase of the wave to fluctuate around the phase φ_0 which would be observed in a uniform plasma at zero temperature. As the wave progresses in time or space this fluctuation φ increases and when $<\varphi^2>^{1/2} = 2\pi$ one may consider the original wave to be essentially damped out.

Let us consider a beam whose cross-section is λ^2. Let this beam be scattered by plasma inhomogeneities whose mean dimension is l (often called "blobs" in the scattering theory). Thus there will be $(\lambda/l)^2$ blobs in each section of the beam. The phase change $\delta\varphi$ caused by the density fluctuation δn in one blob will be

$$\delta\varphi = 2\pi \frac{\delta v_p\, \tau}{\lambda} \left(\frac{l}{\lambda}\right)^2 = 2\pi \frac{\delta v_p}{v_p} \frac{l}{\lambda} \left(\frac{l}{\lambda}\right)^2 \tag{89}$$

where τ is the transit time of the wave over l. Since there are $\left(\frac{l}{\lambda}\right)^2$ scattering centra over the section of the beam the mean root square $\delta\varphi$ over this section will be

$$\langle\delta\varphi^2\rangle^{1/2} = \Delta\varphi = \frac{\lambda}{l}\,\delta\varphi. \tag{90}$$

These $\Delta\varphi$'s will add up statistically over a distance s to an average deviation $\langle\varphi^2\rangle^{1/2}$ from φ_0. Thus

$$\langle\varphi^2\rangle^{1/2} = \left(\frac{s}{l}\right)^{1/2} \Delta\varphi. \tag{91}$$

The distance s will be called the scattering or damping distance when $\langle\varphi^2\rangle^{1/2} = 2\pi$. Then

$$s_0 = l\,\frac{4\pi^2}{(\Delta\varphi)^2} = l\left(\frac{v_p}{\delta v_p}\right)^2 \cdot \left(\frac{\lambda}{l}\right)^4 \cdot \tag{92}$$

We shall apply this formula to a transversal wave. Our results will be approximate since we have not evaluated rigorously the interaction of the fluctuation with the wave throughout the thickness $\frac{\lambda}{2}$ and also because the choise of s_0 as the damping distance is somewhat arbitrary. It is unlikely, however, that these assumptions would cause an order of magnitude error.

The dispersion relation for tranversal waves is (p. 150).

$$v_p^2 = \frac{c^2}{1 - \dfrac{\omega_p^2}{\omega^2}} \tag{93}$$

from which

$$\frac{\delta v_p}{v_p} = \left(\frac{\omega_p}{\omega}\right)^2 \left[1 - \left(\frac{\omega_p}{\omega}\right)^2\right]^{-1} \cdot \frac{\delta n}{n}. \tag{94}$$

Substituting into equation (92) we get for $\omega \ll \omega_p$

$$s_0 = l\left(\frac{\lambda}{l}\right)^4 \left(\frac{\omega}{\omega_p}\right)^4 \left(\frac{n}{\delta n}\right)^2 \tag{95}$$

and using equation (86) for $\frac{\delta n}{n}$ with $\lambda_0 = l$ we have finally

$$s_0 = \frac{3}{64\pi}\,\frac{1}{(e^2/mc)^2\,n} \cdot \left(\frac{l}{d}\right)^2 \cdot \tag{96}$$

The scattering distance for Rayleigh scattering is (ref. 11)

$$s_R = \frac{1}{n\sigma_T} \quad \text{where} \quad \sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2. \tag{97}$$

With this we can write the formula (96) as follows

$$s_0 = \frac{9}{512\pi^2} s_R \left(\frac{l}{d}\right)^2. \tag{98}$$

This shows that the most intense scattering is due to fluctuations whose $l \sim d$ (for $l < d$ our treatment is not valid). This can be confirmed by using directly the Rayleigh formula for s, generalized for scattering blobs whose $l \ll \lambda$, remembering that the scattering cross-section σ_ν corresponds to δN charges and therefore, one would expect that $\sigma_\nu = \sigma_T(\delta N)^2$. Then

$$s_0 = \frac{1}{\nu\sigma_\nu} \quad \text{where} \quad \nu = l^{-3} \quad \text{and}$$

$$\sigma_\nu = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 (\delta N)^2 \tag{99}$$

$$\delta N = \delta n \cdot l^3 = \delta \cdot l^3 \cdot n. \tag{100}$$

Using equation (100) we get

$$s_0 = \frac{l^3}{\sigma_T \cdot (n \cdot l^3 \cdot \delta)^2} = \frac{3}{64\pi^3} s_R \left(\frac{l}{d}\right)^2 \tag{101}$$

which is essentially the same as formula (98).

It is seen that the damping due to scattering is very small. Similar analysis could be done for the scattering caused by velocity fluctuations. The corresponding damping coefficient is of the same order of magitude as s_0 in eq. (98).

5.2. Positive Ion Oscillations

In these oscillations the kinetic energy of positive ions is converted into electromagnetic energy and vice versa. Let the frequency of such conversion be ω_i. A full conversion of ion-energy into an electric field energy is described by

$$\tfrac{1}{2} n_p M \frac{e^2E^2}{M^2\omega_i^2} = \frac{E^2}{8\pi} \tag{102}$$

from which

$$\omega_i = \left(\frac{4\pi e^2 n_I}{M}\right)^{1/2}. \tag{102a}$$

This is the plasma ion-frequency.

A similar order of magnitude argument applies to oscillatory conversions of magnetic energy into the kinetic energy of ions. Thus it is evident that the frequencies at which such types of conversion would be expected to occur are by a factor $\sqrt{M/m}$ lower than those corresponding to electron oscillations.

This implies that during one period of an ion oscillation there will be, generally, enough time for the electrons to reach an equilibrium distribution, almost as if the oscillating electromagnetic field were a steady field for the electrons *.

We shall divide the subject of ion-oscillations into three parts:

a) purely electrostatic oscillations,
b) hydromagnetic oscillations in stationary plasma,
c) hydromagnetic oscillations in neutralized electron streams.

5.2.1. ELECTROSTATIC ION OSCILLATIONS

Let us consider the two-fluid model of plasma in which all quantities depend only on z. As we shall be concerned with longitudinal oscillations the only velocities entering the analysis are v and w_z. Using eqs. (3-56a), (3-57) one obtains (putting $v_z = v$, $w_z = w$, $\gamma \cong 1$, $\boldsymbol{B} = 0$)

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial z} v = -\frac{e}{m} E - \frac{1}{n_e m}\frac{\partial p_e}{\partial z} \tag{103}$$

$$\frac{\partial w}{\partial t} + \frac{\partial w}{\partial z} w = \frac{Ze}{M} E - \frac{1}{n_p M}\frac{\partial p_p}{\partial z}. \tag{104}$$

Let the time variation be harmonic and consider only one Fourier component (whose wave number is k) of any of the oscillating quantities. Assume also an isothermal plasma, i.e., $\dfrac{\partial \overline{u_e^2}}{\partial z} = 0$ Keeping only first order terms we get

$$j\omega v = -\frac{e}{m} E - jk\overline{u_e^2}\,\frac{n_{1e}}{n_e} \tag{103a}$$

* This is not true for some discharges, where the electric field is parallel to the velocity of the electron-flow. There, the electrons may "run-away" (chapter 8).

$$\mathrm{j}\omega w = \frac{Ze}{M} E - \mathrm{j}k\overline{u_p^2} \frac{n_{1p}}{n_p} \tag{104a}$$

where n_{1p} and n_{1e} are the time and space variable components of n_p and n_e.

The equation expressing the dependence of E on n is the div E equation. Thus

$$\frac{\partial E}{\partial z} = 4\pi e(Zn_p - n_e) \tag{105}$$

and from equations of continuity one has

$$-\frac{\partial n_p}{\partial t} = \frac{\partial}{\partial z}(n_p w), \quad -\frac{\partial n_e}{\partial t} = \frac{\partial}{\partial z}(n_e v). \tag{106a,b}$$

From these, the expression occurring on the right hand side of eq. (105) is to first order

$$Zn_p - n_e = -\frac{k}{\omega}(Zn_p w - n_e v)$$

and eq. (105) can be written as

$$-E = \frac{4\pi e}{\mathrm{j}\omega}(Zn_p\omega - n_e v). \tag{105a}$$

Sustituting this equation into eqs. (103a) and (104a) there follows

$$\mathrm{j}\omega v = \frac{4\pi e^2}{\mathrm{j}m\omega}(Zn_p w - n_e v) + \mathrm{j}k^2\overline{u_e^2}\frac{v}{\omega} \tag{103b}$$

$$\mathrm{j}\omega w = -\frac{4\pi e^2 Z}{\mathrm{j}M\omega}(Zn_p w - n_e v) + \mathrm{j}k^2\overline{u_p^2}\frac{w}{\omega} \tag{104b}$$

and the dispersion relation follows from putting the determinant of these two simultaneous and homogeneous equations equal to zero. Remembering that for small amplitudes $n_e \approx Zn_p$, one has

$$\left(-1 + \frac{\omega_{pe}^2}{\omega^2} + \frac{k^2\overline{u_e^2}}{\omega^2}\right)\left(-1 + \frac{\omega_{pp}^2}{\omega^2} + \frac{k^2\overline{u_p^2}}{\omega^2}\right) = \frac{\omega_{pp}^2\omega_{pe}^2}{\omega^4}. \tag{107}$$

For $\omega^2 \ll \omega_{pp}\omega_{pe}$ one gets a dispersion relationship for electron oscillations

$$-1 + \frac{\omega_{pe}^2}{\omega^2} + \frac{k^2\overline{u_e^2}}{\omega^2} \cong 0$$

from which *

$$\omega^2 \cong \omega_{pe}^2 + k^2\overline{u_e^2} = \omega_{pe}^2\,[1 + (kd)^2] \tag{108}$$

* For an adiabatic, rather than isothermic electron gas, one would obtain $\omega^2 = \omega^2_{pe}\,[1 + 3\,(k\,d)^2]$. See p. 161 (also ref. 12).

For an electron gas at zero temperature $u_e = 0$ and one recovers the frequency relation (18).

For $\omega^2 \ll \omega_{pe}^2$ the relation (107) simplifies to

$$\left(1 + \frac{k^2\overline{u_e^2}}{\omega_{pe}^2}\right)\left(-1 + \frac{\omega_{pp}^2}{\omega^2} + \frac{k^2\overline{u_p^2}}{\omega^2}\right) = \frac{\omega_{pp}^2}{\omega^2}.$$

As long as wave length of the oscillations is much larger than the Debye distance the following inequality applies

$$kd = \sqrt{\frac{k^2\overline{u_e^2}}{\omega_{pe}^2}} \ll 1$$

and we have

$$\omega^2 = k^2\overline{u_p^2}\left(1 + \frac{\omega_{pp}^2}{\omega_{pe}^2}\frac{\overline{u_e^2}}{\overline{u_p^2}}\right)$$

or

$$\omega^2 = k^2\overline{u_p^2}\left(1 + \frac{ZT_e}{T_p}\right). \tag{109}$$

The phase velocity of these ion waves is given by

$$v_p^2 = \overline{u_p^2}\left(1 + \frac{ZT_e}{T_p}\right) \tag{109a}$$

and is, therefore, $[1 + (ZT_e/T_p)]^{1/2}$ times larger than the velocity of sound in the ion gas. In the spectrum of ion-frequencies and for $T_e > T_p$ the plasma behaves as a medium whose elasticity is provided by the electron gas, whereas the inertia is due mainly to the ions.

The random velocities of the ions will be also responsible for a Landau damping and a phase mixing mechanism described on pp. 160 and 169 and consequently the wave will be damped. The coefficient of damping can be derived in the same way.

5.2.2. HYDROMAGNETIC OSCILLATIONS IN A STATIONARY INFINITE PLASMA. WAVES ON A PLASMA CYLINDER

If the speed of flow of either of the plasma components is much smaller than the mean thermal speed one may use the fluid equations corresponding to the *one fluid model* of plasma (eqs. (3.62) and (3.63)) and the appropriate Maxwell equations. As we wish to investigate oscillating processes in which ions play a dominant rôle, the frequencies will be of the order of magnitude given by eq. (102) and we can neglect the term representing the displacement flux in Maxwell's first equation.

Again restricting ourselves to a small amplitude analysis we write eqs. (3.62) and (3.63), as

$$\frac{\partial \rho_1}{\partial t} = - \rho_0 \operatorname{div} \boldsymbol{V} \tag{110}$$

$$\rho_0 \frac{\partial \boldsymbol{V}}{\partial t} = \frac{1}{c} \boldsymbol{i} \wedge \boldsymbol{B}_0 - \operatorname{grad} p_1 \tag{111}$$

$$\operatorname{curl} \boldsymbol{B}_1 = \frac{4\pi}{c} \boldsymbol{i} \tag{112}$$

$$\operatorname{curl} (\boldsymbol{V} \wedge \boldsymbol{B}_0) = \frac{\partial \boldsymbol{B}_1}{\partial t} \tag{113}$$

where $\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_1$, $\rho = \rho_0 + \rho_1$, $p = p_0 + p_1$.

These equations are linear in ρ_1 $\boldsymbol{B}_1$ $\boldsymbol{i}$ and $\boldsymbol{V}$ and it follows, therefore, that a complex solution can be constructed from elementary Fourier components having a variation

$$e^{j(\omega t + kz)}.$$

At this point one may note that two types of hydromagnetic waves are evidently possible. Firstly the longitudinal waves in which $V = V_z$ and secondly the transversal waves in which $V_z = 0$.

For the first group the eqs. (110)-(113) have the following form

$$\omega \rho_1 = - \rho_0 k V \tag{114}$$

$$j\omega \rho_0 V = \frac{1}{c} i B_0 - jk p_1 \tag{115}$$

$$jk B_1 = \frac{4\pi}{c} i \tag{116}$$

$$k V B_0 = \omega B_1 \tag{117}$$

where $V = V_z$, $i = i_x$, $B_0 = B_{0y}$, $B_1 = B_{1y}$.

The structure of such a wave can be deduced from the form of these equations (fig. 86).

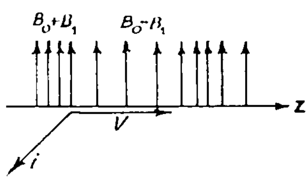


Fig. 86. Vector diagramme for a longitudinal hydromagnetic wave.

One requires an additional equation i.e., one connecting p_1 and ρ_1. Let us employ the equation of state for a monomolecular adiabatic gas with α degrees of freedom. Then

$$\frac{p_1}{p_0} = \left(\frac{\rho_1}{\rho_0}\right)\left(1 + \frac{2}{\alpha}\right). \tag{118}$$

Eliminating $p_1 B_1$ and i from eqs. (115), (116), (117) and (118) one obtains two equations for V and ρ_1

$$\omega\rho_1 + \rho_0 k V = 0 \tag{119}$$

$$k\left(1 + \frac{2}{\alpha}\right)\frac{p_0}{\rho_0}\rho_1 + \left(\omega\rho_0 - k^2\frac{1}{\omega}\frac{B_0^2}{4\pi}\right)V = 0. \tag{120}$$

The dispersion relationship for these longitudinal waves is obtained by putting the determinant of these equations equal to zero. Thus

$$\frac{\omega^2}{k^2} \equiv v_p^2 = \frac{B_0^2}{4\pi\rho_0} + \left(1 + \frac{2}{\alpha}\right)\frac{p_0}{\rho_0}. \tag{121}$$

Interpreting $B_0^2/8\pi$ as the magnetic pressure p_M one has

$$v_p = \sqrt{\frac{2p_M + (1 + 2/\alpha)p_0}{\rho_0}} \tag{122}$$

which can be compared with the equation for the speed of sound in the plasma. Evidently as $p_M/p_0 \to 0$, the velocity of a longitudinal hydromagnetic wave approaches the speed of sound. In the opposite extreme, i.e., when $p_M/p_0 \ll 1$, the velocity v_p approaches that of transversal hydromagnetic waves.

The equations for transversal waves can be appreciable simplified owing to the fact that in transversal waves (ref. 13)

$$\operatorname{div} \boldsymbol{V} = 0$$

$$\operatorname{grad} p_1 = 0.$$

Eqs. (110)-(113) become

$$\rho_0 \frac{\partial \boldsymbol{V}}{\partial t} = \frac{1}{c}\, \boldsymbol{i} \wedge \boldsymbol{B}_0 \tag{123}$$

$$\operatorname{curl} \boldsymbol{B}_1 = \frac{4\pi}{c}\, \boldsymbol{i} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} \tag{124}$$

$$\operatorname{curl} \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}_1}{\partial t} \tag{125}$$

$$\boldsymbol{E} = \frac{1}{c}\, \boldsymbol{V} \wedge \boldsymbol{B}_0. \tag{126}$$

It is instructive to retain the displacement current in the eq. (124) as in this way one is able to appreciate the relation of transversal hydromagnetic waves to transversal electron waves in a gyrotropic plasma (p. 149). If V is perpendicular to the direction of propagation (i.e., z) then it follows from eqs. (123)-(126) that $\boldsymbol{B}_0$ is longitudinal and $\boldsymbol{i}$, $\boldsymbol{E}$ and $\boldsymbol{B}_1$ are transversal (fig. 87).

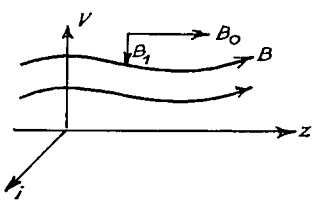


Fig. 87. Vector diagramme for a transversal hydromagnetic wave.

From eqs. (124) and (125) one has

$$\text{curl curl } \boldsymbol{E} = \frac{4\pi}{c^2} \frac{\partial}{\partial t} \boldsymbol{i} + \frac{1}{c^2} \frac{\partial^2 \boldsymbol{E}}{\partial t^2}. \tag{127}$$

Using eqs. (123) and (126), $\boldsymbol{i}$ can be expressed as

$$\begin{aligned} \boldsymbol{i} &= \frac{\rho_0 c}{\boldsymbol{B}_0} \frac{\partial \boldsymbol{V}}{\partial t} \\ &= \frac{\rho_0 c^2}{\boldsymbol{B}_0^2} \frac{\partial \boldsymbol{E}}{\partial t}. \end{aligned} \tag{128}$$

Substituting into eq. (127) one obtains a homogenous wave equation for $\boldsymbol{E}$

$$\text{curl curl } \boldsymbol{E} - \frac{1}{c^2} \left(1 + \frac{4\pi\rho_0 c^2}{\boldsymbol{B}_0^2}\right) \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0. \tag{127a}$$

Comparing this wave equation with that representing electromagnetic waves in a medium whose dielectric constant is ε one can ascribe to plasma (for frequencies lower-than ion-cyclotron frequencies) a dielectric constant

$$\varepsilon = 1 + \frac{4\pi\rho_0 c^2}{\boldsymbol{B}_0^2} \tag{129}$$

and a phase velocity, known as the Alfvén's speed

$$v_p \equiv \frac{c}{\sqrt{\varepsilon}} = \frac{c}{\sqrt{1 + \frac{4\pi\rho_0 c^2}{B_0^2}}}. \tag{130}$$

As $\rho_0 \to 0$ the character of the hydromagnetic wave changes into an electron wave and finally for $\rho_0 = 0$ into a free space electromagnetic wave.

Hydromagnetic waves in a plasma cylinder.

It was shown that electron waves on a plasma cylinder are a mixture of transversal and longitudinal oscillations. The same can be said about hydromagnetic waves.

Let us note that for $B^2/8\pi \gg p_0$ the dispersion relation for longitudinal waves is the same as that for transversal waves and therefore, a plasma cylinder can be compared to a dielectric cylinder whose dielectric constant is isotropic and given by eq. (129). However, this analogy cannot be applied to large amplitude oscillations. This can be understood as follows. In solving the wave equation for propagation on dielectric cylinders one assumes that the dielectric boundary is unperturbed by the electromagnetic field. This is not so for hydromagnetic waves in plasma cylinders, where the boundary moves with the speed $\boldsymbol{v}$ of the oscillating plasma. Apart from the mass motion, there will also be separate motion of the two plasma components, which at a plasma boundary will produce surface charge distribution. These polarisation charges occur on oscillating dielectric cylinders too but are associated with material displacements of only atomic dimensions.

If we restrict ourselves to a small amplitude theory, the results of the dielectric wave guide analysis are applicable to propagation of hydromagnetic waves in a plasma cylinder in a uniform magnetic

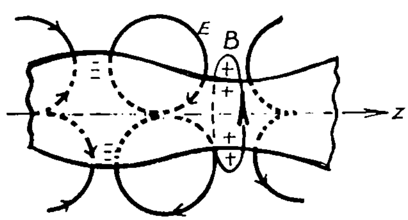


Fig. 88. Rotationally symmetric hydromagnetic wave on a plasma cylinder.

field $B_0 = B_{0z}$ in free space. In particular one finds that for circularly symmetrical modes (fig. 88) there is a cut-off frequency * at (ref. 14).

$$f_{\min} = \frac{2.4c}{2\pi r_0 \sqrt{\varepsilon - 1}}$$

$$= \frac{0.38}{r_0} \sqrt{\frac{B_0^2}{4\pi\rho_0}}. \tag{131}$$

There is no cut-off frequency for the dipole mode (fig. 89).

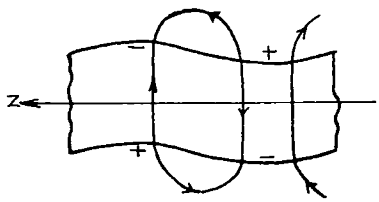


Fig. 89. Dipole hydromagnetic wave on a plasma cylinder.

Even within the limits of small amplitude analysis the dielectric analogy cannot be usefully applied to plasmas in non-uniform magnetic fields, such as, for instance, generated by an axial current in a plasma cylinder. Here the corresponding dielectric constant varies in space and an approximate analysis has to be used. The dielectric analogy fails also in cases in which the plasma pressure p_0 cannot be neglected in comparison with the magnetic pressure $B_0^2/8\pi$ and this is for two reasons. The first is that the dielectric constant is anisotropic, i.e., it depends on the angle between $\boldsymbol{v}$ and $\boldsymbol{B}_0$. The second reason is that the boundary conditions at the plasma surface are no longer a field-matching problem but are also a pressure-matching problem. If the pressure balance at the surface shows a lack of restoring force, a perturbation of the plasma may result in an instability. This problem will be treated in section 5.3.

An interesting characteristic of hydromagnetic waves is that the plasma moves together with the magnetic field lines, i.e., in a hydromagnetic motion *plasma fluid is coupled to the tubes of magnetic flux* or as Alfvén describes it the magnetic field-lines are 'frozen' into the moving plasma.

* I.e., a frequency below which guided waves cannot be propagated.

This follows directly from the theorem of conservation of magnetic moment, (p. 40) according to which a gyrating charged particle encircles always the same amount of magnetic flux. The assumptions, on which the theorem is based, of adiabatic variation in time and space of the magnetic field constitute a limitation of our treatment of hydromagnetic oscillations. Thus, e.g., collisions and large non-uniformities or high rates of change in the field $\boldsymbol{B}$ will permit a magnetic flux tube to penetrate a plasma (see later p. 226).

The first attempts to detect MHD waves in liquid metals did not meet with a great success owing to the large resistivity of these media (ref. 23). The first satisfactory study of these oscillations were made on cylindrical or toroidal plasmas (ref. 24). Transmission of slow transversal hydromagnetic waves along magnetic lines of earth's magnetic field has also been observed — a phenomenon known as whistlers.

5.2.3. HYDROMAGNETIC OSCILLATIONS IN PLASMA STREAMS

The analysis of hydromagnetic propagation on a plasma cylinder in an external axial magnetic field was based on the assumption that the plasma pressure was appreciably inferior to the magnetic pressure and therefore an idealized plasma density distribution could be considered, as the presence of such a tenuous plasma does not affect the externally generated magnetic field.

In situations, where the magnetic field is produced by a current distribution in a plasma one must choose a self-consistent equilibrium distribution of n, $\boldsymbol{v}$ and $\boldsymbol{B}$ in order to analyse the hydromagnetic oscillation of such current-carrying plasma.

In this section we shall consider the well known Bennett distribution in a fully neutralized electron stream. The relationship corresponding to this density distrbution was derived in chapter 4, p. 129.

We shall start from eqs. (4.44a) to (4.49) which still contain the $\partial/\partial t$ operators which are essential for problems concerned with oscillations (ref. 15).

In order to solve these equations we assume that, in spite of small amplitude oscillations, the plasma column does not depart appreciably from the sharply defined cylindrical structure obtained from the steady state analysis (eq. (4.53)).

Let us now describe the motion of a small volume element located at $r = r_0$, the characteristic radius of the Bennett distribution. The equations (4.44a) to (4.47a) become

$$\frac{\partial v_b}{\partial t} = -\frac{e}{m} E_0 + \frac{e}{mc}\frac{\partial A}{\partial t} + \frac{e}{mc} w_b \frac{\partial A}{\partial r}, \tag{132}$$

$$\frac{\partial w_b}{\partial t} = -\frac{e}{Mc}\, v_b \frac{\partial A}{\partial t} + \frac{2k(T_e + T_p)}{Mr_0}, \tag{133}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right) - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = \frac{4n}{c}\, nev_b, \tag{134}$$

where the subscript b denotes the quantities at the $r = r_0$ boundary.

If the Bennett density distribution is to apply at all times then the equation of continuity is satisfied only if

$$w = (r/r_0)w_b.$$

Evidently this does not correspond to a physically realizable model and one expects that for $r > r_0$ the Bennett distribution breaks down. This does not represent an appreciable error owing to the sharp fall in particle density in the region of $r > r_0$.

Eq. (4.50) can be simplified if one makes an approximation, which is consistent with the previous assumptions, that

$$2\pi\int_0^R n\frac{\partial}{\partial r}(rw)\, dr = (\pi w_b/r_0)\int_0^R nr\, dr = 2w_b N/r_0.$$

Then

$$\frac{T}{T_1} = \left(\frac{r_{10}}{r_0}\right)^{4/a} \tag{135}$$

where r_{10} is the initial value of r_0, $T_1 = T_{e1} + T_{p1}$ is the initial temperature of the plasma and α is the number of degress of freedom.

The vector potential A follows directly from eqs. (4.51) and (4.53) rather than from eq. (2a) as in the hydromagnetic approximation the displacement flux $(1/c)\,(\partial A/\partial t)$ is neglected. Thus

$$A = -\frac{e}{c}\, Nv \ln\frac{1 + R^2/r_0^2}{1 + r^2/r_0^2}. \tag{136}$$

From this one may derive both $\partial A_b/\partial t$ and $\partial A_b/\partial r$.

These are

$$\frac{\partial A_b}{\partial t} = -\frac{e}{c}Nv\left\{\frac{w_b}{r_0}\frac{1 - R^2/r_0^2}{1 + R^2/r_0^2} + \frac{1}{v}\frac{dv}{dt}\left[\ln\left(1 + \frac{R^2}{r_0^2}\right) - \ln 2\right]\right\}, \tag{136a}$$

$$\frac{\partial A_b}{\partial r} = \frac{e}{c} - Nv\frac{1}{r_0}. \tag{136b}$$

When one substitutes eqs. (135), 136a) and (136b) into (132) and (133) one obtains (writing now d/dt for $\partial/\partial t$ and r for r_0)

$$\frac{\mathrm{d}v}{\mathrm{d}t}\left\{1+\nu\ln \tfrac{1}{2}\left(1+\frac{R^2}{r^2}\right)\right\}=-\frac{e}{m}E_0+\nu\frac{vw}{r}\frac{2R^2/r}{1+R^2/r^2} \quad (137)$$

$$\frac{\mathrm{d}w}{\mathrm{d}t}=-\frac{m}{M}\frac{\nu v^2}{r}+\frac{2kT_1r_{10}{}^{4/a}}{M}r^{-(4/a+1)}. \quad (138)$$

For large R/r one has

$$\ln \tfrac{1}{2}\left(1+\frac{R^2}{r^2}\right)\approx 2\left(\ln\frac{R}{r}-0.35\right).$$

Let us define a mass $m^* = m\{1 + 2\nu[\ln(R/r) - 0.35]\}$ which is the effective longitudinal mass of the electrons in the stream *.

Provided $\nu \leqslant 1$ and the changes in r_0 are not larger than two orders of magnitude one may assume that

$$\frac{m^*}{m}\approx \text{const.} = \eta^{-1}.$$

The final form of eqs.(137) and (138) is, therefore,

$$\frac{\mathrm{d}v}{\mathrm{d}t}=-\frac{e}{m^*}E_0+2\eta\nu\frac{vw}{r}, \quad (137a)$$

$$\frac{\mathrm{d}w}{\mathrm{d}t}=-\frac{m}{M}\frac{\nu v^2}{r}+ac^2r^{-(4/a+1)} \quad (138a)$$

where $a = 2kT_1r_{10}{}^{4/a}/Mc^2$.

Small-amplitude solution

Eqs. (137a) and (138a) can be solved analytically for small oscillations of the quantities v, w and r_0. Let us put

$$v = v_0 + v_1; \qquad w = w_0 + \dot{r}_1; \qquad r = r_0 + r_1; \qquad E_0 = 0.$$

Neglecting all second-order quantities one has

$$\dot{v}_1 = 2\eta\nu v_0\frac{\dot{r}_1}{r_0} \quad (139)$$

$$\ddot{r}_1=-\frac{m}{M}\nu\frac{v_0{}^2}{r_0}\left(\frac{2v_1}{v_0}-\frac{r_1}{r_0}\right)-\left(\frac{4}{a}+1\right)\frac{ac^2r_1}{r_0{}^{4/a+2}} \quad (140)$$

* The added inertia, represented by the term $2\nu[\ln(R/r) - 0.35]$, results from the participation of each electron in the generation of the magnetic field B_ϕ around the beam.

If the quantities v_1 and r_1 change harmonically with time, i.e., as $e^{j\omega t}$, one obtains from these equations a characteristic equation for ω

$$\omega = \left\{ \frac{m}{M} \left(\frac{v_0}{r_0}\right)^2 (4\eta\nu - 1)\nu + \left(\frac{4}{\alpha} + 1\right) \frac{ac^2}{r_0^{4/a+2}} \right\}^{1/2}. \quad (141)$$

From the original eq. (138a) one gets for the steady state

$$ac^2 r_0^{-(4/a+2)} = \frac{m}{M} \nu \frac{v_0^2}{r_0^2}.$$

Substituting this into eq. (141) the frequency of small amplitude oscillations of the plasma channel becomes

$$\omega = 2 \frac{v_0}{r_0} \nu \sqrt{\left\{ \frac{m}{M} \left(\eta + \frac{4}{\alpha}\right) \right\}}. \quad (141a)$$

The value of α depends on the ratio of collision frequency between plasma particles and the hydromagnetic frequency ω. The value of this ratio for the electron gas is different from that for the positive ion gas. In all cases α lies in the range

$$2 < \alpha < 3.$$

The frequency given by eq. (141a) is, of course, the frequency of the circularly symmetrical mode of infinite wavelength only and in this sense it represents a cut-off frequency, i.e. the lowest frequency at which radial oscillations can be supported by the plasma stream. A true dispersion relationship would be obtained only if the $\partial/\partial z$ operator had been retained in eqs. (4.40) to (4.43). For waves whose guide-wave length $\lambda_g = 2\pi/k$ is shorter than r_0 one can assume that their propagation is similar to that in an infinite plasma, i.e. that their dispersion relationship is that of eq. (130), where B_0 and ρ_0 are to be interpreted as some characteristic mean values of $\partial A/\partial r$ and nM, such as the values of these quantities encountered in the steady state at r_0.

5.3. Growing Waves and Instabilities

An instability of a system of fields and particles has been defined on p. 144 as a unidirectional conversion of one type of energy of this system into another.

Every system can be characterised by a set of parameters ξ_k. Let us consider an arbitrary small change $\delta\xi_k$ in each of these parameters. To each $\delta\xi_k$ corresponds a change δW_k of energy W of the system. The total change δW due to an arbitrary combination of $\delta\xi_k$'s is

$$\delta W = \sum_k \delta W_k.$$

According to the well known principle of virtual displacements a system is in equilibrium if the first variation δW of its total energy is equal to zero.

In order that the above mentioned energy conversion should not occur, i.e. that the system is in stable equilibrium it is necessary that not only the first variation must be equal to zero but also the second variation must be positive.

This can be illustrated in the case of a drop of liquid, having a certain amount of kinetic energy $1/2MV^2$ on a slope of a potential hill (fig. 90). A stable situation, exists evidently, at $W = W_3$ where

$$\frac{\partial W}{\partial \xi} = 0, \quad \frac{\partial^2 W}{\partial \xi^2} > 0 \text{ assuming that } \Delta W > \tfrac{1}{2}MV^2.$$

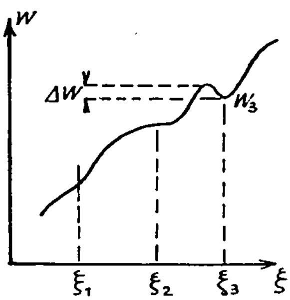


Fig. 90. Energy hill.

One of the most frequently used methods for investigating plasma stability is the "normal mode analysis". This consists of setting up a linearized wave equation in the plasma and outside it. The wave is supposed to have a variation of exp $(j\,\omega t)\,.\,F(\boldsymbol{k}\,.\,\boldsymbol{r})$. The matching of the two wave equations at the plasma boundary yields a dispersion relation

$$\omega = \omega(k).$$

The evidence for stability is that ω is real for all real values of k.

The normal mode method is often applied to systems assumed to be loss-less. However, it is well known in the theory of oscillators, especially microwave oscillators, that oscillations will build up only when the electron current injected into such an oscillator exceeds a certain minimum value, known as the starting current. This corresponds, in plasma physics, to a certain minimum of energy which must be available in the system, before it can be converted into either electromagnetic or kinetic energy of the plasma.

We shall use the normal mode analysis in this section as it follows naturally from the fluid equations developed so far. At the same time we shall try to evaluate a quantity corresponding to a "starting current" or at least point out the energy aspect of instabilities predicted by the normal mode analyses.

This section is divided into sub-sections dealing with different types of energy conversion.

5.3.1. CONVERSION OF KINETIC ENERGY OF PARTICLE STREAMS INTO THE ENERGY OF LONGITUDINAL PLASMA OSCILLATIONS

Let us consider an infinite, neutral and uniform plasma in which the electron gas has a velocity $v_z = v_0$ and the positive ion gas a velocity $w_z = w_0$. Let us assume that both these steady velocities are perturbed by a small amplitude wave, whose space-time variation is $e^{j(\omega t + kz)}$. The situation is then described by eqs. (3.56a), (3.57) which in our case can be linearised and become *

$$\mathrm{j}\omega v_1 + \mathrm{j}kv_1v_0 = -\frac{e}{m}E_1 - \mathrm{j}k\frac{n_{1e}}{n_e}\overline{u_e^2} \tag{142}$$

$$\mathrm{j}\omega w_1 + \mathrm{j}kw_1w_0 = \frac{eZ}{M}E_1 - \mathrm{j}k\frac{n_{1p}}{n_p}\overline{u_p^2}. \tag{143}$$

The relationships for E_1 and n_e and n_p follow from the equation for divergence of E_1 and from the continuity equation (see eqs. (105) (106a,b)). Thus

$$\mathrm{j}kE_1 = 4\pi e(Zn_{1p} - n_{1e}) \tag{144}$$

$$\mathrm{j}\omega n_{1e} = -\mathrm{j}k(n_{1e}v_0 + n_ev_1) \tag{145}$$

$$\mathrm{j}\omega n_{1p} = -\mathrm{j}k(n_{1p}w_0 + n_pw_1). \tag{146}$$

Substituting E_1, n_{1e} and n_{1p} from eqs. (144)-(146) into eqs. (142) and (143) one obtains (choosing $Z = 1$)

$$v_1[\omega_{pe}^2 - (\omega + kv_0)^2 + k^2\overline{u_e^2}] - w_1\omega_{pe}^2\frac{\omega + kv_0}{\omega + kw_0} = 0 \tag{142a}$$

* Compare with eqs. (103), (104). Hypothesis of isothermal plasma retained in eqs. (142) and (143).

$$-v_1\frac{m}{M}\omega_{pe}{}^2\frac{\omega+kw_0}{\omega+kv_0}+w_1\left[\frac{m}{M}\omega_{pe}{}^2-(\omega+kw_0)^2+k^2\overline{u_p{}^2}\right]=0. \quad (143a)$$

The dispersion equation is derived by putting the determinant of this system equal to zero. Thus

$$[-\omega_{pe}{}^2+(\omega+kv_0)^2-k^2\overline{u_e{}^2}]$$
$$\left[-\frac{m}{M}\omega_{pe}{}^2+(\omega+kw_0)^2-k^2\overline{u_p{}^2}\right]=\frac{m}{M}\omega_{pe}{}^4.$$

Putting $(m/M)\ \omega_{pe}{}^2=\omega_{pp}{}^2$, which is the ion plasma frequeney this equation can be written

$$\frac{\omega_{pe}{}^2}{(\omega+kv_0)^2-k^2\overline{u_e{}^2}}+\frac{\omega_{pp}{}^2}{(\omega+kw_0)^2-k^2\overline{u_p{}^2}}=1. \quad (147)$$

This equation is of the 4th order in ω. The condition for stability is that all the four roots $\omega_i(k)$ are real numbers.

Let us consider first a simplified case in which

$$w_0=0, \qquad \overline{u_p{}^2}=\overline{u_e{}^2}=0.$$

Eq. (147) becomes

$$\frac{1}{(\omega+kv_0)^2}+\frac{m}{M}\frac{1}{\omega^2}=\frac{1}{\omega_{pe}{}^2}. \quad (147a)$$

Let us plot the left-hand side of this equation as a function $f(\omega)$ (fig. 91). Imaginary roots appear for

$$f(\omega)_{\min}>\frac{1}{\omega_{pe}{}^2}.$$

where *

$$f(\omega)_{\min}=\left[1+\left(\frac{m}{M}\right)^{1/2}\right]^3(kv_0)^{-2}$$

from which it follows that for

$$k^2<\left[1+\left(\frac{m}{M}\right)^{1/2}\right]^3\frac{\omega_{pe}{}^2}{v_0{}^2} \quad (148)$$

the electron stream will become unstable and convert its kinetic energy into longitudinal electric field E_1. The same results is obtained

* As the minimum is very near to $\omega=0$, we may take as a first approximation $f(\omega)_{\min}\cong\dfrac{1}{k^2v_0{}^2}$. The correcting term $\left(\dfrac{m}{M}\right)^{1/3}$ corresponds to a precise calculation of the minimum $f(\omega)$.

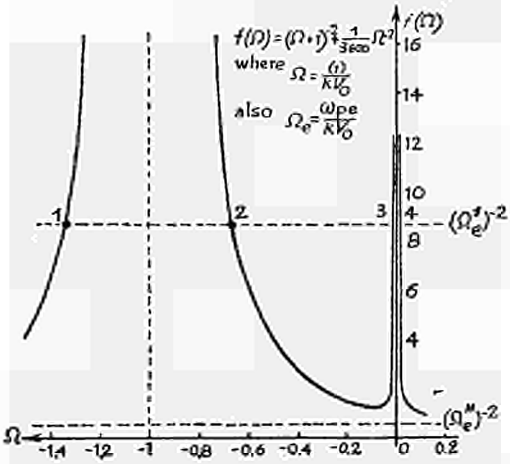


Fig. 91. Graphical representation of eq. (147a).
Ω'_e belongs to a stable mode, Ω''_e to an unstable one.

for a relativistic stream, only m must be interpreted as the longitudinal mass $m_0\gamma^3$. Thus

$$k^2 < \left[\frac{1}{\gamma} + \left(\frac{m_0}{M}\right)^{1/2}\right]^3 \frac{\omega_{pe}^2}{v_0^2}. \tag{148a}$$

This type of instability occurs, therefore, at long wave lengths. It is obvious that the smaller ω_{pe} (i.e., the lower the plasma density) the higher v_0 and the higher γ the longer is the wave length at which instabilities set in.

Instabilities will occur for wave lengths

$$\lambda_{\text{instab}} > 2\pi \frac{v_0}{\omega_{pe}} \left[1 - \frac{3}{2}\left(\frac{m_0}{M}\right)^{1/2}\right] \text{(cm)}. \tag{149}$$

This result is not applicable to a plasma whose dimension D at right angles to v_0, w_0 and E_1 is smaller than $1/2\ \lambda_{\text{instab}}$ as in that case, the frequencies ω_{pe}, ω_{pp} of the longitudinal oscillations depend on k, i.e., the field equation is no longer a simple div E_1 equation. A similar problem, namely one concerned with propagation of electromagnetic waves on plasma cylinder, was treated on p. 155 and the corresponding dispersion relationship found (eq. (50b)). It can be shown that for small k's this dispersion relation is approximately

$$\omega \approx \frac{2\pi r_0}{\lambda} \omega_{pe} \left[\ln \frac{\lambda}{2\pi r_0}\right]^{1/2}. \tag{150}$$

This is the frequency that must be used in eqs. (148)-(149) instead of ω_{pe}. The criterion for instability for cylindrical streams thus becomes

$$\ln \frac{\lambda_{instab}}{2\pi r_0} > \frac{1}{r_0^2} \cdot \frac{v_0^2}{\omega_{pe}^2} \left[1 - 3 \left(\frac{m_0}{M} \right)^{1/2} \right] \tag{151}$$

which in many cases yields λ_{instab} larger than the length of plasma cylinders likely to be used in practice.

So far we have not taken into account the energy losses from the longitudinal mode E_1. These may be represented by a damping coefficient δ operating on the assumed form of space-time variation of E_1. The total variation is then

$$e^{j(\omega t + kz) - \delta t}.$$

It follows that if an instability generating mechanism exists it must be associated with an imaginary component of ω

$$\mathrm{Im}(\omega) > -\delta$$

in order that the corresponding perturbation e^{jkz} can grow inspite of the energy loss.

This energy loss from modes propagated in plasma cylinders may be very large as such modes can be coupled to a suitable waveguide outside the plasma.

5.3.2. RAYLEIGH-TAYLOR INSTABILITY

Of particular interest is the instability in which potential energy of a fluid is steadily converted into its kinetic energy. A classical example of this situation is a perturbed equilibrium of a fluid supported by a lighter fluid against gravitational forces. When the heavy fluid is a plasma (or another electrically conducting fluid) it is possible to substitute for the light fluid a magnetic field. In such a case we require

$$\rho_m = \frac{B^2}{8\pi c^2} \ll n(m + M). \tag{152}$$

As on the boundary the pressure of the plasma is equal to that of the magnetic field it is evident that a non relativistic plasma will always behave as a heavy fluid when supported by a magnetic field.

Let us now consider a slab of a perfectly conducting fluid of thickness a supported against a gravitational field g by a uniform magnetic field $\boldsymbol{B}$ (fig. 92). Let us further suppose the fluid is of uniform density

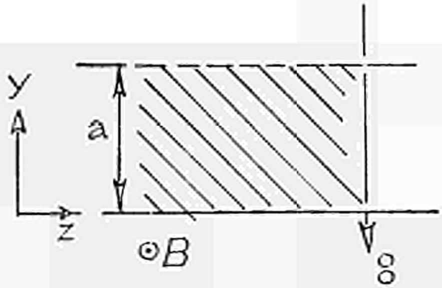


Fig. 92. Conducting fluid layer supported by a magnetic field against a gravitational field g.

ρ (therefore, incompressible). The last assumption is not necessary and is introduced only to simplify the analysis (ref. 25).

The equilibrium requires that

$$\frac{B^2}{8\pi} = p = a\rho g. \tag{153}$$

Let us now consider a perturbation periodic in z, which does not bend lines of magnetic field. It will be shown later that such a perturbation leads often to the so called exchange instability and is generally the most readily growing one. We shall use the method of oscillating modes for the analysis of the stability of our system.

The displacement of an element of the fluid at y, z is then

$$\xi = f(y) \cdot \exp\,[\mathrm{j}(\omega t + kz)] \tag{154}$$

and the function $f(y)$ is equal to ξ_1 at $y = 0$ and ξ_2 at $y = a$.

The velocity in the fluid is $v = \dot{\xi}$. Since initially rot $v = 0$ then owing to the conservation of angular momentum rot $v = 0$ at any time. The velocity field is, therefore, irrotational and can be derived from a potential φ, i.e.,

$$v = -\nabla\varphi. \tag{155}$$

As the fluid is supposed incompressible

$$\operatorname{div} v = 0$$

and consequently

$$\Delta\varphi = 0. \tag{156}$$

The solution of this equation in two dimensions y and z is well known, i.e.,

$$\varphi = (Ae^{ky} + Be^{-ky}) \cdot e^{\mathrm{j}(\omega t + kz)}. \tag{157}$$

Our problem evidently corresponds to the one-fluid-model. The equation of motion related to this model is

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\operatorname{grad}(gy) - \frac{1}{\rho}\operatorname{grad} p. \tag{158}$$

Using eq. (5) we get

$$\frac{\partial \varphi}{\partial t} = gy + \frac{p}{\rho} + C. \tag{159}$$

The constant C can beasily determined, aussuming that originally the perturbation $\varphi = 0$. Then for $y = a$ we have $p = 0$ and $C = -ga$.

Let us write these equations for $y = 0$ and for $y = a$ in order to find the constants A and B.

We get

$$\mathrm{j}\omega\varphi_a + \frac{g}{\mathrm{j}\omega}\left(\frac{\partial\varphi}{\partial \mathrm{y}}\right)_a = 0 \tag{160a}$$

$$-\mathrm{j}\omega\varphi_0 - \frac{g}{\mathrm{j}\omega}\left(\frac{\partial\varphi}{\partial \mathrm{y}}\right)_0 = 0. \tag{160b}$$

Substituting in eq. (160) from eq. (157) for φ have

$$A(\omega^2 - kg)e^{ka} + B(\omega^2 + kg)e^{-ka} = 0 \tag{161a}$$

$$A(\omega^2 - kg) + B(\omega^2 + kg) = 0. \tag{161b}$$

Putting the determinant of this system equal to zero we obtain five solutions

$$k = 0, \qquad \omega^2 = kg, \qquad \omega^2 = -kg.$$

The first one corresponds to a trivial case of a static equilibrium. The second and third solutions represent steady ascillations of the

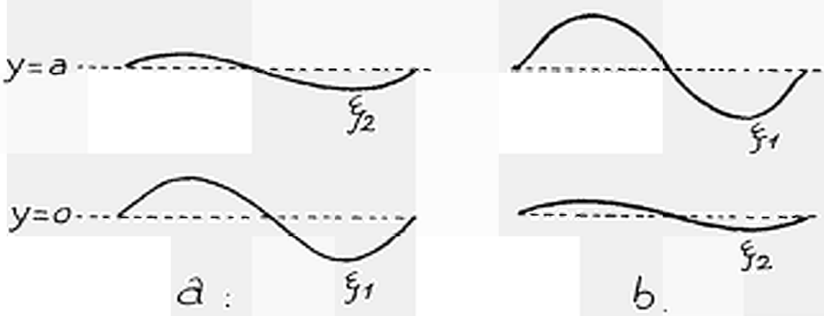


Fig. 93.

a : instability b . surface wave

slab with frequencies $\pm\sqrt{kg}$, therefore, waves propagating either in the $+z$ or $-z$ rirection. The perturbation resembles a surface wave whose amplitude is largest at the upper surface of the fluid (fig. 93b).

The last two solutions, i.e., $\omega = \pm j\sqrt{kg}$ imply a damped and a growing perturbation, the growing one obviously related to an instability of the system in which

$$\xi \propto e^{-ky} \cdot e^{\sqrt{kg} \cdot t}. \tag{162}$$

The perturbation at $y = a$ is, therefore, smaller than that at $y = 0$ (fig. 93a).

In case of a compressible fluid in equilibrium, the density distribution is that of a gaseous atmosphere in a gravitational field, i.e.,

$$\rho = \rho_0 e^{-y/\delta} \tag{163}$$

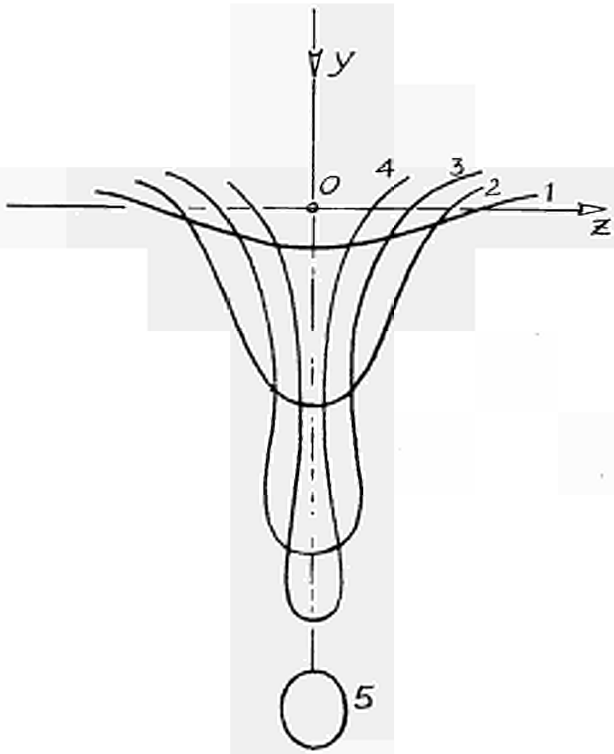


Fig. 94.

where δ is known as Haley's thickness and

$$\delta = \frac{p_0}{\rho_0 g}. \tag{164}$$

The result of the analysis of the Rayleigh-Taylor instability is, in this case, the same as in the case of the incompressible fluid, the distance a must be substituted by δ.

The assumption of small amplitude perturbation always made when the method of oscillating modes is used implies that

$$\xi \ll a \qquad \text{and also} \qquad \xi \ll \frac{\lambda}{2} = \frac{\pi}{k}.$$

This assumption makes it possible to linearise the equations and resolve them analytically. When the amplitude of the unstable mode becomes comparable or larger than $\frac{\lambda}{2}$ the problem has to be resolved by numerical methods, or by experiments. In the case of the Rayleigh-Taylor instability it has been thus demonstrated that the perturbation grows into tongues of fluid which finally form drops (or rather detached cylinders of fluid, fig. 94). The exponential growth-rate (eq. (162)) is no longer applicable and the drops fall freely in the gravitational field.

The $\xi_y(t)$ at the surface of the growing instability is represented by a curve which starts as an exponential and after a time τ of the order of several times $(kg)^{-1/2}$ merges into a linear section $\dot{\xi} \sim gt + \text{const.}$

5.3.3. MAGNETOHYDRODYNAMIC INSTABILITY

This instability has to do with a transformation of thermal (internal) energy of plasma into its kinetic energy and in some cases also into the energy of the magnetic field.

a. Stability of a linear Z-pinch

Let us consider a cylindrical column of plasma of radius r_0, surrounded by a perfectly conducting rigid wall of radius $R = \beta r_0$. There will be three magnetic fields in this system: B_φ and B_{zv} outside the plasma and B_{zp} trapped inside the plasma. Let us also make the following assumptions.

1. All the currents are surface-currents (i.e., the thickness of the surface current-layer is much smaller than r_0).

2. Plasma is a perfect conductor. There will be, therefore, no

diffusion of magnetic fields and we shall be able to use the MHD model.

3. Our pinch is initially in equilibrium (p. 135) and is perturbed by a small amplitude ripple whose form is exp. $j\,(kz + m\varphi + \omega t)$.

4. All dynamical processes are isentropic (no shocks).

We shall use the following nomenclature $B_\varphi = B_0 \dfrac{r_0}{r}$, $B_{zv} = \alpha_v B_0$, $B_{zp} = \alpha_p B_0$. This allows us to measure all fields in terms of B_0, the $B\varphi$ field at the plasma surface.

We shall use the method of normal modes. The result which we shall be looking for is of the form

$$\omega = \omega(\alpha_v, \alpha_p, \beta, k, m) \tag{165}$$

real ω corresponds to stationary oscillation, complex ω to either damped oscillations or to instability.

We shall proceed as follows: corresponding to the form of the perturbation we shall find the perturbed magnetic field outside and the perturbed magnetic field and plasma pressure inside the plasma. Matching the two solutions (outside and inside one) at the plasma boundary gives a characteristic equation and the dispersion relation

Let us now write the equations corresponding to the MHD model and to our case.

First the Maxwell's equations

$$\frac{1}{c}\dot{\boldsymbol{B}} = \operatorname{rot}\left(\frac{1}{c}\boldsymbol{v}\wedge\boldsymbol{B}\right), \qquad \operatorname{rot}\boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{i}, \qquad \operatorname{div}\boldsymbol{B} = 0, \tag{166a,b,c}$$

then the equations of continuity and of plasma motion

$$\dot{\rho} = \operatorname{div}\rho\boldsymbol{v}, \qquad \rho\dot{\boldsymbol{v}} = \frac{1}{c}\boldsymbol{i}\wedge\boldsymbol{B} - \operatorname{grad} p \tag{167a,b}$$

and finally the equation of state in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(p\rho^{-\gamma}) = 0. \tag{168}$$

Let us now find the expression for the perturbed magnetic field in vacuum.

This can be written as

$$\boldsymbol{B} = \boldsymbol{B}_\varphi + \alpha_v\boldsymbol{B}_0 + \delta\boldsymbol{B} = \boldsymbol{F} + \delta\boldsymbol{B}. \tag{169}$$

As there are no currents outside the plasma

$$\text{rot } \delta \boldsymbol{B} = \text{div } \delta \boldsymbol{B} = 0 \tag{170}$$

which allows us to write

$$\delta \boldsymbol{B} = \nabla \psi \qquad \text{and} \qquad \Delta \psi = 0. \tag{171a,b}$$

The solution of this equation in cylindrical geometry is

$$\psi = [a\, J_m(jkr) + b\, H_m(jkr)]\, e^{j(kz+m\varphi)}. \tag{172}$$

The constants a and b are to be determined from the boundary condition

$$\boldsymbol{B} \cdot \boldsymbol{n} = 0 \tag{173}$$

on $r = R$ and on the surface of the plasma, $\boldsymbol{n}$ being the vector normal to these surfaces. On the outer surface we have

$$\delta B_r = \left(\frac{\partial \psi}{\partial r}\right)_{r=\beta r_0} = 0. \tag{174}$$

The plasma surface is described by

$$r = r_0\, [1 + \delta_0\, e^{j(kz+m\varphi)}]. \tag{175}$$

The component of the field normal to the surface is

$$\left(\frac{\partial \psi}{\partial r}\right)_{r=r_0} = jB_0\delta_0\, (kr_0\alpha_v + m)\, e^{j(kz+m\varphi)} = 0. \tag{176}$$

From equations (172)-(176) we have

$$a = B_0 \frac{\dfrac{\delta_0}{k}\,(k\alpha_v r_0 + m)}{J'_m(jkr_0) - H'_m(jkr_0)\,\dfrac{J'_m(jkr_0\beta)}{H'_m(jkr_0\beta)}} \tag{177}$$

$$b = -a\, \frac{J'_m(jkr_0\beta)}{H'_m(jkr_0\beta)}. \tag{178}$$

Knowing $\delta \boldsymbol{B}$ it is easy to calculate the magnetic pressure δp_v from the outside on the plasma surface. Thus

$$\delta p_v = \delta \left(\frac{B^2}{8\pi}\right) = \frac{B_0}{4\pi}\, (\delta B_\varphi + \alpha_v \delta B_{zv} - B_0\delta_0 \cdot e^{j(kz+m\varphi)}). \tag{179}$$

Using equations (171a), (172), (177) and (178) we get

$$\delta p_v = \delta_0 \frac{B_0^2}{8\pi}\, e^{j(kz+m\varphi)} \left\{ (m + \alpha_v Y)^2\, \frac{G_{\beta m}K_m - L_m}{1 - G_{\beta m}} - 1 \right\} \tag{180}$$

where $Y = kr_0$, $K_m = \dfrac{J_m(\mathrm{j}Y)}{\mathrm{j}YJ'_m(\mathrm{j}Y)}$, $L_m = \dfrac{H_m(\mathrm{j}Y)}{\mathrm{j}YH'_m(\mathrm{j}Y)}$

$$G_{\beta m} = \frac{H'_m(\mathrm{j}\beta Y)}{H'_m(\mathrm{j}Y)} \cdot \frac{J'_m(\mathrm{j}Y)}{J'_m(\mathrm{j}\beta Y)}$$

Let us now find the perturbation in the plasma. It is convenient to introduce (as we did on p. 183) a new variable

$$\xi = \int_0^t v \, \mathrm{d}t \tag{181}$$

which represents the displacement of an element of fluid. Taking into account only first order terms, equation (166a) becomes

$$\delta\dot{\boldsymbol{B}} = \mathrm{rot}\ (\boldsymbol{v} \wedge \boldsymbol{B}) \tag{182}$$

and integrating both sides and using equation (181) one has:

$$\delta\boldsymbol{B} = \mathrm{rot}\ (\xi \wedge \boldsymbol{B}). \tag{183}$$

The equation of motion can be written as

$$\rho\ddot{\xi} = \frac{1}{4\pi}\ (\mathrm{rot}\ \delta\boldsymbol{B}) \wedge \boldsymbol{B} - \mathrm{grad}\ \delta p. \tag{184}$$

Substituting eq. (183) into eq. (184) we get

$$\rho\ddot{\xi} = \frac{1}{4\pi}\ [\mathrm{rot}\ \mathrm{rot}\ (\xi \wedge \boldsymbol{B})] \wedge \boldsymbol{B} - \mathrm{grad}\ \delta p. \tag{185}$$

The equation of continuity written in Langrangian coordinates becomes

$$\delta\rho = -\rho_0\ \mathrm{div}\ \xi \tag{186}$$

and the equation of state and eq. (186) gives

$$\delta p = -\gamma p_0\ \mathrm{div}\ \xi. \tag{187}$$

Using eq. (187) the equation of motion becomes finally

$$-4\pi\rho\omega^2\xi = \boldsymbol{B} \wedge \mathrm{rot}\ \mathrm{rot}\ (\boldsymbol{B} \wedge \xi) + 4\pi\gamma p_0\ \mathrm{grad}\ \mathrm{div}\,\xi. \tag{188}$$

The perturbation exp. $j\ (kz + m\varphi)$ is symmetrical in both z and φ and when ω is real, the difference between exp. $(j\omega t)$ and exp. $(-j\omega t)$ is only in the direction of propagation. It is evident that the solution of equation (188) must be ω^2 so that both $+\omega$ and $-\omega$ is a solution. Changing one of the parameters of the perturbation, e.g., k, both roots $+\omega$ and $-\omega$ will move symmetrically along the real axis in the ω-plane (fig. 95). Before the $\omega(k)$ values can become purely imaginary they must pass through $\omega(k) = 0$. This corresponds to the passage from a stably oscillating system $(k < k_0)$ through labile situation $(k = k_0)$,

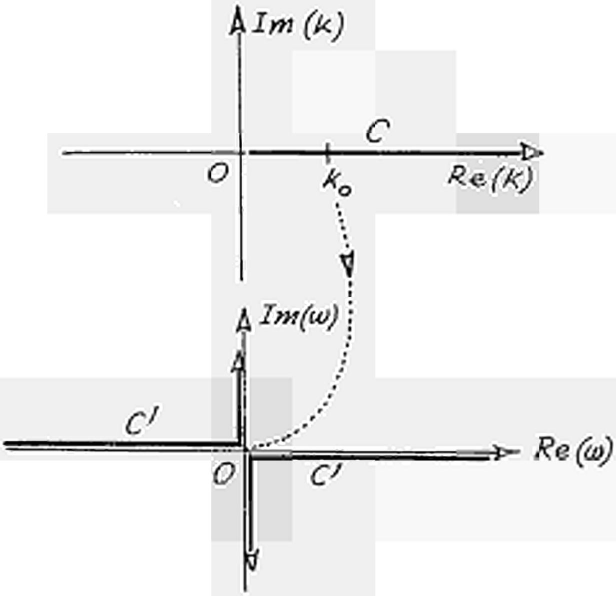


Fig. 95.

to an unstable system ($k > k_0$). This can be represented by the energy well of diminishing depth in fig. 96.

Let us now examine the case $\omega = 0$, which represents the margin

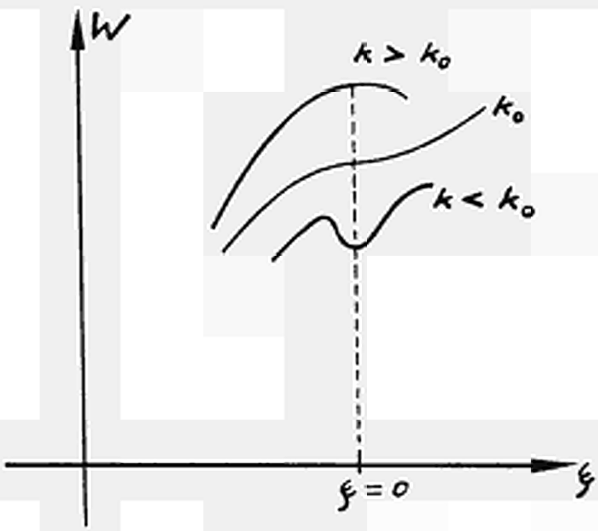


Fig. 96.

of stability. As the solenoidal and irrotational parts of ξ cannot cancell each other, it is obvious from equation (188) that

$$\text{rot } (\xi \wedge B) = 0 \qquad \text{and} \qquad \text{div}\, \xi = 0 \tag{189a,b}$$

which is equivalent to saying that

$$\xi = \text{grad } \phi \qquad \text{where} \qquad \Delta\phi = 0. \tag{190a,b}$$

We have, therefore, in cylindrical geometry, in the region containing the axis,

$$\xi = c \text{ grad } J_m(jkr)\ e^{j(kz+m\varphi)}. \tag{191}$$

The boundary condition on $r = r_0(1 + \delta_0 e^{j(kz+m\varphi)})$ gives

$$\xi_r = r_0\delta_0\ e^{j(kz+m\varphi)} \tag{192}$$

from which

$$c = \frac{r_0\delta_0}{jkJ'_m(jkr_0)}. \tag{193}$$

It is now possible to calculate the change δp_p in the internal pressure on the plasma surface. As the volume of the plasma did not change the kinetic pressure of the plasma remains constant and the δp_p corresponds to only the changed magnetic pressure inside the plasma. Thus

$$\delta p_p = \delta \left(\frac{B^2{}_{zp}}{8\pi}\right)_{r=r_0} = \left|\frac{B_{zp}}{4\pi}\,\delta B_z\right|_{r=r_0} \tag{194}$$

From eq. (183) and with eq. (189b) follows that

$$\delta B_z = (B\,\nabla)\,\xi = B_z \frac{\partial \xi_z}{\partial z} = \alpha_p B_0 \frac{\partial^2 \phi}{\partial z^2}. \tag{195}$$

Using eqs. (190) and (191) we get

$$\delta p_p = \alpha_p{}^2\delta_0 \frac{B_0{}^2}{8\pi}\ e^{j(kz+m\varphi)}\ Y^2 K_m(Y). \tag{196}$$

To a certain amplitude δ_0 of the perturbation corresponds, therefore, a pressure difference

$$\delta p = \delta p_p - \delta p_v. \tag{197}$$

Since our system is labile (indifferent equilibrium, $\omega = 0$) it follows that $\delta p = 0$. For $\delta p > 0$ and $r - r_0 > 0$ the surface will be displaced further and the perturbation is unstable. Using eqs. (180) and (196) we get

$$\delta p \propto \alpha_p^2 Y^2 K_m - (m + \alpha_v Y)^2 \frac{G_{\beta m} K_m - L_m}{1 - G_{\beta m}} + 1 = 0 \qquad (198)$$

and the condition for stability becomes

$$- \alpha_p^2 Y^2 K_m + (m + \alpha_v Y)^2 \frac{G_{\beta m} K_m - L_m}{1 - G_{\beta m}} \geqslant 1. \qquad (199)$$

This criterion can be interpreted as follows. Let us choose the geometry of the perturbation $\left(\text{i.e., } m \text{ and } k = \dfrac{Y}{r_0}\right)$ and the geometry of the magnetic fields (i.e., α_v and α_p). It is then possible to calculate from equation (199) the $\beta = \dfrac{R}{r_0}$ corresponding to a plasma column, stable against the perturbation considered. This gives a curve in the graph α_p^2, β (fig. 97). To every m corresponds a different curve. The simplest graph is obtained for $\alpha_v = 0$. In such a case no equilibrium is possible for $\alpha_p > 1$ (p. 135, eq. (77)). Above each curve (for fixed m) is the

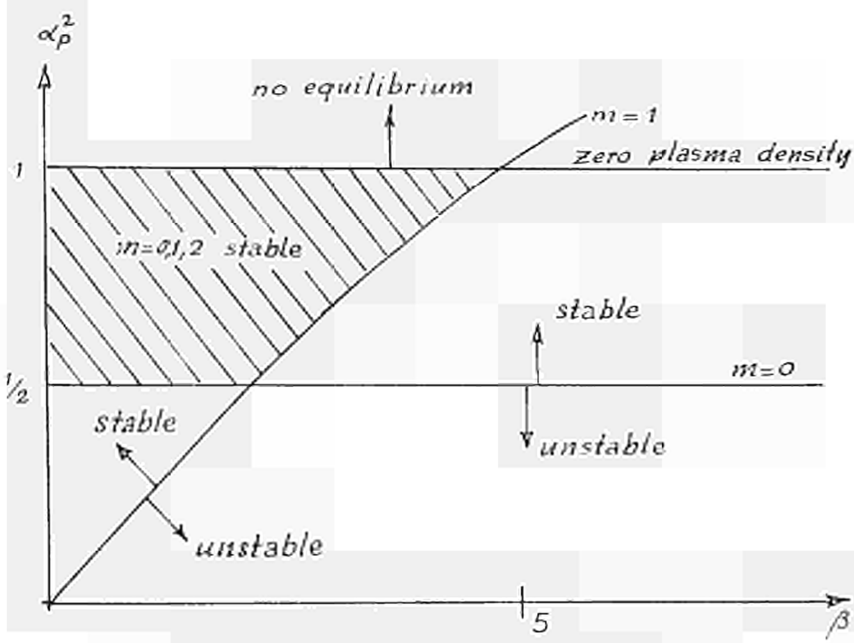


Fig. 97. Stability diagram for $m = 0, 1, 2$ and $\alpha_v = 0$.

zone in which the plasma is stable. The common zone of stability for all m is the zone of absolute stability.

The most dangerous instabilities are the $m = 0$ and $m = 1$. The $m = 0$ is known as sausage instability, whereas the $m = 1$ is called the kink instability (fig. 98). It is seen from fig. 97 that the sausage instability can be suppressed by making $\alpha_p^2 > \frac{1}{2}$. Stability against kinks can be obtained only for tube to plasma radius ratios smaller than 5.

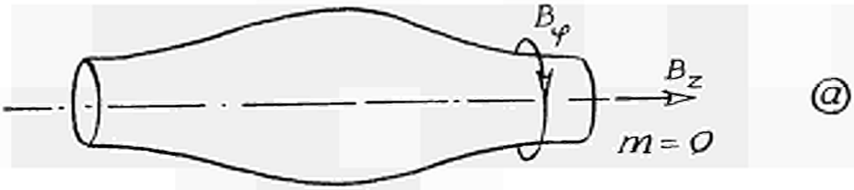


Fig. 98. Deformations corresponding to (a) sausage and (b) kink-instability.

In order to obtain the growth-rates of the various unstable modes it is necessary to resolve eq. (188) for imaginary ω.

This equation can be transformed into

$$a'\xi = 2\mathrm{j}k\boldsymbol{z}_0 \wedge \mathrm{rot}\ \xi - b'\ \mathrm{grad\ div}\ \xi \tag{200}$$

where

$$a' = \frac{8\pi\rho_0\omega^2}{\alpha_p^2 B_0^2}, \quad b' = \gamma(\alpha_v^2 - \alpha_p^2 + 1)\ \alpha_p^{-2}.$$

The components of these equations are (using operators jk, jm)

$$\xi_r = -\frac{\mathrm{j}}{k}\frac{\partial \xi_z}{\partial r} \tag{201a}$$

$$\xi_\varphi = \frac{2k}{a'}\left(\frac{m}{r}\xi_z + k\xi_r\right) + \frac{m}{kr}\xi_z \tag{201b}$$

$$-\frac{a'}{jb'k}\xi_z = \frac{1}{r}\frac{\partial}{\partial r}(r\xi_r) + \frac{jm}{r}\xi_\varphi + jk\xi_z. \tag{201c}$$

Substituting for ξ_r and ξ_φ in the last equation we have from eqs. (201a,b)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \xi_z}{\partial r}\right) + \left[\left(\frac{a'}{b'} - k^2\right) - \frac{m^2}{r^2}\right]\xi_z = 0 \tag{202}$$

the solution of which is

$$\xi = C \cdot J_m\left(\sqrt{\frac{a'}{b'} - k^2} \cdot r\right) \tag{203}$$

and one finds from the first boundary condition $\xi_r = r_0\delta_0$ that

$$C = \frac{\delta_0 r_0 k}{XJ'(jXr_0)} \tag{204}$$

where $X^2 = k^2 - \dfrac{a'}{b'}$.

From the second boundary condition, i.e., from

$$\delta p_p = \delta p_v$$

we get the dispersion relationship $\omega = \omega(k,m)$

$$-\alpha_p^2\left(1 - \frac{a'}{2X^2}\right) r_0^2X^2 K_m(jXr_0) +$$

$$+ (m + \alpha_v Y)^2 \frac{G_{\beta m}(Y)K_m(Y) - L_m(Y)}{1 - G_{\beta m}(Y)} = 1 \tag{205}$$

from which the growth-rate of any mode (k,m) can be calculated. This equation merges into equation (199) when $\omega = 0$, i.e., when $X = k$.

b. Systems with diffused (volume) currents

Let us consider two adjacent flux tubes, containing plasma. We shall assume that the tubes are locally parallel to each other, i.e., there is no shear in the lines of force (fig. 99). Let the tubes be tied to the plasma, which for grad $|B| \neq 0, \infty$ leads to a continuous current-distribution.

Let l_1 A_1 Ω_1 B_1 N_1 and ϕ_1 be the length, cross-section, volume, field, total number of ions and flux in the first tube (system S_1) and l_2 A_2 Ω_2 B_2 N_2 and ϕ_2 the corresponding quantities in the second tube (system S_2).

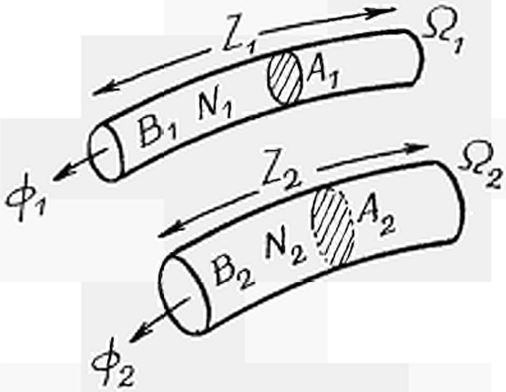


Fig. 99. Adjacent parallel flux tubes.

We shall calculate the energy ΔW involved when the systems S_1 and S_2 exchange places, i.e. when the flux ϕ_1 and particles N_1 are fitted into Ω_2 and ϕ_2 and N_2 are put into Ω_1. Evidently

$$\Delta W = \Delta W_1 + \Delta W_2 \tag{206}$$

where ΔW_1 corresponds to the first operation, ΔW_2 to the second. It will be assumed that the changes are made adiabatically. When $\Delta W > 0$, the exchange $S_1 \rightleftarrows S_2$ can be made only if an energy. ΔW is supplied from external sources, i.e., our configuration of magnetic field and plasma is stable against $S_1 \rightleftarrows S_2$. If, on the contrary $\Delta W < 0$, energy is liberated. This energy can be transformed in the energy of motion of the plasma and the configuration is unstable against the exchange considered.

The three basic equations are those of conservation of number of particles, of flux and of energy:

$$n_1\Omega_1 = n'_1\Omega_2 = N_1, \quad n_2\Omega_2 = n'_2\Omega_1 = N_2 \tag{207a,b}$$

$$B_1A_1 = B'_1A_2 = \phi_1, \quad B_2A_2 = B'_2A_1 = \phi_2 \tag{208a,b}$$

$$\frac{T'_1}{T_1} = \left(\frac{\Omega_1}{\Omega_2}\right)^{\gamma-1}, \quad \frac{T'_2}{T_2} = \left(\frac{\Omega_2}{\Omega_1}\right)^{\gamma-1} \tag{209a,b}$$

where $\gamma = 1 + \dfrac{2}{\alpha}$ and α is the number of degrees of freedom of the plasma particles.

The two energy differences are

$$\Delta W_1 = 3N_1k\Delta T_1 + \frac{1}{8\pi}\,(B'^2_1\Omega_2 - B_1^2\Omega_1) \tag{210a}$$

$$\Delta W_2 = 3N_2 k \Delta T_2 + \frac{1}{8\pi}(B'^2_2 \Omega_1 - B_2^2 \Omega_2). \qquad (210b)$$

From eqs. (206), (208) and (209) we get

$$\Delta W = 3k\left\{N_1 T_1\left[\left(\frac{\Omega_1}{\Omega_2}\right)^{\gamma-1} - 1\right] + N_2 T_2\left[\left(\frac{\Omega_2}{\Omega_1}\right)^{\gamma-1} - 1\right]\right\} + $$
$$+ \frac{1}{8\pi}\left[\left(\phi_1^2 \frac{l_2}{A_2} - \frac{l_1}{A_1}\right) + \phi_2^2\left(\frac{l_1}{A_1} - \frac{l_2}{A_2}\right)\right]. \qquad (211)$$

The simplest case to which we can apply our formula is a perturbation of a free-surface plasma *. We shall consider an exchange of flux and plasma, both occupying the same volume $\Omega = \Omega_1 = \Omega_2$. We have, therefore, $\phi_1 = 0$ and $N_2 = 0$.

From eq. (211) we get

$$\Delta W = \frac{1}{8\pi} \frac{\phi_2^2}{\Omega}(l_1^2 - l_2^2) \qquad (212)$$

and the system is stable if $l_1 > l_2$. This is possible only if *the free surface is concave* (fig. 100).

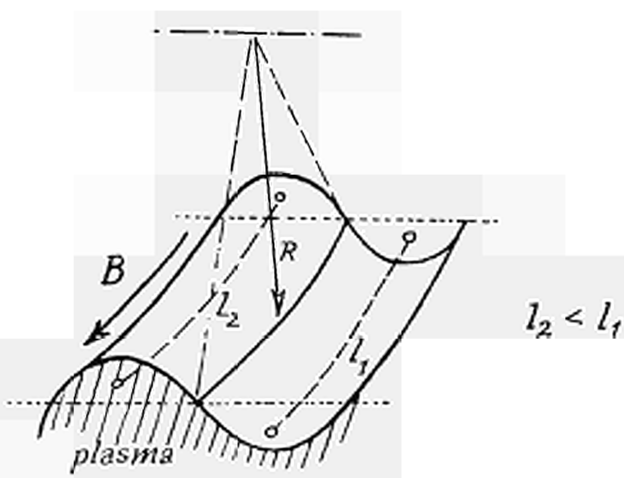


Fig. 100. An example of a stable plasma surface.

* This our first example still corresponds to surface currents rather than to volume currents.

Let us now consider two adjacent flux tubes, containing the same flux but different number of particles. Thus $\phi_1 = \phi_2$ and from eq. (211) we get

$$\Delta W = 3kN_1T_1 \left[\left(\frac{\Omega_1}{\Omega_2}\right)^{2/3} - 1\right] + 3kN_2T_2\left[\left(\frac{\Omega_2}{\Omega_1}\right)^{2/3} - 1\right] \quad (213)$$

where we have assumed that $\gamma - 1 = 2/3$, corresponding to 3 degrees of freedom. Substituting for $2kN_1T_1 = p_1\Omega_1$

$$2kN_2T_2 = p_2\Omega_2$$

we have

$$\Delta W = 3/2\Omega_1 p_1 \left[\left(\frac{\Omega_1}{\Omega_2}\right)^{2/3} - 1\right] + 3/2\Omega_2 p_2 \left[\left(\frac{\Omega_2}{\Omega_1}\right)^{2/3} - 1\right]. \quad (214)$$

Since the tubes are adjacent

$$\Omega_2 = \Omega + \delta\Omega, \qquad p_2 = p + \delta p \quad (215)$$

where we put $\Omega = \Omega_1$, $p = p_1$. Expanding $\left(\dfrac{\Omega + \delta\Omega}{\Omega}\right)^{2/3}$ we obtain

$$\Delta W = 3/2\Omega p \left[\frac{10}{9}\left(\frac{\delta\Omega}{\Omega}\right)^2 + 2/3\frac{\delta p}{p}\frac{\delta\Omega}{\Omega}\right] = \Omega^{-\gamma}\delta\ (p\Omega^{\gamma})\delta\Omega. \quad (216)$$

In some cases it is of interest to consider ΔW resulting from the exchange of two tubes all along the length l of the system containing plasma. Then

$$\Omega = \int_l A\ \mathrm{d}l. \quad (217)$$

As the flux in the tube is constant, $\phi = A \cdot B$ and

$$\Omega = \phi\int_{-l'}^{+l}\frac{\mathrm{d}l}{B} \quad (218)$$

where l and l' are the lengths in the $+$ and $-$ axial direction up to those points of the tube beyond which the configuration is known to be stable or not to be of interest any longer.

It can be shown that the term $\Gamma = \delta(p\Omega^{\gamma})$ in eq. (216) is always negative:

$$\frac{\Gamma}{p} = \left(\frac{\delta p}{p} + \gamma\frac{\delta\Omega}{\Omega}\right)\Omega^{\gamma}. \quad (219)$$

In the outer regions of the confined plasma $\delta p < 0$ and $p \to 0$, whereas Ω does not change rapidly. Therefore,

$$\left|\frac{\delta p}{p}\right| > \left|\gamma\,\frac{\delta\Omega}{\Omega}\right| \tag{220}$$

and

$$\Gamma\Omega^{-\gamma} < 0. \tag{221}$$

In order that the system is stable, i.e., that $\Delta W > 0$ it is, therefore, necessary that

$$\delta\Omega < 0 \tag{222}$$

or

$$\delta\int_{l}\frac{\mathrm{d}l}{B} < 0. \tag{223}$$

Let us apply this criterion to a simple symmetric magnetic bottle (fig. 101). The variational form of the stability criterion can be transformed as follows

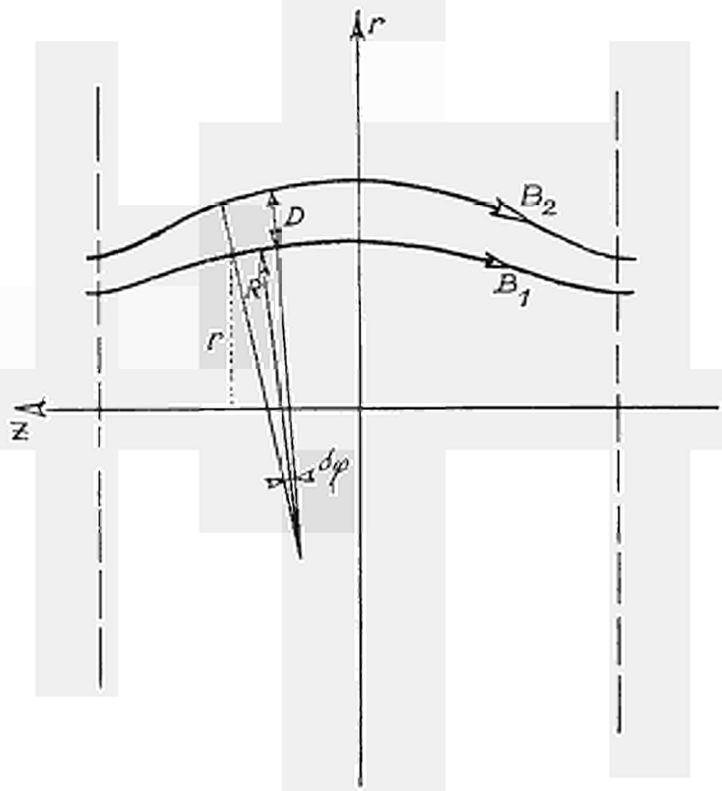


Fig. 101.

$$\delta \int_l \frac{dl}{B} = -\int_{l_1} \frac{dl}{B} + \int_{l_2} \frac{dl}{B} = \int_l B\delta \left(\frac{1}{B^2}\right) dl \tag{224}$$

where l_1 and l_2 are the lengths of two adjacent tubes, whose separation is $D(l)$. If the pressure of the plasma is relatively small i.e., if $\beta = \dfrac{8\pi p}{B^2} \ll 1$ we can assume that rot $\boldsymbol{B} \simeq 0$ and we get

$$\frac{\delta B}{B} = -\frac{D}{R} \tag{225}$$

and noting that the flux between the two lines B_1 and B_2 is

$$\Delta\phi = 2\pi D \cdot rB = \text{const.} \tag{226}$$

the above criterion (eq. (223)) can be written

$$\int_l \frac{dl}{rRB^2} > 0. \tag{227}$$

The sign of the integral depends on R. In the concave part of the bottle, $R > 0$; in the convex part $R < 0$. However, as $rB^2 \propto \dfrac{\phi^2}{r^3}$ it follows that the integrand is largest for $z = 0$ and therefore, the negative contribution of the central part of the bottle predominates and the simple magnetic bottle is thus unstable against flute (exchange) instabilities.

c. Systems of confinement and their stability

The stabilized linear Z-pinch is a typical representative of plasma confinement in which the plasma pressure is, at least partially, balanced by self-fields. Using the model of surface currents we have derived the criteria for the stability. According to this (ref. 16), this stability against the most dangerous mode, i.e., the sausage instability, is assured when $\dfrac{B_0}{B} \leqq \sqrt{2}$ for $B_{zv} = 0$, where B_0 is the intensity of the B_φ field at the plasma surface. The kink instability can be stabilized only if the conducting wall is near enough to the plasma, so that the Foucault currents (image currents) are able to repel a growing kink (fig. 102). This is expressed by requiring that $\beta < 5$. If the modes $m = 0,1$ are thus stabilized, the higher modes will not trouble the confinement. It has been found that the stability of a laboratory plasma

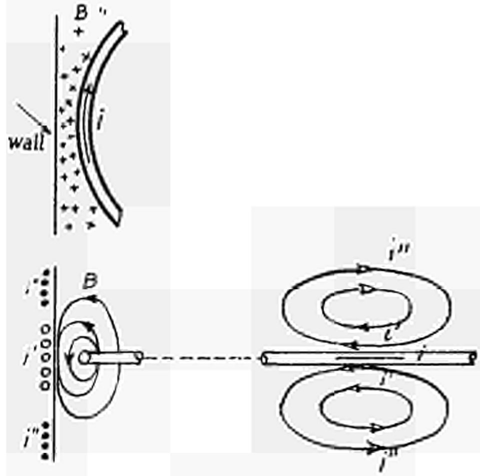


Fig. 102.

cannot be achieved by satisfying these criteria for stability. This is due to principally three reasons:

a) the i_z, i_φ currents are never surface currents, but are diffused throughout the pinch,

b) the cooling effects of electrodes and plasma radiation loss,

c) two-stream and other "micro" instabilities tend to destroy the configuration even if it is hydromagnetically stable.

In 1958 Suydam has generalized the stability criterion, taking into account the diffuse nature of the currents i_z and i_φ. This criterion can be written as

$$\frac{4\pi}{B_z^2}\frac{\mathrm{d}p}{\mathrm{d}r}+\frac{r^3}{4}\left(\frac{B_z}{B_\varphi}\right)^2\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{B_\varphi}{rB_z}\right)\right]^2>0. \tag{228}$$

In order that a diffuse Z-pinch should be stable at every point, the criterion (228) implies existence of a central B_z reversed with respect to B_z in the outer regions of the pinch. It has been shown later that the above criterion has to be made still more stringent, since in the above written form it is a necessary but not a sufficient condition for stability.

The effect b) responsible for the destruction of stability has been observed experimentally (ref. 17, fig. 103) and in the long run cannot be entirely eliminated, even so as effect c). These two effects will cause a loss of plasma from the confined region, however, may not lead

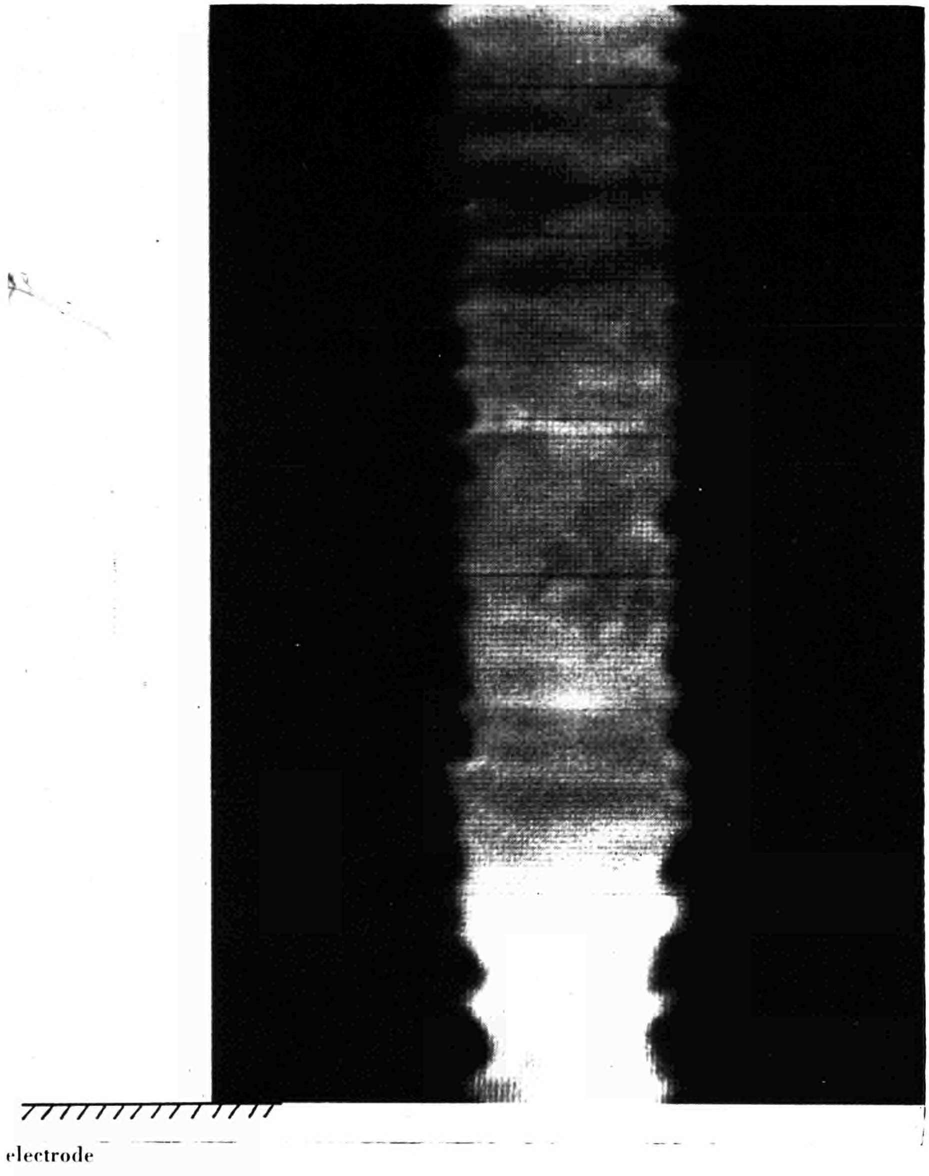


Fig. 103
Development of sausage instability near an electrode.

to a lasting gross instability. In such a case the duration of the configuration may be infinite, provided energy and new plasma particles are continuously supplied. The typical time corresponding to such a confinement is known as the energy-confinement time and corresponds to the recycling of the energy originally contained in the confined plasma (see fig. 103).

The simple magnetic bottle is an example of configurations without self-fields. Although the criterion (227) shows that it is unstable against flutes, experiments habe been made in which no gross instability has been observed (ref. 18). It has been suggested that this apparent stability is due to the anchoring of the ends of the flux tubes on conductors at the ends of the bottle. Even if there are no conductors, insulating surfaces in vacuum and in contact with the plasma may become temporarily good conductors. If such flux-trapping existed it may make te exchange of two-flux tubes impossible and the criterion (227) becomes too pessimistic.

However, for long term confinement this fortuitous effect is insufficient and it is necessary to find magnetic field geometries which are basically stable. A forerunner of these is the magnetic cusp (ref. 19). Experiments have demonstrated that this bottle is hydromagnetically stable, however it is relatively leaky. This is due to the zone of nearly zero magnetic field at its centre, in which particles tied normally to flux tubes can become untied and slip onto tubes leading the particles out of the bottle (non-adiabatic orbits).

This defect can be eliminated by superposing on a linear cusp-field (B_c) an orthogonal (B_z) axial field (fig. 104). This configuration

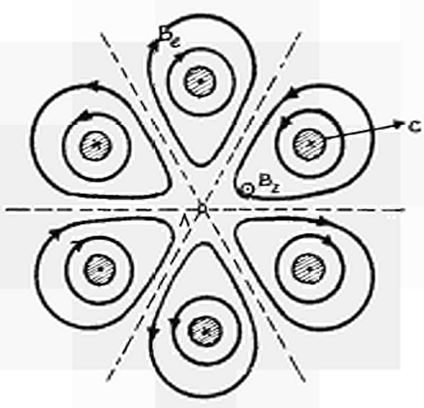


Fig. 104.

is known as the stuffed cusp or the Ioffe's bottle. It has been shown later (ref. 20) that Ioffe's bottle is only a special case of a class of similar configurations, now known as the non-zero minimum B-configurations.

In order to eliminate particle end-losses from an open magnetic bottle, efforts have been made to close it into a torus. An example of such a trend is the Stellarator geometry. However, many variants of such a toroidal system either do not possess an equilibrium, or the lines of the magnetic field intersect the toroidal vacuum envelope, or the system is not stable. One of the best compromises is shown in fig. 105 (ref. 21) in which $\delta \oint \frac{dl}{B} < 0$. The system, known as toroidal multipole, does not possess a local stability everywhere — as there are zones in which B is convex.

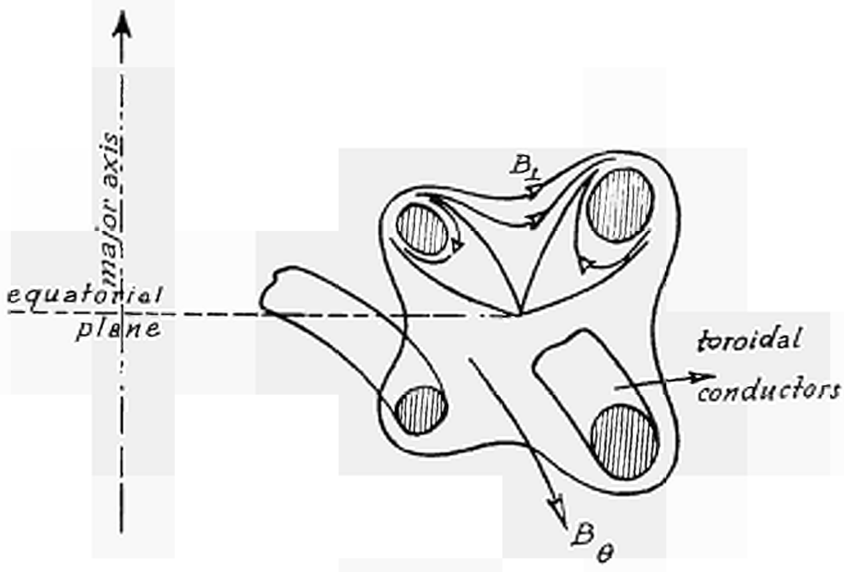


Fig. 105.

5.3.4. HYDRODYNAMIC INSTABILITY

We have seen in section 5.3.1. how a kinetic energy of a neutralized electron stream can be converted into the energy of growing longitudinal electron oscillations. We have also shown in section 5.3.2 how the energy of a magnetic field of a current flow can become a source for the flute instability. In this section we shall analyse an hydrodynamic instability in electron streams in which the *kinetic energy* of the stream is converted into the energy of *transversal motion*, a mechanism related to the well known kink instability. This type of energy conversion becomes important when the kinetic energy of an electron stream is much larger than the magnetic energy associated with it. The kinetic energy per unit length of the stream is

$$W_k = (m_0\gamma c^2 - m_0 c^2)N \tag{229}$$

where N is the linear density of electrons. The corresponding magnetic energy for a plasma cylinder of uniform density whose radius is r_0 is

$$W_M = e^2 N^2 \left(\ln \frac{R}{r_0} + \frac{1}{4}\right) \cdot \frac{v^2}{c^2} \tag{230}$$

and where one assumes that the magnetic field extends radially to a radius R. The ratio of these energies is *

$$\kappa = \frac{W_k}{W_M} = \frac{\gamma - 1}{\nu\left(\ln \dfrac{R}{r_0} + \dfrac{1}{4}\right)} \frac{c^2}{v^2} \tag{231}$$

where $\nu = (e^2/mc^2)N$.

The hydrodynamic instability becomes an important feature of a neutralized electron stream when

$$\kappa \gg 1. \tag{232}$$

It can be shown that this criterion can be also derived from a comparison between hydrodynamic and magnetic forces driving a kink instability.

In order to do this let us consider an electron beam (fig. 106)

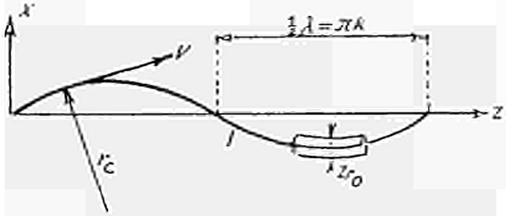


Fig. 106. Hydrodynamic kink-instability.

* Compare with a related equation (4.57) for a non relativistic Bennett distribution.

constrained to move along a sinusoidal curve $L \equiv x = x_m \sin kz$. The centrifugal force on a unit segment of the beam is

$$F_h = \frac{Nm\gamma v^2}{r_c} \tag{233}$$

where r_c is the radius of curvature of the curve L at the segment considered. If $x_m \ll 1/2\lambda$ the radius becomes

$$r_c = \frac{1}{\dfrac{\partial^2 x}{\partial z^2}} = -x_m^{-1}k^{-2}(\sin kz)^{-1}$$

and

$$F_h = k^2 x_m \gamma N m v^2 \sin kz. \tag{233a}$$

The magnetic force F_m is found by using the Biot-Savart law. Thus the force d^2F_m exerted by a current element dI_1 on a current element dI_2 is

$$d^2\boldsymbol{F}_m = ds_2 \frac{d\boldsymbol{I}_1 \wedge \boldsymbol{r}_{12}}{r^3_{12}} I.$$

The total force F_m on a unit length element dI due to all other current elements of the sinusoidal beam is then obtained by integration

$$F_m = k^2 x_m \frac{I^2}{c^2}\left[\ln \frac{\lambda}{2\pi r_0}\right] \sin kz. \tag{234}$$

If a conducting cylindrical wall exists at $r = R$ then owing to the appearance of image-currents we must put $\dfrac{\lambda_{max}}{2\pi} = R$.

Let us form a ratio F_h/F_m using eqs. (233a) and (234)

$$\frac{F_h}{F_m} = \frac{\gamma}{\nu \ln \dfrac{R}{r_0}}. \tag{235}$$

It is, therefore, obvious that if inequality (232) is obeyed, the hydrodynamic forces play a more important rôle than the self-magnetic forces in the excitation of instabilities.

Let us use an idealised model for a plasma stream, consisting of two cylindrical beams of radius r_0, one formed by the fast electrons, the other by almost stationary positive ions (these will be considered to be protons). This model is often referred to as a two-string model of a neutralized (or even partially neutralized) electron beam. In an unperturbed state the axes of these beams coincide; however, when transversal forces are applied the electron beam can partially separate from the positive ion beam (fig. 107). Such a separation is an important feature of energetic electron beams in plasmas.

Let us assume that no redistribution of charges occurs in the two beams. Then a displacement x of the axis of one of the two cylinders gives rise to surface charges $+\sigma$ and $-\sigma$, which generate a field E inside the still neutral plasma. This field causes an attraction force to appear between the two beams whose magnitude is

$$F_e = \pi r_0^2 neE \tag{236}$$

where $E = 2\pi\bar{\sigma}$ and $\bar{\sigma} = (1/\pi)nex$.

With these, F_e becomes

$$F_e = 2\pi e^2 n^2 r_0^2 x = \frac{2e^2}{\pi} \frac{N^2}{r_0^2} x. \tag{236a}$$

It is interesting to estimate to what extent the motion of the electron beam is decoupled from the shape of the positive beam. In order to do this let us consider a sharp bend in the positive beam (fig. 108).

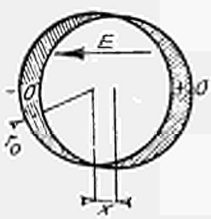


Fig. 107. Electric field induced by a displacement of the "strings".

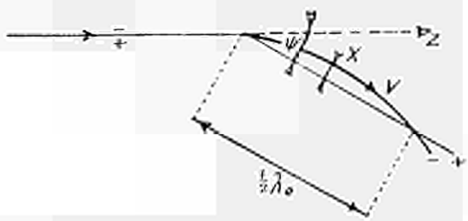


Fig. 108. The cohesion of a positive and negative string.

The fast electron beam will shoot off at a tangent making initially an angle ψ wth the new direction of the positives. Taking into account only F_h and F_e forces we have for the electric stored energy resulting from the separation of the two beams

$$W_e = \int_0^x F_e \, \mathrm{d}x$$

$$W_e = \pi e^2 n^2 r_0^2 x^2.$$

The initial kinetic energy available for movement in the x-direction at $x = 0$ is

$$W_k = \tfrac{1}{2}\pi r_0^2 nm\gamma(\beta c\psi)^2. \tag{237}$$

The distance x_m of maximum separation is obtained from

$$W_e = W_k.$$

Thus

$$x_m = \sqrt{\tfrac{1}{2}\pi\beta\psi r_0}\ \sqrt{\gamma/\nu}. \tag{238}$$

The wave length λ_e corresponding to one full oscillation of the electrons about the positive beam is

$$\lambda_e = 2\pi \frac{x_m}{\psi} = \sqrt{2}\,\pi^{3/2}\beta r_0\ \sqrt{\gamma/\nu}. \tag{238a}$$

From this it follows that any perturbation of the positive beam, whose wave length is smaller than λ_e will not be impressed on the electron beam.

This relationship can be also derived from the equations of motion for the two strings. These follow directly from the two-fluid equations

$$\gamma N m_0 \left(\frac{\partial v_x}{\partial t} + v \frac{\partial v_x}{\partial z} \right) = - F_e \tag{239a}$$

$$NM \left(\frac{\partial w_x}{\partial t} + w \frac{\partial w_x}{\partial z} \right) = F_e. \tag{239b}$$

In terms of the deflections x_1 and x_2 of the two strings, eqs. (239a,b) become

$$\gamma N m_0 \left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial z} \right)^2 x_1 = a(x_1 - x_2) \tag{240a}$$

$$NM \left(\frac{\partial}{\partial t} + w \frac{\partial}{\partial z} \right)^2 x_2 = a(x_2 - x_1). \tag{240b}$$

Let us consider a Fourier component of the actual perturbation, i.e., let us study x_1 and x_2 of the form $e^{j(\omega t \div kz)}$. Then eqs. (240a,b) become algebraic equations and by putting their determinant equal to zero one derives the dispersion equation

$$\frac{\Omega_1^2}{(\omega - vk)^2} + \frac{\Omega_2^2}{(\omega - wk)^2} = 1, \tag{241}$$

where

$$\Omega_1^2 = \frac{2\nu}{\pi\gamma} \left(\frac{c}{r_0} \right)^2, \quad \Omega_2^2 = \frac{2}{\pi} \frac{m_0}{M} \nu \left(\frac{c}{r_0} \right)^2$$

which has the same form as eq. (147).

The criterion for the onset of instabilities will have, therefore, the same form as eq. (148a) and for $w = 0$ one obtains for the longest stable wave length

$$\lambda_{\max} = \sqrt{2}\pi^{3/2} \left[1 + \left(\gamma \frac{m_0}{M} \right)^{1/2} \right]^{-3/2} \beta r_0 \sqrt{\frac{\gamma}{\nu}}. \tag{242}$$

Remember that in evaluating the coupling force F_e between the strings we have taken as our model beams of equal characteristic radius r_0. If one considers a cold-core model (p. 135) the dimension of the electron stream is considerably smaller than that of the positive ion stream. In that case, in order to evaluate F_e, one should consider only that portion N_p of the positive cloud which is located inside the electron beam. It has been shown (eq. (4.73a)) that for a flat stream

$$N/N_p = \gamma^2$$

and from eq. (236a) one has for F_e

$$F_e = \frac{2e^2}{\pi} \frac{N^2}{\gamma^2 r_0^2} x. \tag{243}$$

Substituting this into eqs. (239a,b) one obtains for the longest stable wave length

$$\lambda_{\max} = \sqrt{2}\pi^{3/2} \left[1 + \left(\gamma \frac{m_0}{M} \right)^{1/2} \right]^{-3/2} \beta r_0 \sqrt{\frac{\gamma^3}{\nu}} \tag{242a}$$

which suggests considerable stability.

Evidently the criterion of stability depends on the structure of the neutralized beam, but one may expect that the corresponding value of $\lambda_{\max}$ will lie somewhere between the values given by eq. (242) and eq. (242a).

The behaviour of a relativistic beam immersed in a plasma is more stable (ref. 22).

REFERENCES TO CHAPTER 5

1) E. Fünfer and G. Lehner.
2) M. A. Held and C. B. Wharton, Plasma diagnostics with microwaves, John Wiley and Sons, N.Y. (1965).
3) W. O. Schumann, Berichte Bayer. Ac. d. Wiss. p. 255 (1948).
   G. Bitner, *Zeit. Angen. Phys.*, *10*, p. 117 (1958).
4) K. D. Sinelnikov *et al.*, Proc. Gen. Conf. 1958, vol. *31*, p. 292, Unit. Nations, N.Y.
   Izvestia VUZ, *Radiofisika*, *3*, p. 725 (1960) and *4*, p. 1078 (1961).
   Trivelpiece and Gould, *JAP*, *30*, p. 1784 (1959).

5) J. G. Linhart and W. Zych, Proc. conf. ion. phen. in gases, p. 778, Uppsala (1960).
6) L. Landau, *J. Phys.* USSR, *10*, p. 25 (1946).
7) H. Weitzner, *Phys. Fl.*, *6*, p. 1123 (1963).
8) F. Berz, Proc. Phys. Soc. B, *69*, pp. 939-952 (1956).
9) R. D. Present, Kinetic theory of gases, p. 164, Mc Graw Hill Co. Inc. (1958).
10) Th. O'Neil, *Phys. Fl.*, *8*, p. 2255 (1965).
J. H. Malmberg and C. B. Wharton, *Phys. Letters*, *19*, p. 775 (1967).
11) E. E. Salpeter, *Phys. Rev.*, *120*, p. 1528 (Dec. 1960).
Jour. Geophys. res., *63*, p. 1321 (1963).
J. A. Fejer, *Can. Jour. Phys.*, *38*, p. 1114 (1960).
12) D. Bohm and E.P. Gross, Phys. Rev., *75*, p. 1851 and p. 1834 (1949).
13) H. Alfen and C.G. Falthammar, Cosmical electrodynamics, p. 100, Clarendon Press (1963).
14) A. Sommerfeld, Elektrodynamik, p. 191, DVB (1948).
15) J. G. Linhart, Proc. Roy. Soc., *249* A, p. 318 (1959); R. V. Polovin and N.L. Tsintsadze, Transl. JETP, pp. 1354-61 (1957).
16) M.N. Rosenbluth, Proc. 3rd conf. ionized gases, Venice 1957, p. 903.
17) R. Latham and A. Folkierski, *Nucl. Fus.* 1962 Suppl. p. 627 (Conf. Salzburg 1961).
18) R. F. Post *et al.*, *Phys. Rev. Letters*, *4*, p. 166 (1960).
19) E. Persico and J. G. Linhart, *Nuovo Cim.*, p. 740 (1958), T. k. Allen *et al.*, *Nucl. Fus.* 1962 Suppl., p. 67 (Conf. Salzburg).
20) M. S. Joffe, *Nucl. Fus. 1962, Suppl.*, p. 1045 (Conf. Salz. 1961).
J. Andreoletti, *C. R. Ac. Sc. Paris*, *257*, p. 1235 (1963).
21) T. Okhawa *et al.*, Plasma phys. and nucl. fusion, IAEA Vienna 1966, *2*, p. 53 (Conf. Culham, 1965).
22) R. C. Mjolsness *et al.*, *Phys. Fl.*, *6*, p. 1741 (1963).
23) S. Lundquist, *Nature*, *164*, p. 145 (1949).
24) D. F. Jephcott and P. M. Stocker, *J. Fl. Mech.*, *13*, p. 587 (1962).
25) E. G. Harris, *Phys. Fl.*, *5*, p. 1057 (1962).

List of symbols used in Chapter 5

A	vector potential or surface
$B, \mathbf{B}$	magnetic field strength
c	velocity of light
d	distance, Debye length
e	charge of electron
$E, \mathbf{E}$	electric field strength
f	distribution function, frequency
F	force
g	gravitational field
H	Hamiltonian
i	electron current density
I	current
j	$\sqrt{-1}$
k	wave number or Boltzmann's constant
K	abs. value of the imaginary part of the wavenumber
l	dimension
m, M	particle mass
n	particle density
N	linear density
p	pressure
r, s	coordinates, distance
R	radius
t	time
u, v, w	velocity
v_p	phase velocity

W	energy density
x, y, z	coordinates
Z	atomic number
α	number of degrees of freedom
$\beta =$	v/c
$\gamma =$	$(1 - v^2/c^2)^{-1/2}$ or c_p/c_v
δ	skin depth, damping coefficient or mean fluctuation
ε	dielectric constant
λ	wave length
λ_D	Debye length
ν	frequency
φ	phase angle or potential
θ	coordinate
θ_0	Brewster's angle
ρ	density
σ	surface charge density
σ_T	Thomson cross-section
τ	period of oscillation
ω	angular frequency
ω_c	cyclotron frequency
ω_p	plasma frequency
ξ	displacement
ϕ	flux
ψ	potential
Ω	volume
ϕ	magnetic flux

CHAPTER 6

SHOCK WAVES IN PLASMA

Introduction

In the previous chapter we have described various energy conversion processes which give rise to oscillations in plasma. The propagation of small amplitude oscillations through plasma as investigated by solving the linearized wave equations for a particular wave form, whose space-time variation was assumed to be exp $[j(\omega t + kz)]$. It was argued that any small amplitude perturbation could be expressed by its Fourier components and, therefore, its progress in space and time can always be found from the dispersion relation $\omega(k)$ for the individual components. Thus if the perturbation has a form

$$f_0(z) = \int_{-\infty}^{+\infty} F(k) e^{jkz}\, dk \tag{1}$$

at the time $t = 0$, it will be transformed into

$$f(z) = \int_{-\infty}^{+\infty} F(k) e^{j[kz - \omega(k)t]}\, dk \tag{1a}$$

at some later time t.

It was also mentioned that for most types of oscillations, the dispersion relationship $\omega = \omega(k)$ is valid only within certain frequency range, e.g., for hydromagnetic waves the validity of the dispersion relation is in doubt for wave lengths λ_{min} shorter than the gyration radius of the ions. Thus only propagation of perturbations whose initial distribution $f_0(z)$ does not contain harmonics whose wave length is shorter than λ_{min} can be described by the dispersion relations derived in chapter 5.

When the amplitude of a wave becomes so large that the quadratic and higher terms in n_1, v_1, w_1 cannot be neglected the fluid equations cannot, as a rule, be resolved analytically. No simple dispersion equation is obtainable, in fact, the frequency of an oscillation will depend on its amplitude and consequently its phase and group velocity will also depend on the amplitude.

It is obvious that, owing to the above mentioned two limitations, propagation of strong disturbances in a plasma cannot be described using the dispersion relations derived for small amplitude waves.

Nevertheless, certain concepts in the propagation of large amplitude waves resemble the concept of group velocity. This can be demonstrated in a simple way for disturbances having a plane geometry. The equations of an ideal fluid in absence of external forces are (from eqs. (2.59a) and (2.62)).

$$\frac{\partial \rho}{\partial t} = - \frac{\partial}{\partial x} (v\rho) \tag{2}$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = - \frac{1}{\rho} \frac{\partial p}{\partial x} \tag{3}$$

$$p\rho^{-\gamma} = \text{const.} \tag{4 *}$$

Let us define a new variable

$$f = \int_{\rho_0}^{\rho} c \frac{d\rho}{\rho} \qquad \text{where} \qquad c^2 = \gamma \frac{p}{\rho}$$

and ρ_0 is the undisturbed fluid density.

It follows that

$$f = \frac{2}{\gamma - 1} (c - c_0). \tag{5}$$

Using this quantity eqs. (2) and (3) can be now written as

$$\frac{\partial f}{\partial t} + c \frac{\partial v}{\partial x} + v \frac{\partial f}{\partial x} = 0 \tag{6}$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + c \frac{\partial f}{\partial x} = 0. \tag{7}$$

Adding or subtracting these equations we obtain

$$\frac{\partial}{\partial t} (f + v) + (v + c) \frac{\partial}{\partial x} (f + v) = 0 \tag{8}$$

$$\frac{\partial}{\partial t} (f - v) + (v - c) \frac{\partial}{\partial x} (f - v) = 0. \tag{9}$$

These equations have the form of wave equations for the quantities f + v and $f - v$. The velocity of propagation of these waves is $v + c$

* $\gamma = 1 + \frac{z}{\alpha}$ where α is the number of degrees of freedom.

and $v - c$. Thus for an observer travelling with one of these speeds one of the quantities $f + v$, $f - v$ remains invariant. These quantities are known as Riemannian invariants and the curves in the x, t space on which $f \pm v = \text{const.}$ are called the characteristics. The two Riemann invariants are equivalent to the following two invariants

$$R_1 = \frac{2}{\gamma - 1}\, c + v, \qquad R_2 = \frac{2}{\gamma - 1}\, c - v \tag{10}$$

where c has the meaning of a local speed of sound.

Let us now consider the propagation of a plane fluid distribution shown in fig. 109 for $t = 0$.

If at this time the p, ρ and v are known functions of x, it is possible to find the angles of characteristics in the x, t plane for $t = 0$. It is evident that the characteristic passing through the point A corresponds to a faster signal than that passing through B. At some later time t_s these characteristics will intersect suggesting that a zone is being created where the values $R_1(A) = R_1(B)$. As this is physically impossible a discontinuity in p, ρ and v must exist at the point C or possibly even at an earlier time. In this case the family of characteristics between A and B has an envelope Σ which represents the formation of such a discontinuity. The fluid region at Σ is known as the shock front (ref. 1).

Since the occurence of shocks is related to intersecting characteristics let us derive a criterion for the intersection of two adjacent characteristics $v + c$. The difference of speeds of two points following these curves is

$$\delta = \delta v + \delta c = \frac{\partial v}{\partial \rho}\, \delta\rho + \tfrac{1}{2} c \left(\frac{\delta p}{p} - \frac{\delta \rho}{\rho} \right) \tag{11}$$

as $p \sim \rho^\gamma$ we have $\dfrac{\delta p}{p} = \dfrac{\gamma}{\rho}\, \delta\rho$ and

$$\delta = \frac{\partial v}{\partial \rho}\, \delta\rho + \frac{\gamma - 1}{2}\, c \cdot \delta\rho. \tag{12}$$

If the characteristics are to intersect we must have $\delta > 0$ and, therefore,

$$\left(\frac{\partial v}{\partial \rho} + \frac{\gamma - 1}{2}\, c \right) \delta\rho > 0. \tag{13}$$

If the opposite is true the zone near the two characteristics is developing a rarefaction wave.

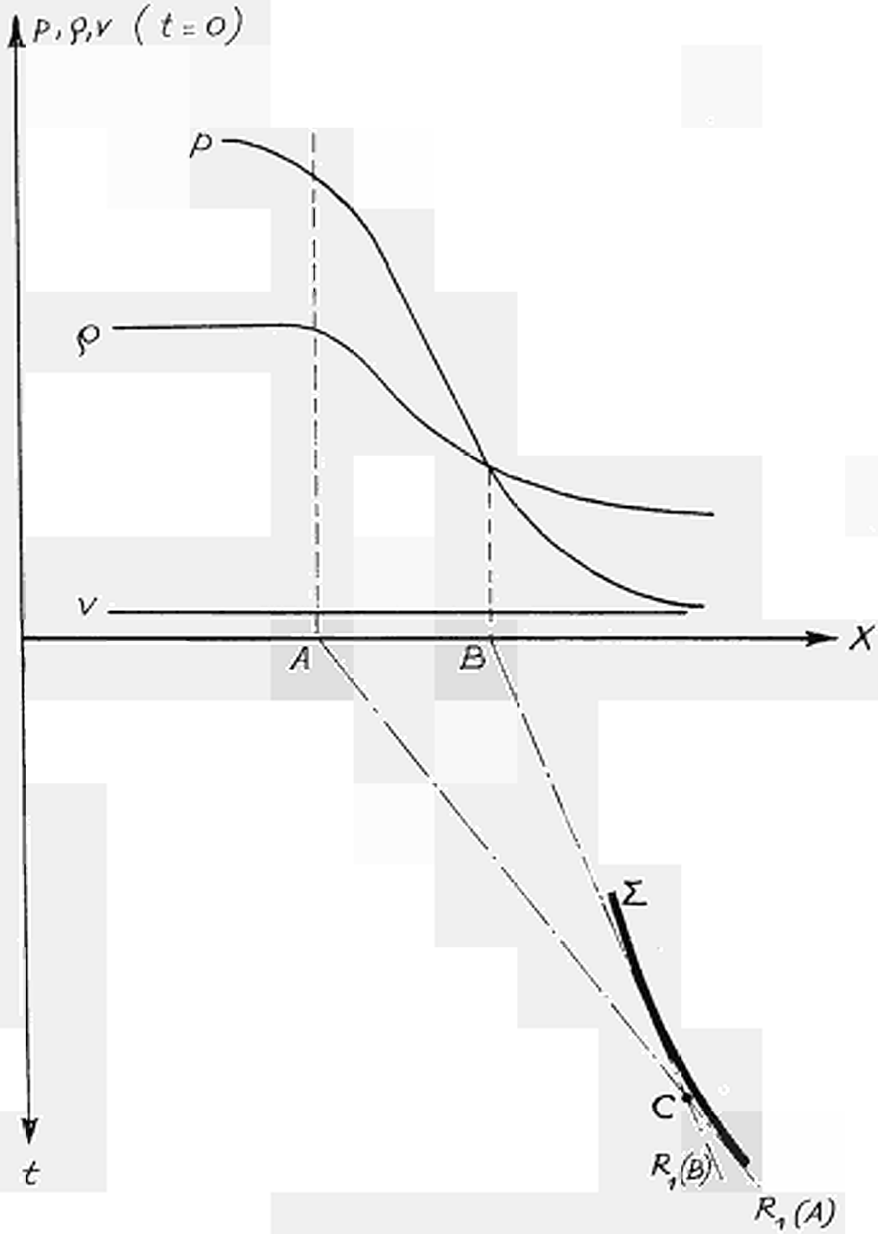


Fig. 109. Formation of a shock.

Most of these observations are true for a plasma in a magnetic field, the sound speed c must be in such a case interpreted as the speed of longitudinal hydromagnetic waves.

6.1. Relations of Rankine-Hugoniot. Shock-speed

The behaviour of a plasma invested by a shock-wave can be described by the equations of continuity, conservation of momentum, magnetic flux and energy. We shall use the model of one fluid with infinite electrical conductivity. We have (see eqs. (3.59a), (3.60) and (3.66))

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho v) \quad \text{eq. of continuity} \tag{14}$$

$$\rho \frac{\partial v}{\partial t} + \rho v \frac{\partial v}{\partial x} = -\frac{1}{4\pi} B \frac{\partial B}{\partial x} - \frac{\partial p}{\partial x} \quad \text{eq. of cons. of momentum} \tag{15}$$

$$\frac{\partial}{\partial x}(vB) = \frac{\partial B}{\partial t} \quad \text{eq. of cons. of mag. flux} \tag{16}$$

$$\left(\frac{\partial v}{\partial x} + \frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)\left(\frac{p}{\gamma - 1} + \tfrac{1}{2}\rho v^2\right) + \frac{\partial (pv)}{\partial x} + \frac{1}{8\pi}\frac{\partial (vB^2)}{\partial x} = 0 \quad \text{energy flow.} \tag{17}$$

Let us assume that some time τ after the passage of the shock-front Σ the fluid settles into a new equilibrium state.

This usually implies the existence of some dissipation mechanism such as viscosity. As τ can be chosen very large, the damping mechanism can be so weak that it does not has to be represented by a special term in the above equations.

In order to eliminate the time dependence, let us transform our equations into a frame of reference in which Σ is stationary. Thus, if in the laboratory frame of reference the undisturbed region in front of Σ (upstream) is stationary, in our new frame of reference the fluid speed there will be equal and opposite to the shock speed v_s in the laboratory frame.

It is now possible to choose two points x_1 and x_0, the first situated in the shocked, steady flow behind Σ, the second in the still undisturbed plasma in front of Σ. Let us integrate the eqs. (14) and (15) between x_1 and x_0. As according to our model $\frac{\partial}{\partial t} = 0$ we get

$$\rho_1 v_1 = \rho_0 v_0 \tag{18}$$

$$p_1 + \frac{B_1^2}{8\pi} - p_0 - \frac{B_0}{8\pi} = \tfrac{1}{2}\int_{x_1}^{x_0} \rho \frac{\partial v^2}{\partial x}\, dx. \tag{19}$$

Integrating by parts and using eq. (17) one has

$$\frac{1}{2}\int_{x_1}^{x_0} \rho \frac{\partial v^2}{\partial x}\,dx = -\rho_1 v_1^2 + \rho_0 v_0^2 \tag{20}$$

and, therefore,

$$\rho_1 v_1^2 + p_1 + \frac{B_1^2}{8\pi} = \rho_0 v_0^2 + p_0 + \frac{B_0^2}{8\pi}. \tag{21}$$

Eq. (16) gives

$$B_1 v_1 = B_0 v_0. \tag{22}$$

The equation of energy transport can be written

$$\frac{\partial}{\partial x}\left[v\left(\frac{p}{\gamma - 1} + \frac{1}{2}\rho v^2\right) + pv + v\frac{B^2}{8\pi}\right] = 0 \tag{23}$$

which can be directly integrated. Remembering that $\rho_1 v_1 = \rho_0 v_0$ we get

$$\frac{1}{2}v_1^2 + \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} + \frac{B_1^2}{4\pi\rho_1} = \frac{1}{2}v_0^2 + \frac{\gamma}{\gamma - 1}\frac{p_0}{\rho_0} + \frac{B_0^2}{4\pi\rho_0}. \tag{24}$$

Eqs. (18), (21), (22) and (24) are the generalised Rankine-Hugoniot relations. These equations were orignally derived for shocks in a gas, in which case their form is obtained by putting $B_0 = B_1 = 0$ in the eqs. (21), (22) and (24).

We have now four equations for ρ_1, v_1, p_1 and B_1 and it is, therefore, possible to determine any of these in terms of ρ_0, v_0, p_0 and B_0. Let us eliminate p_1, v_1 and B_1. The remaining equation will contain v_0, p_0, B_0, ρ_0 and ρ_1. Solving this equation for v_0, we obtain the shock-speed $v_0 = -v_s$ in terms of p_0, ρ_0, B_0 and ρ_1/ρ_0. It reads

$$v_s = c_s\left\{\frac{2}{\gamma + 1}\;\frac{1 + \dfrac{B_0^2}{4\pi\gamma\rho_0 c^2}\left[1 + \left(1 - \dfrac{\gamma}{2}\right)\left(\dfrac{\rho_1}{\rho_0} - 1\right)\right]}{\dfrac{\rho_0}{\rho_1} - \dfrac{\gamma - 1}{\gamma + 1}}\right\}^{1/2}. \tag{25}$$

When ρ_1/ρ_0 tends to unity the shock becomes "weak" and v_s tends to v_A, the speed of sound waves in a gyrotropic plasma. If, at the same time, $B_0 \to 0$ then $v_s \to c_s$.

On the other hand, as ρ_1/ρ_0 increases and tends to $\dfrac{\gamma + 1}{\gamma - 1}$ the shock speed tends to infinity, showing that for strong shocks the density ratio across the shock-front cannot exceed the value of $\dfrac{\gamma+1}{\gamma-1}$ (fig. 110). A convenient measure of the strength of the shock is then $M_s = \dfrac{v_s}{v_A}$, the Mach number of the unsteady flow, rather than ρ_1/ρ_0. In absence of the magnetic field $M_s = \dfrac{v_s}{c_s}$ and for strong shocks we get

$$M_s = \left[\frac{2}{(\gamma+1)\dfrac{\rho_0}{\rho_1} - (\gamma-1)}\right]^{1/2}. \tag{26}$$

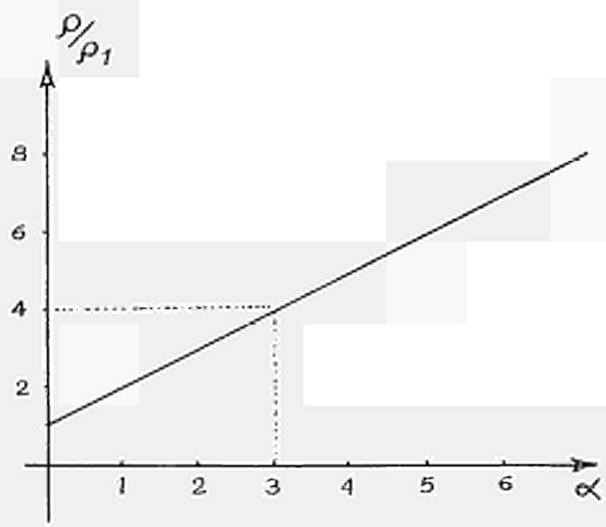


Fig. 110. Dependence of density step on number of degrees of freedom (α).

6.2. Structure of the Shock Front in Absence of Magnetic Field

The simplest demonstration of a shock is the old school-experiment (fig. 111) with beads of equal mass m suspended by threads of equal length l in which the separation of adjacent beads is λ. If

$$\lambda \ll l$$

then the momentum mv is nearly constant during the transit of a bead from zero to λ with respect to its equilibrium position. The transit time is then

$$\tau \approx \lambda/v$$

and the speed of propagation v_s of a shock is

$$v_S \approx v,$$

provided the time consumed during a collision τ_0 between two beads can be neglected in comparison with τ.

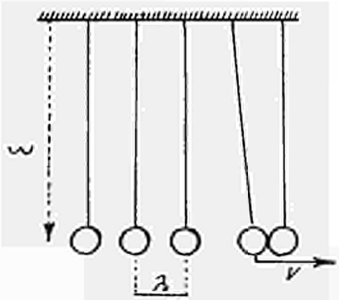


Fig. 111. Transmission of a shock in a mechanical model.

The width of the shock front in the experiment is obviously λ, the distance between two collisions.

This mechanism operates in the same manner in monomolecular gases, in which a layer of the gas has been set into motion with the speed v. The shock front is a few mean free-paths wide. The main difference between the school-experiment and the shock in a gas is the number of degrees of freedom. In gas the molecules involved in the transport of momentum from the shocked region into the as yet undisturbed gas do not meet generally in head-on collisions and therefore, some of the energy of the directed flow behind the shock front is converted in increasing the enthalpy of the gas invested by the shock. The greater is the number of degrees of freedom of the molecules the more rapidly the shock looses its strength.

In plasma a shock corresponds to a momentum transfer in two fluids, electrons and ions, whose motion is coupled by an electric field arising from non-identical density distributions in these fluids. Owing to this coupling the width of the shock front corresponds to the mean free path of the ion-ion collisions; the shock form being impressed on the electron gas through an electric field of the shock front. As, in absence of a magnetic field, the transfer of momentum in plasma is effected through ion-ion collisions it follows that their temperature rises rapidly immediately behind the shock front. The electrons are heated only in contact with this hot ion gas and consequently their temperature T_e will tend to T_i only after a relatively long time $\tau = \tau_{ii}\sqrt{\dfrac{M}{m}}$ (see later p. 267). On the other hand the heat conduction in the eletron gas will tend to diffuse the electron heat by δ_h well ahead of the point A (fig. 112).

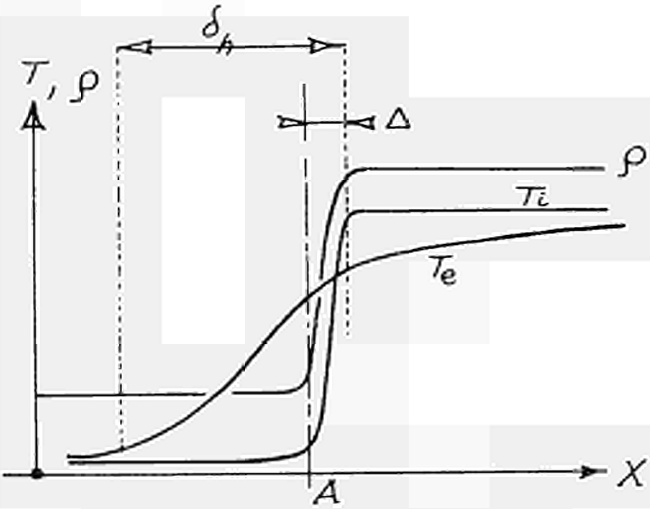


Fig. 112. A typical structure of a shock wave in plasma.

The heat skin depth δ_h is

$$\delta_h = \sqrt{\frac{2\kappa\,\tau_h}{nk}} \tag{27}$$

where τ_h is the duration of the diffusion heating, before the $T_i \to T_e$ transfer takes over. Thus

$$\tau_h = \frac{\delta_h}{v_s} \tag{28}$$

and therefore,

$$\delta_h = \frac{2\kappa}{nkv_s}. \tag{29}$$

Using expressions to be derived in chapter 8 (eqs. (71a,b)) we get putting $\frac{1}{2}Mv_s^2 \approx 3kT_i$

$$\frac{\delta_h}{\Delta} = \frac{\kappa}{v_s^2\tau} \approx k \cdot n \tag{30}$$

suggesting that for plasma densities $n \gg 10^{16}$ the diffusion of heat will broaden considerably the shock front.

6.3. Shocks in a Gyrotropic Plasma

If the direction of propagation of the shock in an infinite and uniform plasma is parallel to the magnetic field $\boldsymbol{B}$, the mechanism of shock propagation described for a magnetic field-free plasma remains essentially unchanged. However, if the direction of propagation, i.e., the shock velocity is at right angles to $\boldsymbol{B}$ a new situation arises.

Let us consider a layer of plasma whose mass velocity is v. The movement of this layer causes a compression of magnetic field in front of the layer. The kinetic energy of the layer may thus be converted into a stored energy of a magnetic field layer, which in turn

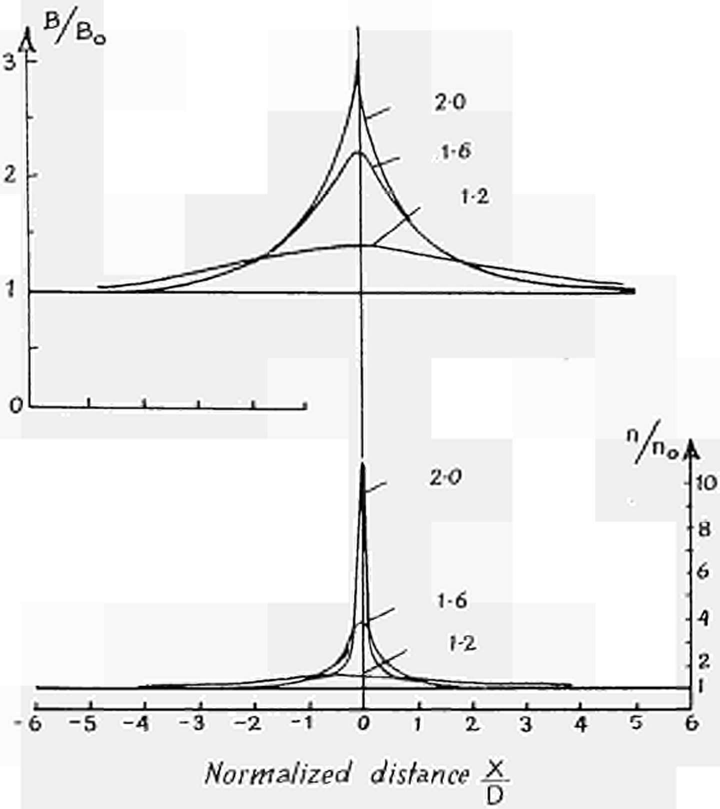


Fig. 113. Isolated magnetic pulse. Curves for $Ms = 1.2, 1.6, 2$.

can exercise pressure on a plasma layer further ahead. Consequently the transfer of momentum can be effected even in absence of collisions. The mechanism of such energy and momentum transmission has not been adequately described so far, although a large amount of valuable work has been published.

The most relevant discovery was made by Adlam and Allen (ref. 2) who described how a single pulse (or a series of pulses) of magnetic field can propagate in collision-less plasma. Choosing a coordinate system travelling with the pulse they have found a non-linear solution of the two-fluid model corresponding to an accumulation of magnetic flux held stationary by the flow of the plasma across it. The particles must possess large enough speeds so that the ion Larmor radius is larger than the effective width D of the magnetic pulse and that the electric field of the ions is able to pull the electrons across D. The particle density and the structure of the magnetic pulse is shown in fig. 113. It is found that if $v^2 \gg u^2$, the width D is of the order of $\frac{c}{\omega_p}$, the collision-less skin depth (see p. 153) and that the strength of the magnetic pulse depends on $M_s = \frac{v}{v_A}$. The analysis is valid only for $1 \leqslant M_s \leqslant 2$, however, it is not unthinkable that more complicated solutions exist even for $M_s > 2$.

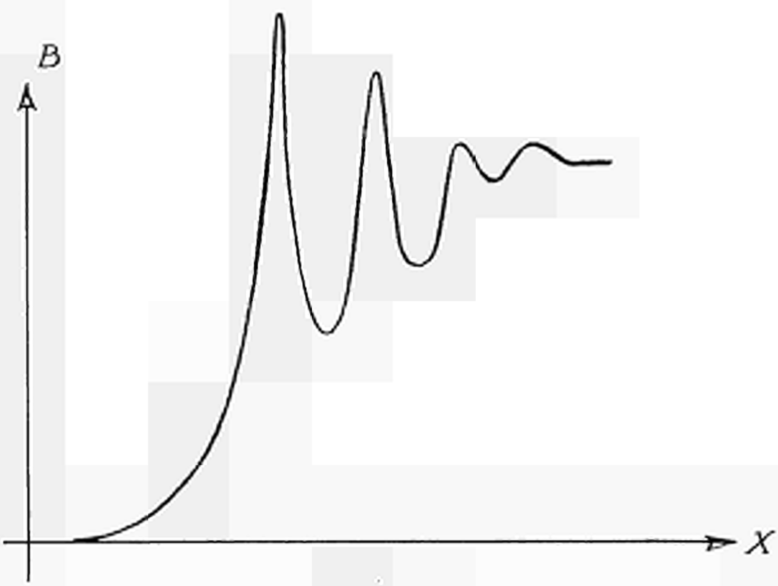


Fig. 114.

It has been shown recently (ref. 3) that M_s of such pulses is limited by the development of a two-stream instability with a subsequent generation of turbulence.

It is plausible that the structure of a collision-less shock in a gyrotropic plasma resembles a succession of damped Adlam-Allen waves, as has been shown theoretically in a special case of $B_0 = 0$ (ref. 4), fig. 114).

6.4. Diverging and Converging Shocks

So far we have considered shocks having a plane geometry, whereas shocks generated in plasmas are mostly either spherical or cylindrical. Typical examples of diverging spherical shock waves are provided by point-like explosions, e.g., supernovae (ref. 5) or an A-bomb explosion (ref. 6). Cylindrical shocks can be of the diverging type in exploding wire experiments or of the converging type in rapid implosions in Z or θ-pinches. Let us treat first the diverging shocks.

6.4.1. DIVERGING SHOCKS

It is always gratifying and helpful if in treating a non-linear phenomenon one is able to discover some characteristic feature which is independent of amplitude. Such a feature in the theory of the shock waves is known as the similarity of flow distributions. In diverging shocks this is exemplified by solutions of the type

$$F(r,t) = R^n(t) \cdot f\left(\frac{r}{R}\right) \tag{31}$$

where F is one of the quantities characterising the flow (such as p, ρ or v), R is the radius of the expanding shock front and r is the radial coordinate. If solutions of this type exist, then the profile of F behind the shock front remains functionaly the same and the distribution of F at t_1 is similar to that at t_2, i.e.,

$$\frac{F_1(r_1)}{F_2(r_2)} = \left[\frac{R(t_1)}{R(t_2)}\right]^n, \qquad \text{where} \qquad \frac{r_1}{R_1} = \frac{r_2}{R_2} = \xi. \tag{32}$$

It is not difficult to find what the exponent n should be in the case of a point-like liberation of an energy W. At a time t after the explosion this energy must be found as the internal and kinetic energy of the expanding plasma. Therefore,

$$\frac{1}{4\pi} W = \int_0^R \left(\frac{p}{\gamma - 1} + \tfrac{1}{2} \rho v^2 \right) r^2 \, dr. \tag{33}$$

Since we know that in strong shocks the ρ_1/ρ_0 across the shock front is equal to $\frac{\gamma + 1}{\gamma - 1}$ it must be

$$\frac{\rho}{\rho_0} = \psi \left(\frac{r}{R} \right) \tag{34}$$

without depending explicitly on $R(t)$. Putting $\frac{p}{p_0} = R^n \cdot \Theta$ and $v = R^{n/2} \cdot \varphi$ we get

$$\frac{W}{4\pi} = R^{n+3} \int_0^1 \left(\frac{p_0}{\gamma - 1} \Theta + \tfrac{1}{2} \rho_0 \psi \varphi^2 \right) \xi \cdot d\xi. \tag{35}$$

As only R^{n+3} contains time and as $W = \text{const.}$, it follows that

$$n = -3.$$

Let us now try to substitute the expressions for p, ρ and v in the three equations of the one-fluid model (without electromagnetic forces)

$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial r} + \rho \left(\frac{\partial v}{\partial r} + \frac{2v}{r} \right) = 0 \quad \text{eq. of continuity} \tag{36}$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} = - \frac{1}{\rho} \frac{\partial p}{\partial r} \quad \text{eq. of motion} \tag{37}$$

$$\left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial r} \right) (p \rho^{-\gamma}) = 0 \quad \text{eq. of energy transport.} \tag{38}$$

If these are to be satisfied by solutions of the type (31) the following must be true

$$\frac{dR}{dt} = A \, R^{-3/2} \tag{39}$$

or

$$R = (5/2 \, A \cdot t)^{2/5} \tag{40}$$

$$\text{and } \frac{p}{p_0} \propto t^{-6/5}, \quad v \propto t^{-3/5}. \tag{41}$$

In order to reduce eqs. (36), (37) and (38) to a non-dimensional from let us put

$$\Theta = \theta \frac{c^2}{A^2}, \qquad \phi = \frac{\varphi}{A} \qquad \text{where} \qquad c^2 = \gamma \frac{p_0}{\rho_0}.$$

Then

$$\frac{\psi'}{\psi} = \frac{\phi' + 2\phi\xi^{-1}}{\xi - \phi} \tag{42}$$

$$\phi\,(\xi - \phi) = \frac{1}{\gamma}\,\frac{\theta'}{\psi} - 3/2\phi \tag{43}$$

$$3\theta + \xi\theta' + \gamma\frac{\psi'}{\psi}\theta(\phi - \xi) - \phi\theta' = 0. \tag{44}$$

Substituting into the last one from eqs. (42) and (43) we get

$$\theta' = \frac{-3\xi + \phi\left(3 + \dfrac{\gamma}{2}\right) - 2\gamma\phi^2\xi^{-1}}{\theta(\xi - \phi)^2 - \psi^{-1}}. \tag{45}$$

Thus knowing θ, ϕ and ψ at one point it is possible to find by numerical methods these functions everywhere.

Such a starting point is on the shock front, i.e., $\xi = 1$. Then θ_1, ϕ_1 and ψ_1 are given by the Hugoniet relations (18), (21), (22) and (24) which for strong shocks become

$$\theta_1 = \frac{2\gamma}{\gamma + 1},\quad \psi_1 = \frac{\gamma + 1}{\gamma - 1},\quad \phi_1 = \frac{2}{\gamma + 1}. \tag{46}$$

The numerical solution of eqs (42), (43) and (44) has been obtained for plasma ($\gamma = 5/3$) and the form of θ, ϕ and ψ is shown in fig. 115 (ref. 6).

It is seen that in the central region, i.e., for $\xi < \frac{1}{2}$ the pressure ($\propto \theta\ (\xi)$) is approximately constant and, therefore, the internal energy density $\dfrac{p}{\gamma - 1}$ is also nearly constant. Since the density decreases with decreasing ξ (for $\xi = 0$, $\psi = 0$), the temperature must tend to infinity near the centre of the explosion. Similar analysis for diverging cylindrical shocks shows that $n = 2$ and $R = \sqrt{2At}$; $p \propto R^{-2}$, $v \propto R^{-1}$. The functions θ, ψ and ϕ show similar behaviour as those of spherical shocks (ref. 7).

6.4.2. CONVERGING SHOCKS

The method of similarity solutions can be used for converging shocks too. This has been done by Guderley and Somon (refs. 8 and 9). The latter finds that for $\gamma = 5/3$ the pressure behind the converging shock front increases as $R^{-0.905}$ for spherical shocks and as $R^{-0.452}$ for cylindrical shocks. As the denstiy behind a strong shock front is always

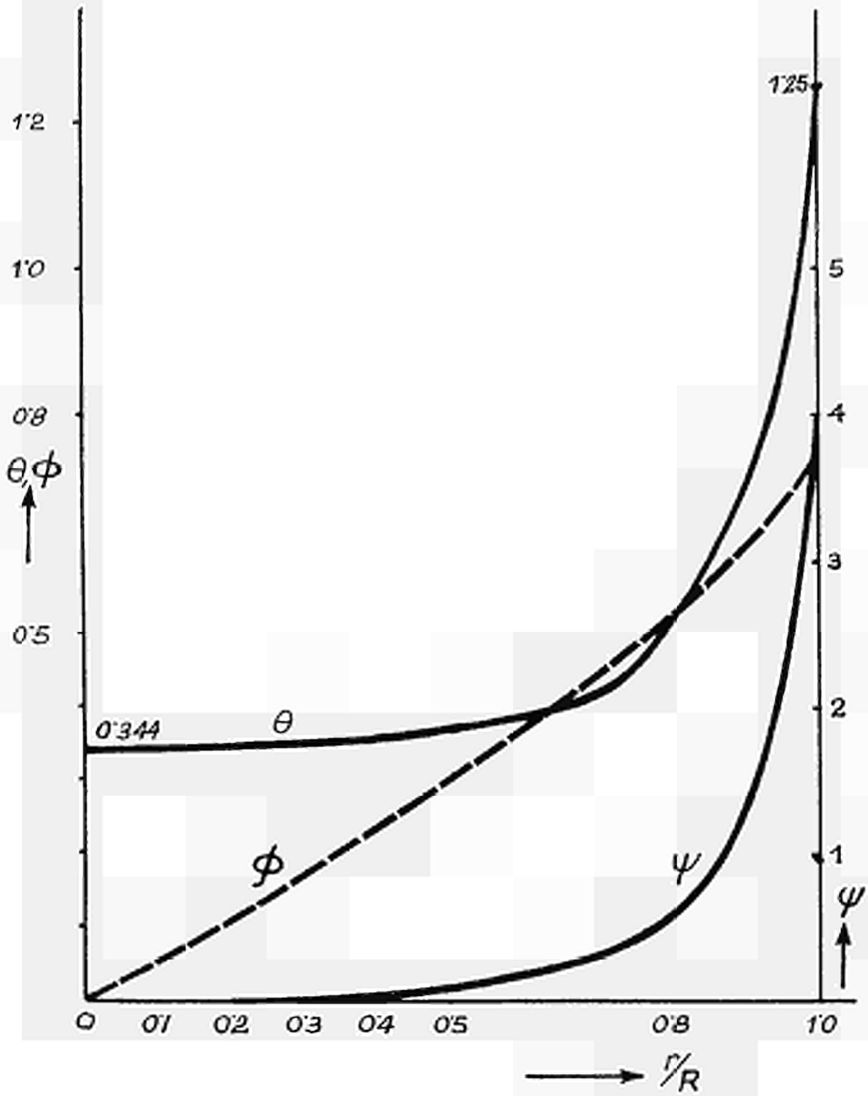


Fig. 115.

approximately $\frac{\gamma+1}{\gamma-1} = 4$ it follows that the temperature of the plasma behind the shock front increases as the pressure does. This suggests that converging shock waves could be used as a mechanism to heat plasma to very high temperatures. This is generally not true owing to three reasons. Firstly, the efficiency of such heating is low—most of the shock energy is used up to heat the bulk of the plasma to only moderate temperatures. Secondly, the radial collapse is usually destroyed by an instability (overstability) of the shock front. Thirdly the convergence will cease to incéease the temperature when the shock front will approach the centre of implosion to less than a few mean free paths. This third limitation, the mildest of all three, gives for maximum temperature in a spherical implosion of the Guderley type

$$T = T_0 \left(\frac{R_0}{R}\right)^{0.905} .$$

Using for R the expression for mean free path (also the thickness of the shock front, pp. 223 and 265) we get an approximate criterion

$$T \approx \left[\left(\pi \frac{e^4}{k^2} \ln \Lambda\right) \cdot T_0 R_0 n\right]^{1/3} . \qquad (47)$$

Thus, e.g., for deuterium and $T_0 = 10^5$, $R_0 = 100$ we get $T \approx 10 \cdot n^{1/3}$. Obviously, thermonuclear temperatures can be reached only at solid state densities.

Although the Guderley solution corresponds to only a special case of a converging wave it has been shown that a converging shock distribution tends to transform itself into a self-similar distribution of the Guderley type (ref. 9) and thus it appears that self-similar solutions are the basic shock distributions.

Converging cylindrical shocks have been produced in a conventional shock tube behind a pear-shaped obstacle (ref. 10, fig. 116). A plane

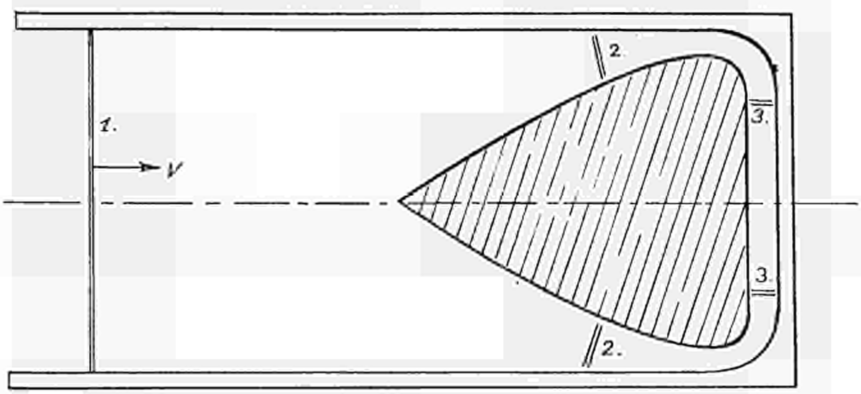


Fig. 116.
1. Original plane shock;
2. Split plane shock;
3. Converging cylindrical shock.

shock front is split and guided round the corner of the pear. A well defined converging shock front was produced only when the original Mach number of the plane shock was smaller than 1.7. At higher Mach numbers the bending of the shock round the corner did not prove feasible. The temperatures reached in this device are also limited by the proximity of the walls of both the pear and the shock tube.

Strong converging cylindrical shocks are produced in some fast pinches especially in Z-pinches (in which the converging shock does not encounter any magnetic field). The current layer driven by the $i_z B_\varphi$ Lorentz forces acts as piston and drives a cylindrical shock towards the axis. The progress of both, the piston and the shock-front can be observed by means of a smear camera, which also shows the collision of the front on the axis (ref. 11).

REFERENCES TO CHAPTER 6

1) L. Landau and E. M. Lifshitz, Fluid mechanics, Pergamon Press 1959, pp. 310-319.
2) J. H. Adlam and J. E. Allen, *Phil. Mag.*, *3*, p. 448 (1958).
3) G. Bardotti *et al.*, *Nucl. Fus.*, *6*, p. 46 (1966).
4) P. L. Auer *et al.*, *Phys. Fluids*, *4*, p. 1105 (1961).
5) M. J. Lighthill, *Mem. Manch. Lit. Phil. Soc.*, ***101***, p. 7 (1958).
6) G. I. Taylor, Proc. Royal Soc. A, ***201***, p. 159 and p. 175 (1950).
7) S. C. Lin, *J. Appl. Phys.*, ***25***, p. 54 (1954).
K. Oshima, *J. Phys. Soc. Jap.*, ***12***, p. 1421 (1957).
8) G. Guderley, *Zeit. Luftfahrtforsch.*, ***19***, p. 302 (1942).
9) J. P. Somon *et al.*, Nucl. Fusion 1962 suppl., p. 717 (Proc. Salzburg conf. 1961).
L. I. Sedov: Similarity and dimensional methods in mechanics, Infosearch Ltd., London 1959.
10) R. W. Perry and A. Kantrowitz, *J. Appl. Phys.*, ***22***, p. 878 (1951).
11) I. F. Kvartskava *et al.*, Nucl. Fus. Suppl. 1962, p. 535 (Proc. Salzburg conf. 1961).

List of symbols used in Chapter 6

B	magnetic field strength	u, v	thermal and flow velocity
c	speed of sound	x, y, z	coordinates
k	wave number or Boltzmann's constant	W	energy
l	length	γ	specific heats rapport
M, m	mass	δ	depth
M_s	Mach number	λ	wave length or width of a shock front
n	particle density	ρ	mass density
p	pressure	$\xi\ \psi\ \varphi$	reduced variables
R	radius	κ	heat conductivity
t	time	τ	time interval
T	temperature	ω	angular frequency

CHAPTER 7

PLASMA DYNAMICS

Introduction

Having studied shocks we have already encountered several typical problems and concepts of plasma dynamics. In this chapter we shall study the motion of plasma bunches and the steady plasma flow, choosing these as representatives of two contrasting dynamical situations.

7.1. Plasmoids

Individual clouds of plasma are known as plasmoids. Plasmoids can be generated by electrical discharges and may possess an internal distribution of currents, charges and electric and magnetic fields. Such distributions are the result of the birth-conditions of the plasmoid and of its interaction with the external fields.

Let us discuss first the interaction of a nearly spherical plasmoid with a magnetic field. We shall assume that the random velocities of the particles in the plasmoid are very small compared with their directed velocity. In that case the plasmoid expands by only a small fraction of its diameter whilst it traverses a distance of several diameters. Let such a plasmoid be shot into a magnetic field $\boldsymbol{B}$ (fig. 118). As the plasma is good conductor the magnetic field will penetrate only to a certain skin depth, which we shall assume to be small compared with the diameter of the plasmoid.

The plasmoid will, therefore, expell the magnetic field from a certain volume Ω. The magnetic field $\boldsymbol{B}$ on the surface of the plasmoid

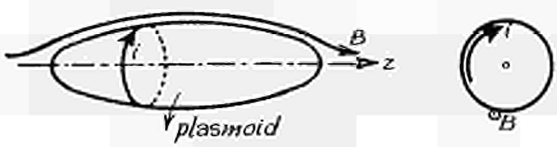


Fig. 118

will induce surface currents $\boldsymbol{i}$ and compress the plasmoid by a pressure

$$p = \boldsymbol{i} \wedge B.$$

The same pressure will also compress the magnetic field. The energy expended in these compression processes is of the order of $\Omega(B^2/8\pi)$.

This energy has to be supplied from the kinetic energy of the plasmoid. Thus on entering the magnetic field the plasmoid will be decelerated and compressed. Part of the lost kinetic energy will be converted into the stored energy of the magnetic field and part into the heating of the plasmoid. Depending on the shape of the lines of force the plasmoid may, of course, expand in the direction parallel to these lines and change its shape from that of a sphere to that of an elongated ellipsoid (fig. 118).

Let us now observe two such plasmoids, each possessing a system of surface currents, in a magnetic field. It is evident that these two plasmoids can interact through their surface currents. As long as such currents are induced by the magnetic field, the plasmoids may pass through each other, as the lines of magnetic field between them can slip to the side and the currents can join (fig. 119). During the passage of these plasma clouds through each other the collisions between the positive ions and electrons will convert part of their kinetic energy into heat *. This may result in the plasmoids being unable to separate from each other after the collision and thus lead to a formation of a single slow plasmoid. That plasmoids can interact in such a way has been demonstrated in experiments on the formation of plasma rings out of several colliding plasmoids (ref. 1, fig. 120).

Hydromagnetic elements of a turbulent gyrotropic plasma are probably structures similar to plasmoids. Let us consider a plasmoid shot

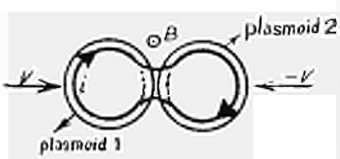


Fig. 119. Collision of two plasmoids.

Fig. 120. Formation of a plasma ring from 4 colliding plasmoids as evidenced by light emitted at 2, 4 and 6 μsec.
Source: W. H. Bostick, *Phys. Rev.*, **106** (1957) 409, fig. 13.

* Apart from binary interactions cooperative interactions may be responsible for such conversion.

into a viscous, gyrotropic plasma. The viscous forces cause an instability of the boundary of the plasmoid (chapter 8, p. 277). This instability gives birth to a new generation of small plasmoids (fig. 121) which in their turn produce others. In this way a whole spectrum of plasmoids having smaller and smaller dimensions is produced. The kinetic energy distribution in this spectrum will probably resemble the Kolmogoroff spectral law for eddies in the turbulent state of a magnetic field-free plasma (ref. 2).

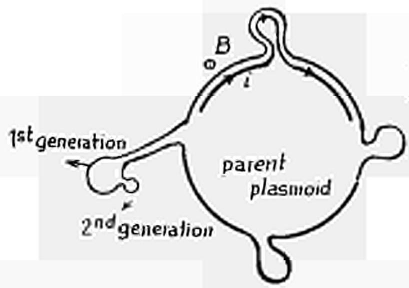


Fig. 121. Formation of a new generation of plasmoids.

We have mentioned that plasmoids of sufficiently large kinetic energy can move across a magnetic field. If, therefore, plasmoids are generated in a plasma immobilized by an external magnetic field, one may expect that some of these will break away from the main body of the confined plasma. The loss of particles due to such a process can be substantially higher than the diffusion loss discussed in chapter 8. The theory of such a disassembly of a turbulent plasma in a magnetic field has not yet been worked out.

When plasmoids are accelerated or decelerated by a magnetic piston it is convenient to consider an equivalent situation of a plasma in a strong gravitational field g supported by a magnetic field. This model has been already used in chapter 5 (p. 189) where it has been demonstrated that in a plane geometry one can expect the plasma-magnetic field interface to be unstable. Before an instability develops, the plasma will reach an equilibrium density distribution which will follow the barometric law, i.e.,

$$n = n_0 \cdot \exp\left(-\frac{x}{\delta}\right) \tag{1}$$

where

$$\delta = \frac{k(T_e + T_i)}{g(M + m)}, \quad n_0 = \frac{B^2}{8\pi k(T_e + T_i)}. \tag{2}$$

The thickness δ is also known as the Haley thickness.

Plasmoids are accelerated in devices known as plasma guns. These can be divided in two classes, those in which the plasmoid is in contact with electrodes and induction or electrodeless ones. The representatives of the first group are the T-tube, the rail- and the coaxial gun. The devices of the second group are all derivatives of the thetatron guns.

The T-tube (fig. 122). A breakdown between two electrodes generates a current channel in a plasma. The magnetic field of this current is stronger on the side of the back-conductor (back-strap) S than on the external side of the circuit and the force $F = \int_\Omega B_\varphi \cdot i_z d\Omega$ (Ω is the volume of the channel) drives the plasma out of circuit, forming thus a fast plasmoid which may or may not have an internal trapped

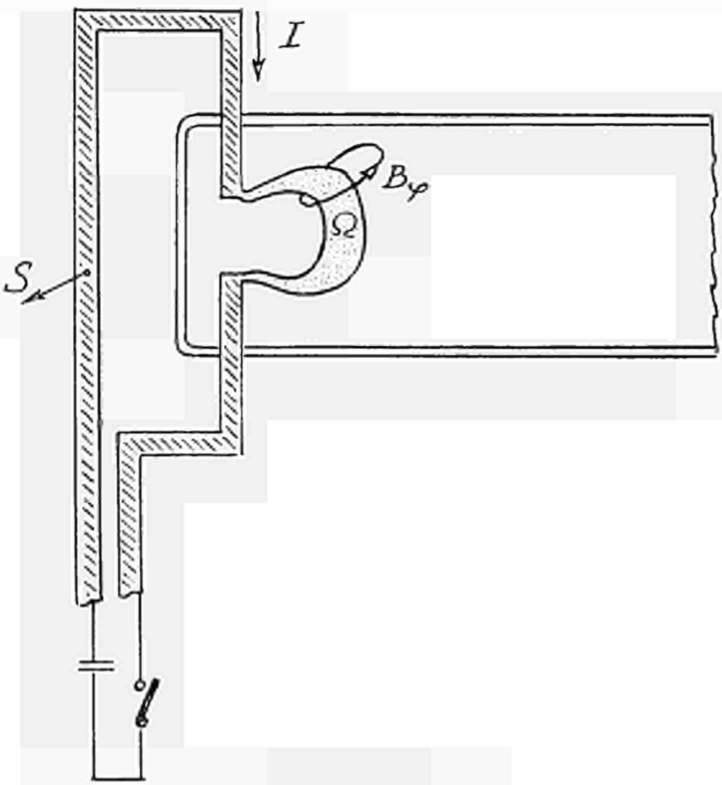


Fig. 122. The T-tube.

magnetic field. This plasmoid may be then guided by an auxiliary magnetostatic field and kept away from the walls of the vacuum vessel.

The rail-gun (fig. 123). In the *T*-tube the action of the magnetic field is limited in space and, therefore, also in time. In order to provide a prolonged acceleration of the plasma, rail-electrodes are used. In most plasma guns the plasma in formed by ionizing an injected gas cloud. Such inpection is achieved by fast gas valves (ref. 3) in which case it is almost always difficult to generate well localized plasmas (ref. 4). In rail-guns the plasma to be accelerated has been often generated by an electrically exploded thin wire connected to the two rails (ref. 5).

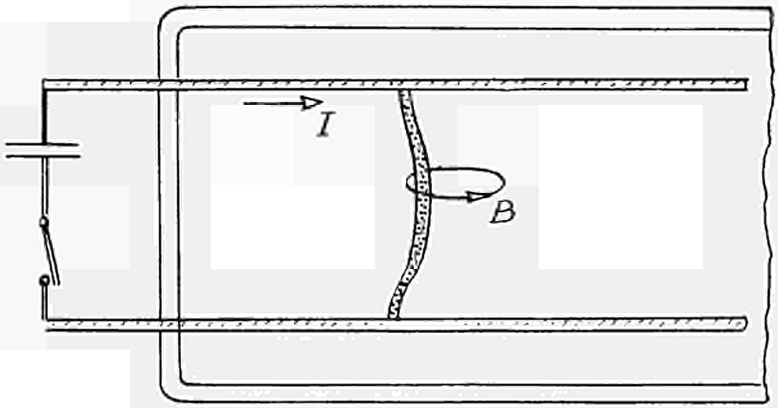


Fig. 123. The rail-gun.

The coaxial gun (fig. 24). This represents an improvement on the rail-gun as far as the efficiency of propulsion is concerned. Here all the magnetic field is behind the plasma. In the rail-gun part of the magnetic field was being used in a *Z*-pinch of the driven plasma, in the coaxial gun all of it is used for propulsion. On the other hand the force $\frac{B^2}{8\pi}$ is not uniform behind the plasma, it has its maximum on the inner conductor (ref. 6).

Topologically related to the coaxial gun is a cylindrical gun in which a plasma layer is driven by an azimuthal magnetic field (a dynamic hollow *Z*-pinch, ref. 7).

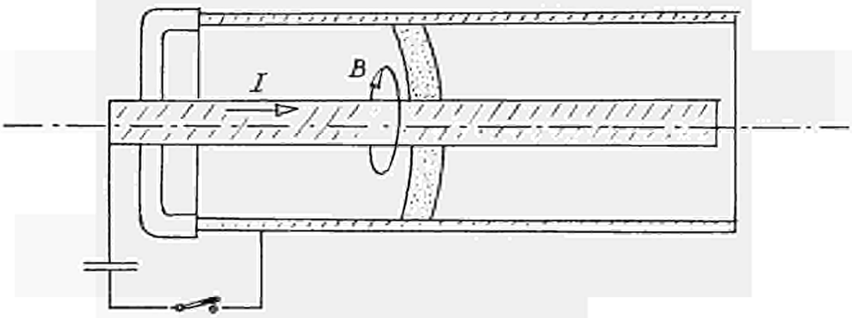


Fig. 124. The coaxial gun.

We shall give a short analysis of such a magnetically driven cylindrical layer of plasma as this is a configuration often approximated by experimental arrangements.

The motion of a cylindrical plasma shell due to a magnetic field B of the current I flowing through the plasma in the direction of axis of symmetry (fig. 125) can be described by the following set of equations

$$lM_S\ddot{r} = 2\pi \frac{B_\theta^2}{8\pi} l \tag{3}$$

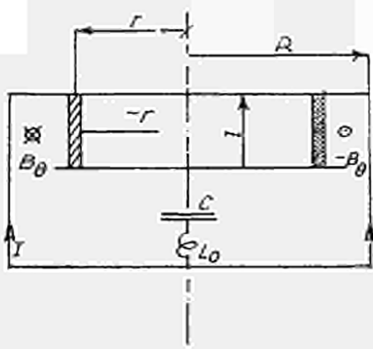


Fig. 125. A cylindrical shock magnetically driven into vacuum.

where

M_S is the mass of the plasma shell per cm length
l is the length of the shell
B_θ is the self-magnetic field at the shell and it is

$$B_\theta = \frac{0.2I}{r} \tag{4}$$

where I is the total current in amps.
and therefore, eq. (3) can be written as

$$r\ddot{r} = -\frac{I^2}{100M_S} \text{ (cm, cm/sec}^2\text{, amps, g).} \tag{5}$$

The equation for the current I is

$$I = -C\frac{\partial V}{\partial t} = -C\frac{\partial^2}{\partial t^2}(LI) \tag{6}$$

where

C is the capacity of the condenser bank (F)
V is the voltage across the same bank (volt)
L is the total inductance of the circuit (H).

The latter is

$$L = L_0 + 2 \times 10^{-9} l \ln \frac{R}{r} \text{ (henry).}$$

In many important cases the variable part of the inductance is comparable or smaller than L_0 and also owing to the logarithmic dependence on R/r the inductance is a slowly varying function the shell-radius r. We shall, therefore, assume that for $r/R > 0.5$ the $L = \text{const.} = L_0(1+f)$ and the $f \lesssim 1$. In that case eq. (6) can be written as

$$\omega^2 I = -I \tag{6a}$$

here

$$\omega^2 = \frac{1}{CL_0(1+f)}.$$

The solution of eq. (6a) is

$$I = I_0 \sin \omega t. \tag{7}$$

As at $t = 0$, one has $I = 0$, $V = V_0$ it follows that

$$I_0 = \omega C V_0 = \sqrt{\frac{C}{L_0(1+f)}}\,V_0. \tag{7a}$$

As a result of the assumption about L and L_0 the equations (5) and (6) are not coupled. The discharge is current-fed and its motion does not influence the current flow appreciably. Substituting, therefore, eq. (7) into eq. (5) one obtains a simple differential equation for the radial motion

$$-r\ddot{r} = \frac{CV_0^2 \sin 2\omega t}{100M_S L_0(1+f)}. \tag{8}$$

If r varies slowly with regard to time as compared to $\check{r}$ one may investigate the equation (8) for $r > \frac{1}{2}R$ by assuming that

$$r = \text{const.} \approx \tfrac{3}{4}R.$$

Thus

$$\ddot{r} = -a \sin^2 \omega t \tag{8a}$$

where

$$a = \frac{CV_0^2}{100ML_0(1+f)\frac{3}{4}R}.$$

The solution is

$$-\dot{r} = \tfrac{1}{2}a\left\{t - \frac{1}{2\omega}\sin 2\omega t\right\} \tag{9a}$$

$$R - r = \tfrac{1}{2}\left\{\tfrac{1}{2}t^2 - \frac{1-\cos 2\omega t}{(2\omega)^2}\right\}a. \tag{9b}$$

Let us calculate $\dot{r}$ and r at $t = \pi/\omega$, i.e., at the first reversal of current I. These are

$$\dot{r}_1 = \frac{\pi CV_0^2\sqrt{CL_0(1+f)}}{150M_S L_0(1+f)R} \approx 10^{-2}\frac{2C^{3/2}V_0^2}{\sqrt{L_0(1+f)}M_S R} \text{ (cm/sec)} \tag{10a}$$

$$\Delta r = R - r_1 = \frac{\pi^2 C^2 V_0^2}{300 M_S R} \text{ (cm)}. \tag{10b}$$

Example: $C = \frac{1}{2}10^{-5}$ (F), $V_0 = 10^4$ (volts), $R = 10$ (cm), $L_0(1+f) = 10^{-7}$ (H), $M \cong 10^{-6}$ g. Then:

$R - r_1 = 8$ (cm), $\dot{r}_1 = 2/3 \times 10^7$ (cm/sec),

$$t = \pi\sqrt{CL_0(1+f)} = 2\ (\mu\text{sec}).$$

The major loss of electromagnetic energy from the L, C circuit is in the conversion to the kinetic energy of the driven plasma. This kinetic energy is

$$W_{\text{kin}} = \tfrac{1}{2}M_S\dot{r}^2. \tag{11}$$

After the first half cycle of current there is

$$W_{\text{kin}} = \tfrac{1}{2}lM_S\dot{r}_1^2 = \tfrac{1}{2}lM_S\left[10^{-2}\frac{2C^{3/2}V_0^2}{\sqrt{L}M_S R}\right]^2$$

i.e.,

$$W_{\mathrm{kin}} = 2 \times 10^{-11} \frac{C^3}{M_S L} \frac{l}{R^2} V_0^4 \text{ (joule).} \tag{11a}$$

Whilst the plasma shell travels between the two disc-electrodes it liberates and ionizes gas molecules from the electrode surfaces. These new ions may slow down the radial motion of the plasma and in some experiments this appears to be a dominant feature. As this effect depends on the surface conditions and other factors, it seems that the analysis of our problem need not be refined by removing the assumptions we have made, i.e., the assumptions about L_0 and the slow variation of r.

Thetatron-gun (fig. 126). Relies on the one-sided pressure developed by a fast rising cusp-like field, the breakdown of the injected gas being effected by the azimuthal electric field $E_\varphi = \dfrac{1}{2\pi c r}\dfrac{\partial\phi}{\partial t}$, where ϕ is the magnetic flux threading the E_φ loop. The required convergence of the field lines can be achieved by making the θ-tron coil conical or by inserting a cone like conductor into a cylindrical coil (ref. 8).

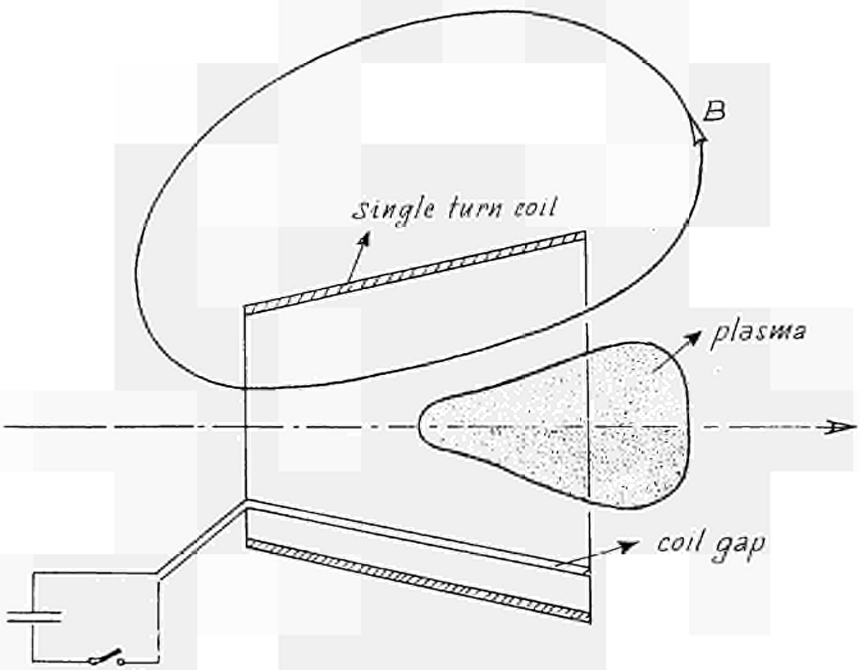


Fig. 126. The thetatron-gun.

7.2. Steady Plasma Flow

Let us imagine that a high pressure reservoir of plasma A is connected to a low pressure reservoir B through a region bounded by electrodes or magnetic flux tubes. The plasma will begin to flow, building up in certain cases a steady distribution of ρ, v and p. Such a flow consists generally of two basic regimes related to the classical examples of viscous flow between parallel plates in a magnetic field: the Poiseuille and the Couette flows.

The Poiseuille flow

In this type of flow the force is provided by a pressure gradient directed along the stream lines. The resistance to the flow is the Lorentz force and the friction between the fluid and the walls. In the one-fluid model approximation we have

$$\frac{\partial}{\partial x}(\rho v) = 0 \tag{12}$$

$$\frac{\partial p}{\partial x} = i_y B_z - \mu \frac{\partial^2 v}{\partial y^2}. \tag{13}$$

The only electromotive force capable of driving i_y is $\frac{1}{c} v_x B_z$. This can locally induce

$$i_y = \frac{1}{c} \eta v B_z \tag{14}$$

provided lossless i_z currents can exist so that div $\boldsymbol{i} = 0$. Assuming that $\frac{\partial \rho}{\partial x} = 0$ and that $\frac{\partial p}{\partial x} = P$ is given, these equations give us $v = v(y)$. We have

$$\frac{\partial^2 v}{\partial y^2} - \frac{\eta B_z^2}{c\mu} \cdot v = P/\mu \tag{15}$$

whose solution is

$$v = \alpha \cdot \exp(ay) + \beta \cdot \exp(-ay) - \frac{Pc}{\eta B_z^2} \tag{16}$$

where $a^2 = \frac{\eta B_z^2}{c\mu}$.

If we require that $v = 0$ at the walls ($y = \pm D$) we get

$$v = \frac{-P}{B_z^2\eta}\left[1 - \frac{\cosh(ay)}{\cosh(aD)}\right]. \tag{17}$$

In absence of magnetic field ($a = 0$) the velocity distribution becomes parabolic

$$v = -\frac{PD^2}{2\mu}\left[1 - \left(\frac{y}{D}\right)^2\right] \tag{18}$$

a result well known from hydrodynamics of viscous fluids. In fig. 127 are plotted three profiles of $v(y)$ for aD = 0, 2.5 and 5. The presence of the magnetic field flattens the v-distribution and diminishes the rate of flow.

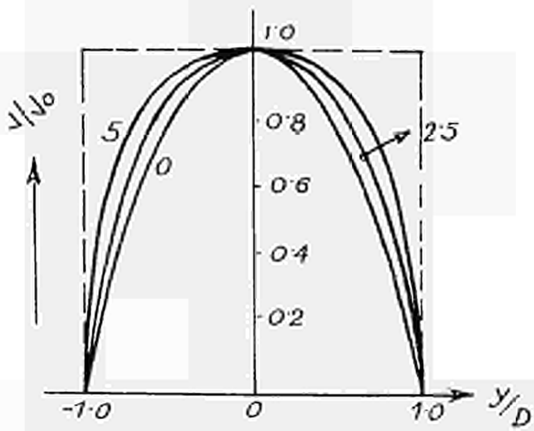


Fig. 127. Velocity profiles for $aD = 0, 2.5, 5$ (Poiseuille flow).

The Couette flow

This flow is driven by the viscous shear alone. There is no pressure gradient. The equations are

$$i_y B_z = \mu \frac{\partial^2 v}{\partial y^2} \tag{19}$$

$$i_y = \frac{\eta}{c} v B_z \tag{20}$$

having made again the assumption that the continuity of current flow is made possible by dissipationless i_z currents and that $\dfrac{\partial p}{\partial x} = 0$.

Then

$$\frac{\partial^2 v}{\partial y^2} - \frac{\eta B_z^2}{c\mu} = 0 \tag{21}$$

whose solution is

$$v = v_D \frac{\sinh(ay)}{\sinh(aD)} \tag{22}$$

where $a^2 = \dfrac{\eta B_z^2}{c\mu}$ and v_D is a given velocity at $y = D$. In absence of magnetic field, i.e., for $a = 0$ we have the classical linear law (fig. 128),

$$v = v_D \cdot \frac{y}{D}\,.$$

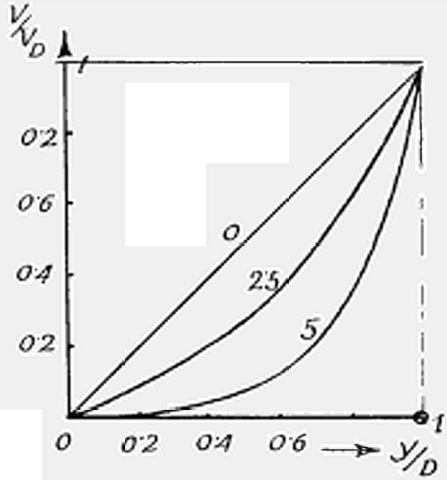


Fig. 128. Velocity profiles for $aD = 0$, 2.5, 5 (Couette flow).

More realistic results would have been obtained if the equation for the current density were written as

$$i(R + R_p) = \frac{B_z}{c} \int_0^D v \, \mathrm{d}y \tag{23}$$

which respects directly the current continuity and the resistance R of an external circuit.

Magnetohydrodynamic propulsion

The force $\boldsymbol{i} \wedge \boldsymbol{B}$ can be applied equally well for steady deceleration as for steady acceleration of plasma. This momentum-transmission property of magnetic fields in conducting fluids is of importance in magnetic pump (ref. 9) and explains also the distribution of angular momentum in an evolving solar system where the central solar mass transmits most of its angular momentum to the planets (ref. 10).

REFERENCES TO CHAPTER 7

1) W. H. Bostick, *Phys. Rev.*, ***104***, pp. 292-299 (1956); and ***106***, pp. 404-412 (1957).
2) I. Kolmogorov, *Comptes R. Ac. Sci. USSR* (DOKLADY), ***30***, p. 301 (1940).
3) B. Gorowitz *et al.*, *Rev. Sci. Instr.*, ***31***, p. 146 (1960).
4) W. H. Bostick, *Phys. Fluids*, ***6***, p. 1598 (1963).
L. Liebing, *Phys. Fluids*, ***6***, p. 1035 (1963).
5) L. A. Artzimovich *et al.*, *JETP*, ***6***, p. 1 (1958).
6) J. Marshall, *Phys. Fluids*, ***3***, p. 134 (1960).
7) Ch. Maisonnier *et al.*, *Nucl. Fus.* 1962 Suppl., p. 727 (Proc. Salzbourg Conf. 1961).
8) F. Waelbroeck *et al.*, *Nucl. Fus.* 1962 Suppl., p. 675 (Proc. Salzburg Conf. 1961).
9) L. R. Blake, Proc. Inst. El. Eng., ***104***, part A (1957).
10) F. Hoyle, *Quar. Jour. Roy. Astro. Soc.*, ***1***, p. 28 (1960).

List of symbols used in Chapter 7

B, $\boldsymbol{B}$	magnetic field strength	R	radius
c	velocity of light	t	time
C	capacity	T	temperature
D	distance	u, v	velocity
e	charge of electron	V	voltage
E	electric field strength	W	energy
i	current density	x, y, z	coordinates
I	current	Z	atomic number
k	wave number or Boltzmann's constant	α	number of degrees of freedom
l	length	δ	depth
L	self-inductance	η	conductivity
m, M	mass	μ	coefficient of viscosity
n	particle density	ϕ	magnetic flux
N	linear density	ρ	mass density
p	pressure	τ	time interval
r	coordinate	ω	angular frequency

CHAPTER 8

COLLISION AND RELAXATION PROCESSES

Introduction

This chapter is devoted to the study of collisions in plasma and their effect on the density distribution $f(q_i v_i)$.

There are three important types of collisions. The first and simplest is a binary collision of two electrically charged particles such as an electron-positive ion (e — p), electron-electron (e — e) or an ion-ion (p—p) collision.

The second type of collision belongs to the class of collective interactions in which a single charged particle is influenced by many other charged articles.

The third is a collision between plasma bunches, such as two plasmoids or two electron bunches. Such an event will be called a coherent collision (ref. 1).

We shall study mainly the binary and collective collisions. The coherent collisions have been treated already in the chapter on plasma dynamics and will be mentioned here only in connection with the mechanism of turbulence.

We shall describe first the elementary dynamics of a binary collision. As we shall see binary collisions produce a diffusion of particles, both in velocity-space and in configuration-space. We shall, therefore, derive an equation governing the diffusion in the velocity-space. As an example of the application of this equation we shall calculate the electric current induced by an electric field in an infinite plasma. Another example of diffusion in velocity space is the heat transfer between the electron and the positive ion gas.

The collision-induced diffusion in ordinary space will be studied for three particular cases: the diffusion of particles, of heat, and of electric charge, in a non-uniform plasma.

The transfer of momentum by viscous effects will be also discussed.

The chapter will be terminated by a few remarks on turbulence.

8.1. Dynamics of a Collision of two Charged Particles

In a two component plasma one is concerned with either collisions between like particles, i.e., particles having equal masses and charges, or unlike particles, i.e., electron-positive ion collissions.

Both these types belong to the simplest binary collisions as in the first case both participants in the collision suffer the same deflection and in the second case the electron is deflected, whereas the position of the heavy positive ion, is virtually unchanged during the collision.

The angle of deflection χ follows from a Keplerian analysis of a related gravitational problem and is given by

$$\operatorname{cotg}\frac{\chi}{2} = \begin{cases} \dfrac{mp(\boldsymbol{v}-\boldsymbol{w})^2}{Ze^2} & \text{for e — p collisions} \quad (1a) \\[2ex] \dfrac{Mp(\boldsymbol{w}_1-\boldsymbol{w}_2)^2}{2Z^2e^2} & \text{for p — p collisions.} \quad (1b) \end{cases}$$

Here p is the collision (impact) parameter, i.e., the smallest distance between the trajectories of the two particles were they not affected by the Coulomb forces. The actual distance p' of closest approach is smaller than p in e-p collisions (fig. 129) but larger than p in like particle collisions (fig. 130) which is due respectively to the attractive, repulsive force active in such collisions.

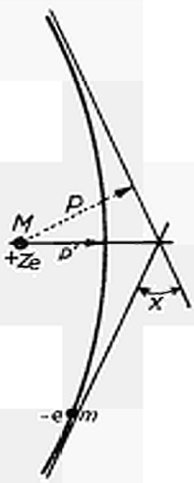


Fig. 129. Geometry of an electron-nucleus collision.

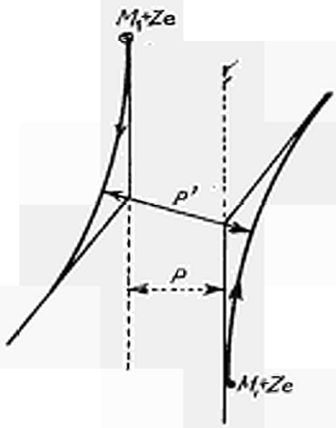


Fig. 130. Ion-ion collision.

Let us find the cross-section $d\sigma$, corresponding to a differential range of scattering angles χ, $\chi + d\chi$. For small angles χ one has, for e—p collisions

$$\chi = \frac{2Ze^2}{m(v-w)^2} p^{-1} \tag{2}$$

and

$$\frac{d\chi}{dp} = -\frac{2Ze^2}{m(v-w)^2} p^{-2} = -\frac{m(v-w)^2}{2Ze^2} \chi^2.$$

The cross-section is, therefore, (fig. 131)

$$\begin{aligned} d\sigma &= 2\pi p \, dp \\ &= \frac{8\pi Z^2 e^4}{m^2(v-w)^4} \frac{d\chi}{\chi^3}. \end{aligned} \tag{3}$$

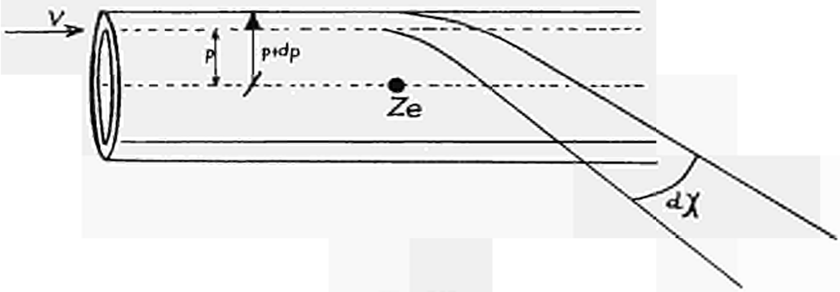


Fig. 131.

Evidently, as $\chi \to 0$ so p and $d\sigma \to \infty$. In plasma there exists an upper limit to p which is approximately equal to the Debye distance λ_D *. A collision whose impact parameter p is larger than λ_d cannot be regarded as a binary collision, because of the screening (polarizing) effect of charges surrounding the target particle. In spite of this cut-off in the collision parameter, the form of eq. (3) suggests that small angle collisions will influence the trajectory of a charged particle more than comparatively rare large angle scatterings.

The effect of these multiple collisions may be taken into account as follows. Considering only small scattering angles (for which a deflection is an event independent of the previous collisions (fig. 132) the mean square deflection after N collisions is

* λ_d must be redefined for a fast incident particle. However, as will be seen, $\langle\chi^2\rangle$ is not critically dependent on λ_d.

$$\langle \chi^2 \rangle = \sum_{i=1}^{N} \langle \chi_i^2 \rangle \tag{4}$$

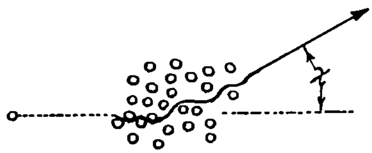


Fig. 132. Cumulative effect of small angle scatterings.

where $\langle \chi_i^2 \rangle$ is the mean square deflection in a single collision and is

$$\langle \chi_i^2 \rangle = \int_{A_{min}}^{1} \chi^2 \, dA, \qquad \chi = \chi[p(A)]$$

The probability dA of a collision in the interval p, p + dp is a simple ratio of areas

$$dA = \frac{2p \, dp}{p^2_{max} - p^2_{min}}. \tag{5}$$

Restricting ourselves to e—p collisions for which $v \ll w$ and expressing χ in terms of p we get

$$\langle \chi_i^2 \rangle = \frac{2}{p^2_{max} - p^2_{min}} \int_{p_{min}}^{p_{max}} \left(\frac{2Ze^2}{mv^2 p} \right)^2 p \, dp$$

$$\cong \frac{8Z^2 e^4}{m^2 v^4 p^2_{max}} \ln \frac{p_{max}}{p_{min}}. \tag{6}$$

As statistically all the N collisions are equivalent

$$\langle \chi^2 \rangle = N \langle \chi_i^2 \rangle \tag{4a}$$

where N is equal to the number of positive ions in a cylinder of radius p_{max} and length l. This number is

$$N = nl\pi p^2_{max}.$$

Therefore,

$$\langle \chi^2 \rangle = \frac{8\pi nlZ^2 e^4}{m^2 v^4} \ln \frac{p_{max}}{p_{min}}. \tag{4b}$$

As we have postulated only small-angle deflections let us have $p_{\min}$ correspond at most to $\chi = 1$. Thus *

$$p_{\min} = \frac{2Ze^2}{mv^2}.$$

The $p_{\max}$ must be comparable with the Debye shielding distance ** d (eq. (1.4a)). With this eq. (4b) can be written

$$\langle \chi^2 \rangle = \frac{8\pi Z^2 e^4}{m^2 v^4} \, ln \ln \left[d \, \frac{mv^2}{2Ze^2} \right]. \tag{4c}$$

The value of the logarithm lies in the range between 1 and 20 in most cases in which we are interested.

The corresponding diffusion coefficient $\langle (\Delta v_\perp)^2 \rangle$ is the mean square of the perpendicular velocity change per second and is (i.e., $l = v$)

$$\langle (\Delta v_\perp)^2 \rangle \cong \langle (\chi v)^2 \rangle.$$

Thus

$$\langle (\Delta v_\perp)^2 \rangle = \frac{8\pi Z^2 e^4}{m^2 v} \, n \ln \Lambda, \tag{7}$$

where

$$\Lambda = 5 \left(\frac{T}{n} \right)^{1/2} \frac{mv^2}{Ze^2}.$$

A similar coefficient can be defined for the diffusion parallel to $\boldsymbol{v}$. This is usually denoted by $\langle \Delta v_\parallel \rangle$ and has been called by Chandrasekhar (ref. 2) the coefficient of dynamical friction.

For electrons this coefficient follows from the energy-conservation principle,

$$-\tfrac{1}{2} m \langle (\Delta v_\perp)^2 \rangle = mv \langle \Delta v_\parallel \rangle$$

and using the expression (7) for $\langle (\Delta v_\perp)^2 \rangle$ one gets

$$\langle \Delta v_\parallel \rangle = - \frac{4\pi Z^2 e^4 n \ln \Lambda}{m^2 v^2}. \tag{8}$$

The third important coefficient is $\langle (\Delta v_\parallel)^2 \rangle$ representing dispersion in velocity in the direction parallel to v. This can be shown to be

$$\langle \Delta v_\parallel^2 \rangle = \frac{4\pi e^4 Z^2 n \ln \Lambda}{m^2 v^3} \, w^2. \tag{9}$$

* In plasmas where the minimum interaction distance is governed by quantum mechanics $p_{\min} = \lambda\!\!\!^{-}_B = \dfrac{h}{2\pi\sqrt{3mkT}}$ (see p. 15).

** Unless $l = n^{-1/3} > d$ (see footnote p. 13)

Such a dispersion is due to the motion of the target particles. In case of fluids the target particles are called the *field particles*, the incident particles are called *test particles*. If the mean speed of the field particles is w, then in some collisions w is added, in others w is substract-ed from the vector v of the test particles. This results in dispersion above and below the arc A (fig. 133) representing the constant speed $|v|$ in the velocity space.

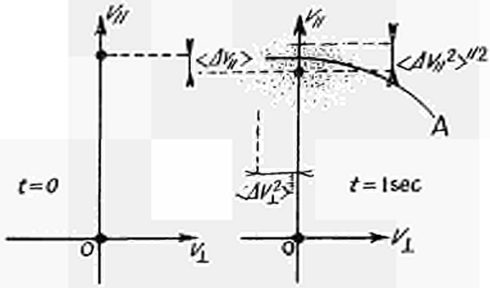


Fig. 133. Diffusion in velocity space caused by collisions after 1 sec.

With the aid of these three coefficients it is possible to describe the diffusion of a group of particles (test particles) in velocity space resulting from their collisions with another group of particles (field particles). The meaning of all these coefficients is illustrated in fig. 133.

8.2. Fokker-Planck Equation

Having defined the friction coefficient $\langle \Delta v_{\parallel} \rangle$ and the dispersion coefficients $\langle (\Delta v_{\parallel})^2 \rangle$, $\langle (\Delta v_{\perp})^2 \rangle$ we shall formulate mathematically the problem of scattering of many particles by many particles as a problem of diffusion in a phase space q_i, v_i. Following the notation of chapter 3, we shall define $f(q_i, v_i)$ as the particle density in the phase space.

Liouville's theorem is not valid for an ensemble of colliding particles and we must add a term to eq. (3.8b) expressing the dispersing action of collisions. This can be done formally as follows (in non-relativistic approximation)

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial q_i} + \frac{F_i}{m} \frac{\partial f}{\partial v_i} = \left(\frac{\partial f}{\partial t} \right)_{\text{coll.}} \tag{10}$$

Let us consider first an infinite and homogeneous plasma in the absence of external forces.

Eq. (10) must be applied to electrons and ions separately. Thus

$$\frac{\partial f_e}{\partial t} = \left(\frac{\partial f_e}{\partial t}\right)_{ee} + \left(\frac{\partial f_e}{\partial t}\right)_{ep} \tag{10a}$$

$$\frac{\partial f_p}{\partial t} = \left(\frac{\partial f_p}{\partial t}\right)_{pp} + \left(\frac{\partial f_p}{\partial t}\right)_{pe} \tag{10b}$$

where the subscript denotes the type of collision responsible for the $(\partial f/\partial t)$.

Let us now consider a small group of particles located between v and $v + \Delta v$ at $t = 0$ (fig. 134) and assume that all quantities depend only on a single velocity component v.

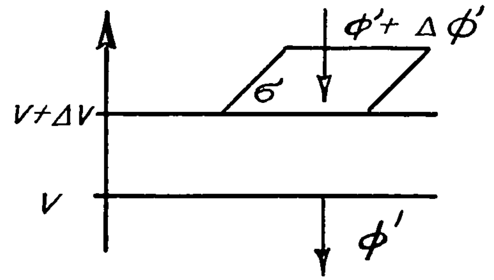


Fig. 134. -

The flux of particles across a unit cross-section σ of the surface at v caused by dynamical friction is

$$\phi' = f \cdot \langle \Delta v \rangle$$

where $\langle \Delta v \rangle$ is related to the concept of $\langle \Delta v_{\parallel} \rangle$ (eq. (8)).

The divergence of this flux in the velocity space and, therefore, the change in the number of particles in a unit velocity volume per second is

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v} \left[f \langle \Delta v \rangle \right] . \tag{11}$$

A change in the number of particles in our velocity volume can also be caused by diffusion processes, characterised by the diffusion coefficients (eqs. (7) and (9)). In our one-dimensional case let us denote this coefficient by $\langle \Delta v^2 \rangle$.

The diffusion equation in the velocity space must have the form

$$\frac{\partial f}{\partial t} = \operatorname{div} \phi''$$

where $\phi'' = \operatorname{grad}\ (f.D)$, D being the diffusion coefficient defined in all theories of diffusion processes as the mean square displacement per unit time. In our case

$$D = \langle \Delta v^2 \rangle$$

and consequently the divergence of the flux ϕ'' is

$$\frac{\partial \phi''}{\partial v} = \frac{\partial^2}{\partial v^2}\ [f\,\langle \Delta v^2 \rangle]. \tag{12}$$

The outflow per unit volume of the velocity space due to the combined effect of friction and dispersion is from eqs. (10), (11) and (12)

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial \boldsymbol{v}}\,(f\langle \Delta v \rangle) + \frac{\partial^2}{\partial v^2}\,(f\langle \Delta v^2 \rangle). \tag{13}$$

Generalizing this expression for all three velocity components and for both like and unlike particle collisions one obtains for electrons

$$\frac{\partial f_e}{\partial t} = -\frac{\partial}{\partial v_j}\{f_e(\langle \Delta v_j \rangle_{ee} + \langle \Delta v_j \rangle_{ep})\}$$
$$+\frac{1}{2}\,\frac{\partial^2}{\partial v_j \partial v_k}\{f_e(\langle \Delta v_j \Delta v_k \rangle_{ee} + \langle \Delta v_j \Delta v_k \rangle_{ep})\} \tag{13a}$$

and a similar equation for the positive ions. These equations are known as the Fokker-Planck equations.

The coefficients of friction and dispersion in these equations are, of course, not the simple ones defined by expressions (7), (8) and (9) but must include the scattering action of all other particles on the particles at v. They must, therefore, involve integartion over the entire velocity space of expressions like $f\langle \Delta v^2 \rangle$.

In order to do that let us first consider the effect of friction of field particles of one type (β) in a velocity volume $d\pi_\beta$ on test electrons (α) in a volume $d\pi_a$. For simplicity sake let us consider only two velocity components v_j and v_k (fig. 135). The friction coefficient for the α particles is (eq. (8))

$$d\langle \Delta v \rangle_{a\beta} = \frac{4\pi Z^2 e^4 \ln \Lambda}{m_a{}^2 |\boldsymbol{v}_a - \boldsymbol{v}_\beta|^2} \cdot f_\beta d\pi_\beta\,.$$

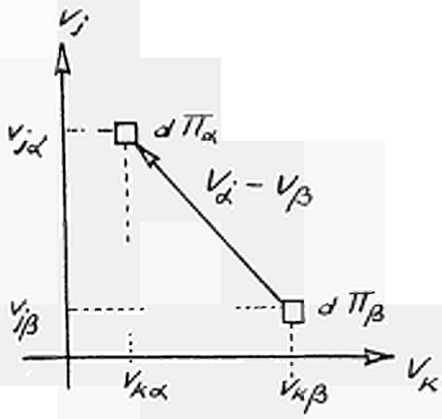


Fig. 135.

This friction coefficient expresses the mean spread of the α particles in one second and in the direction parallel to the vector $\boldsymbol{v}_\alpha - \boldsymbol{v}_\beta$. Putting $\dfrac{4\pi Z^2 e^4}{m_\alpha^2} \ln \Lambda = \Gamma_\alpha$, the component of this spread parallel to v_j is evidently

$$\mathrm{d}\langle \Delta v_j \rangle_{\alpha\beta} = - \Gamma_\alpha \frac{v_{j\alpha} - v_{j\beta}}{|\boldsymbol{v}_\alpha - \boldsymbol{v}_\beta|^3} f_\beta \cdot \mathrm{d}\pi_\beta \,. \tag{14}$$

In order to take into account the effect of the β field-particles in all the π_β space on the α test-particles at $\mathrm{d}\pi_\alpha$ we must integrate the eq. (14) over all π_β. Thus

$$\langle \Delta v_j \rangle_{\alpha\beta} = - \Gamma_\alpha \int_{\pi_\beta} f_\alpha(v_\beta) \frac{v_{j\alpha} - v_{j\beta}}{|\boldsymbol{v}_\alpha - \boldsymbol{v}_\beta|^3} \mathrm{d}\pi_\beta$$

or

$$\langle \Delta v_j \rangle_{\alpha\beta} = \Gamma_\alpha \frac{\partial}{\partial v_{j\alpha}} \int_{\pi_\beta} \frac{f_\beta(v_\beta)}{|\boldsymbol{v}_\alpha - \boldsymbol{v}_\beta|} \mathrm{d}\pi_\beta \tag{14a}$$

Γ_α multiplied by the integral is designated by $H_{\alpha\beta}(v_\alpha)$ and called the first potential in the velocity space. When the field particles are not infinitely heavy, as was postulated in deriving the eq. (8), the expression (14a) must be written as

$$\langle \Delta v_j \rangle_{\alpha\beta} = \frac{m_\alpha + m_\beta}{m_\beta} \frac{\partial}{\partial v_{j\alpha}} H_{\alpha\beta}(v_\alpha). \tag{14b}$$

Using similar reasoning and analysis as above we can obtain the expression for $\langle \Delta v_j \Delta v_k \rangle_{\alpha\beta}$. The result is

$$\langle \Delta v_j \Delta v_k \rangle_{\alpha\beta} = \Gamma_\alpha \frac{\partial^2}{\partial v_{j\alpha} \partial v_{k\alpha}} \int_{\pi_\beta} |\boldsymbol{v}_\alpha - \boldsymbol{v}_\beta| \, f_\beta(v_\beta) \mathrm{d}\pi_\beta$$

$$= \frac{\partial^2}{\partial v_{j\alpha} \partial v_{k\alpha}} G_{\alpha\beta}(v_\alpha) \qquad (15)$$

$G_{\alpha\beta}$ being known as the second potential in velocity space (ref. 3).

The form of the potential $H_{\alpha\beta}(v)$ suggests an analogy from electrostatics, where the field strength $\boldsymbol{E}$ and the electrostatic potential $\phi(\boldsymbol{r})$ are expressed as

$$\boldsymbol{E} = -\operatorname{grad} \phi, \; \phi(\boldsymbol{r}) = \int \frac{\rho(\boldsymbol{r}', t)}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}x' \, \mathrm{d}y' \, \mathrm{d}z'$$

and $\rho(\boldsymbol{r}', t)$ is the density of electric charge.

Thus in our case $\langle \Delta v_j \rangle_{\alpha\beta}$ is analogous to $\boldsymbol{E}$, $H_{\alpha\beta}(v)$ to $\phi(\boldsymbol{r})$, $f_\beta(v_\beta)$ to $\rho(\boldsymbol{r}')$ and $|v_\alpha - v_\beta|$ to $|\boldsymbol{r} - \boldsymbol{r}'|$.

In chapter 3 on the fluid description of a plasma we have shown that the integration of the Boltzmann equation over velocity space yields fluid equations which describe the transport of particle density, momentum or energy. Let us now integrate over velocity space the Boltzmann equation containing the collision terms of the Fokker-Planck equation, multiplied by a quantity Q characterising some particle property. In order to obtain an equation for momentum transfer we take $Q = mv$. The integration of the left hand side of the Boltzmann equation proceeds as on p. 100 i.e., independently of the effets of collisions. The right hand side involves integrals such as

$$-\int_\pi m_\alpha \boldsymbol{v} \frac{\partial}{\partial v_j} \left[f_\alpha(v) \frac{\partial}{\partial v_j} H_{\alpha\beta}(v) \right] \mathrm{d}\pi$$

$$+ \tfrac{1}{2} \int_\pi m_\alpha \boldsymbol{v} \frac{\partial^2}{\partial v_j \partial v_k} \left[f_\alpha(v) \frac{\partial^2}{\partial v_j \partial v_k} G_{\alpha\beta}(v) \right] \mathrm{d}\pi.$$

It can be appreciated that encounters between like particles do not alter the total momentum of the parent gas. Similarly the dispersion in velocity space (the dispersion coefficients refer only to $\langle v^2 \rangle$) leaves the momentum of a gas element unchanged.

As a result of these considerations the equations of momentum transfer become:

for the electron gas

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \operatorname{grad} \boldsymbol{v} + \frac{e}{m} (\boldsymbol{E} + \boldsymbol{v} \wedge \boldsymbol{B}) + \frac{\operatorname{grad} p_e}{n_e m}$$

$$= \frac{1}{n_e} \int_\pi f_e(\boldsymbol{v}, t) \operatorname{grad}_v H_{ep} \, d\pi \qquad (16)$$

and for the positive ion gas

$$\frac{\partial \boldsymbol{w}}{\partial t} + w \cdot \operatorname{grad} \boldsymbol{w} - \frac{e}{M} (\boldsymbol{E} + \boldsymbol{w} \wedge \boldsymbol{B}) + \frac{\operatorname{grad} p_p}{n_p M}$$

$$= \frac{1}{n_p} \int_\pi f_p(\boldsymbol{v}, t) \operatorname{grad}_v H_{pe} \, d\pi. \qquad (17)$$

It is evident that one must know f_e and f_p before these equations can be solved. However, in many cases the solutions are not very sensitive to the choice of these two functions and an approximate form of these follows from order of magnitude physical considerations.

8.2.1. CONDUCTION OF ELECTRICITY IN PLASMA — CONDUCTION OF ELECTRICITY IN A GYROTROPIC PLASMA

From the generalised Ohm's law (eq. (3.64)), derived in chapter 3 it follows that an electrical current in a plasma can be induced by electric and magnetic fields and by pressure gradients.

In this section we shall discuss the induction of electric current by an electric field in the absence of a magnetic field in a spatially homogeneous plasma and later study the influence of a magnetostatic field on a current flow.

In the limit of strong electric fields the effect of electron-ion encounters may be considered as a small perturbation on the motion which the electrons and ions execute in the applied electric field. To a good approximation the electrons and ions are accelerated independently (and at a constant rate) and owing to collisions between like particles their velocity distributions will tend asymptotically in time to Maxwellian distributions which are centered about the drift velocities.

This consideration leads us to the model (fig. 136) of the *displaced* Maxwellian distribution (ref. 4)

$$f_a(\boldsymbol{r}, \boldsymbol{v}, \boldsymbol{v}_a(t)\,) = n_a(\boldsymbol{r}) \left[\frac{\beta_a}{\pi}\right]^{3/2} \exp\left(-\beta_a[\boldsymbol{v} - \boldsymbol{v}_a(t)]^2\right) \qquad (18)$$

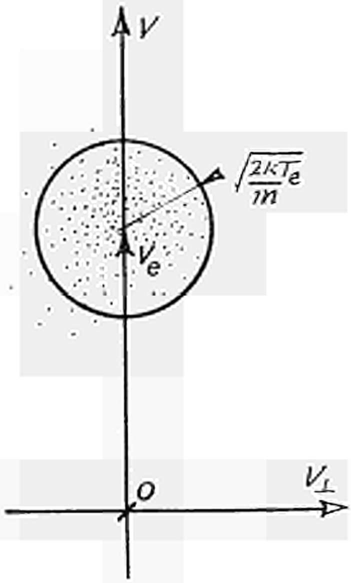


Fig. 136.

where $\beta_\alpha = m_\alpha/(2kT_\alpha)$ and α is either e or p, as an approximate solution which on the average satisfies the Boltzmann equation. Subsequent analysis shows that this distribution leads to many correct results even in the limit of weak electric fields. The $H_{e,p}$ function required for the solution of eqs. (16) and (17) may now be evaluated by substituting eq. (18) into eq. (14). Straightforward integration results in

$$H_{e,p} = n\Gamma_e \frac{m+M}{M} \frac{\Phi(\beta_p^{1/2}q)}{q} \tag{19}$$

where

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\, dt, \qquad q = |\boldsymbol{v}_e - \boldsymbol{v}_p|, \qquad \beta_p = \frac{M}{2kT_p}.$$

In most problems the average random electron speed greatly exceeds the random ion speed, and we can simplify $H_{e,p}$ appreciably by assuming that the ion gas is at zero temperature. Thus

$$H_{ep} = \frac{n\Gamma_e}{v_e} \tag{19a}$$

and after another integration equation (16) takes the form

$$m \frac{\partial v_e}{\partial t} + eE = - eE_0\psi(z) \tag{20}$$

where

$$\psi(z) = \frac{\Phi(z) - z\dfrac{d\Phi(z)}{dz}}{z^2} \tag{21}$$

$$z = \beta_e^{1/2}\,|v_e|, \quad \beta_e = \frac{m}{2kT_e}, \quad E_0 = n\frac{m}{e}\Gamma_e\beta_e.$$

The ψ function (fig. 137) expresses the variation with relative drift velocity of the dynamical friction force exerted by the ions on the electron gas. The total magnitude of this force depends also upon the coefficient E_0 which, as we shall demonstrate shortly, plays the rôle of a critical electric field parameter.

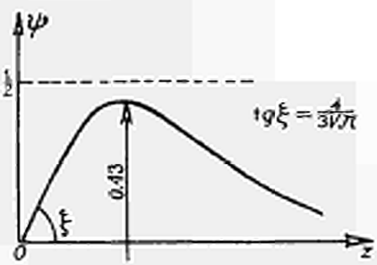


Fig. 137. The "runaway" potential.

In the limit of small z

$$\psi(z) \approx \frac{4}{3\sqrt{\pi}}\, z. \tag{21a}$$

Solving eq. (20) for this value of $\psi(z)$ one has

$$-v(t) = \frac{3\sqrt{\pi}}{4}\,\frac{E}{E_0}\,\beta_e^{-1/2}\left[1 - \exp\left(-\frac{4}{3\sqrt{\pi}}\,\frac{e}{m}\,E_e\beta_e^{1/2}t\right)\right]. \tag{22}$$

After a time

$$t \cong \frac{3\sqrt{\pi}}{4}\,\frac{m}{e}\,(E_0^2\beta)^{-1/2} \tag{23}$$

the drift speed of the electrons saturates and the current density becomes

$$j = \frac{3m}{16\sqrt{\pi}} (Z^2 e^2 \beta_e^{3/2} \ln \Lambda)^{-1} E. \tag{24}$$

The conductivity thus calculated is

$$\sigma = \frac{3m}{16\sqrt{\pi} Z^2 e^2 \ln \Lambda} \left(\frac{2kT_e}{m}\right)^{3/2} \tag{25}$$

which is about ½ of the σ calculated on the basis of a perturbational theory (ref. 5). The assumptions on which this theory is based are more appropriate for $E < E_0$ than the rather crude assumption of a displaced Maxwellian distribution. The value of conductivity derived by the perturbational method for a hydrogen plasma is

$$\sigma = 1.53 \times 10^{-4} \frac{T^{3/2}}{\ln \Lambda} \text{ (mho/cm)}. \tag{25a}$$

In the limit of large z eq. (21) gives

$$\psi(z) \approx \frac{1}{z^2}. \tag{26}$$

The maximum of $\psi(z)$ occurs for

$$z = 1$$

i.e., for a drift speed equal to the mean random speed. Substituting expression (26) into eq. (20) we see at once that when

$$E > E_0 \psi(z)$$

$v(t)$ starts increasing monotonically with t. As $\psi_{\max}(z) = \psi(1) \approx 0.43$ the above inequality becomes

$$E > 0.43 E_0 \cong 2 \times 10^{-8} \frac{n}{T_e} \text{ (volt/cm)}. \tag{27}$$

This instability of the $f_e(v)$ distribution in the velocity space is known as the runaway effect. As the electrons run away more readily than the positive ions one speaks often of "runaway electrons".

Let us now consider the problem of runaways not as a movement of a compact Maxwellian distribution given by eqs. (18) and (20), but rather as a distortion of this velocity distribution. To this end, let us evaluate the friction force experienced by a small sub-group of the electron population in the velocity space. This force $\boldsymbol{F}$ is parallel to $\boldsymbol{v}$ and consists of the friction forces between electrons of our sub-group and all the other electrons and all the positive ions. Thus

$$\boldsymbol{F} = \boldsymbol{F}_{ee} + \boldsymbol{F}_{ep}.$$

We shall neglect $\boldsymbol{F}_{ee}$ as it is initially smaller than $\boldsymbol{F}_{ep}$. The $\boldsymbol{F}_{ep}$ will be evaluated for ions at zero temperature. Then from eq. (8) and for $Z = 1$:

$$F_{ep} = \frac{4\pi e^4 \ln \Lambda}{mv^2}\, n_p. \tag{28}$$

The necessary condition for our sub-group running away is that the component of F parallel to E should be smaller than eE. Thus

$$E \geqslant 4\pi \frac{e^3 \ln \Lambda}{m}\, n_p \frac{v_z}{(v_z^2 + v_x^2 + v_y^2)^{3/2}}. \tag{29}$$

This determines a curve C in the $v_z v_x$ plane. That portion Δn of the electron population situated outside the curve C will, therefore, according to this criterion, begin running away in the v_z-direction.

Although the simple criterion (29) can be refined in several ways *, the number Δn is insensitive to the shape of curve C at large v_z. In calculating this number we shall assume, in fact, that the boundary of the run-away population is v_z = const. = $v_{z\,\max}$ where $v_{z\,\max}$ corresponds to the peak of the C curve and follows from eq. (29) by putting $v_x = v_y = 0$. Thus

$$v_{z\,\max} = \left[\frac{4\pi e^3 \ln \Lambda}{mE}\right]^{1/2}. \tag{30}$$

Let us define a field E_c for which $v_{z\,\max} = \sqrt{2kT_e/m}$.

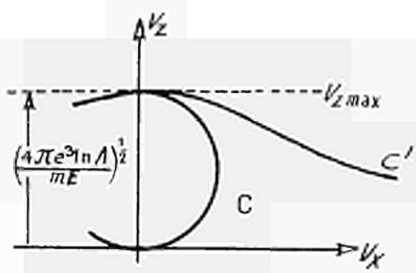


Fig. 138. The contour C above which electrons may runaway.

* Thus one should stipulate that the movement of runaway electrons in the velocity space must not lead to recrossing of the boundary C. This gives a new, more realistic, boundary C′ (fig. 138).

For this E the run-away process begins to resemble the mass-run-away described by eqs. (20) and (27). Thus

$$E_c = \frac{2\pi e^3 \ln \Lambda}{k} \, \frac{n_p}{T_e}$$

or in practical units and for $n_p = n_e = n$

$$E_c = 1.5 \times 10^{-8} \frac{n}{T_e} \text{ (volt/cm).} \tag{31}$$

Let us now calculate the portion Δn of n which is decoupled from the Maxwell distribution by a field E. This is

$$\Delta n = \int_{v_{y\,\max}}^{\infty} \int_{v_{r\,\min}}^{\infty} f(v_z, v_r) 2\pi v_r \, dv_r \, dv_z.$$

Assuming that $f(v)$ was originally Maxwellian one obtains (ref. 6)

$$\Delta n = \tfrac{1}{4} \left[1 - \phi\left(\sqrt{\frac{E_c}{E}}\right)\right] \cdot n \tag{32}$$

where ϕ is the error function (fig. 139).

Owing to collisions, the velocity distribution of those electrons that did not runaway will not remain cut off by the surface C′; these electrons will cross the surface C′ and they too will be decoupled by the field E.

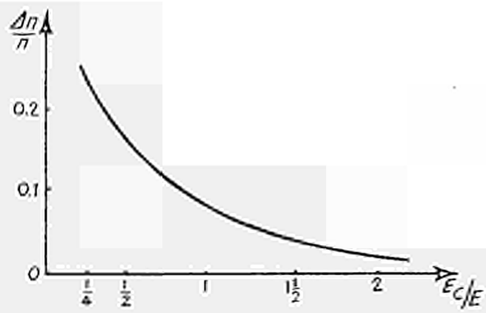


Fig. 139. Relative fraction of runaway electrons as a function of the ratio of the critical field E_c to the applied electric field E.

Let us assume that the surface C′ is a hemisphere, whose radius is $v_{r\,\max} > \sqrt{2kT_e/m}$. Then the flux ψ of electrons out of this hemisphere is a sign of a tendency to reestablish the high energy tail of a Maxwellian distribution and it follows by integrating the Fokker-Planck equation over the volume of the sphere,

$$\psi = -2\pi \int_0^{v_{z\,\max}} v^2 \frac{\partial f}{\partial t}\, dv$$

$$= 2\pi v_z{}^2{}_{\max} \{ (f\langle \Delta v_\| \rangle_{ee} + f\langle \Delta v_\| \rangle_{ep}) + \frac{\partial}{\partial v}(f\langle \Delta v_\|{}^2 \rangle_{ee} + f\langle \Delta v_\|{}^2 \rangle_{ep}) \}_{v=v_{z\,\max}} \tag{33}$$

As the surface C' was defined in such a way that the dynamical friction was balanced by the action of the electric field E, the $\langle \Delta v_\| \rangle$ terms can be ignored.

The positive ions were assumed to have zero random velocity, and therefore, in a spherically symmetrical electron distribution their contribution to ψ must be zero. Thus using eqs. (15) and (33) one has (putting $v_{z\,\max} = v_0$)

$$\psi = 2\pi v_0{}^2 \left[\frac{\partial}{\partial v} (f\langle \Delta v_\|{}^2 \rangle_{ee}) \right]$$

$$\approx \tfrac{1}{2}\pi^{-3/2} \frac{\omega_p{}^2}{v_t{}^3} \cdot \exp\left(-\frac{v_0{}^2}{v_t{}^2}\right) \cdot \ln \Lambda \tag{33a}$$

where $v_t{}^2 = \beta_e{}^{-1}$.

This is the runaway flux that can be drawn from a quasi Maxwellian plasma by weak electric fields, i.e., by $E \ll E_c$ (ref. 7).

In bounded plasma the runaway process will not be adequately described by the analysis developed here and this for several reasons of which we shall mention the two most important ones. The first is that as E_c depends on n and T_e, there may be portions of plasma yielding more runaways than others. This will lead to accumulation of space-charges and to a redistribution of the field E (ref. 8).

The second factor of importance has to do with the two stream instability mechanism mentioned in chapter 5. Thus the kinetic energy of runaway electrons can be converted into a h.f. electromagnetic field. Such a conversion implies a new friction force, which in many cases can be larger than the F_{ep} taken into account so far (ref. 9).

Conduction of electricity in a gyrotropic plasma

In a homogeneous plasma a uniform magnetostatic field has no influence on the conduction of electricity parallel to itself. This follows directly from Boltzmann equation in which $\boldsymbol{E} \,//\, \boldsymbol{B}$.

When the applied electric field $\boldsymbol{E}$ is at right angles to $\boldsymbol{B}$, the equation for the single fluid model of plasma (eq. (3.64)) gives

$$\nu \boldsymbol{j} = \frac{e^2 n}{m} \left(\boldsymbol{E} + \frac{1}{c} \boldsymbol{V} \wedge \boldsymbol{B} \right) - \frac{e}{m} \operatorname{grad} p_p. \tag{34}$$

In a homogeneous plasma grad $p_p = 0$ and in the absence of collisions $\nu = 0$. Thus the mass-velocity $\boldsymbol{V}$ is

$$\boldsymbol{V} = - c \frac{\boldsymbol{E} \wedge \boldsymbol{B}}{B^2}$$

which is the drift velocity of both electrons and ions in crossed $\boldsymbol{E}$ and $\boldsymbol{B}$ fields and no current flows in the direction perpendicular to $\boldsymbol{B}$.

Let us now imagine that the flow $\boldsymbol{V}$ is stopped, e.g., by confining the plasma by a plane perpendicular to $\boldsymbol{E} \wedge \boldsymbol{B}$.

From eq. (34), written in component form, follows

$$\nu_z j_z = \frac{e^2 n}{m} E_z \tag{34a}$$

$$\nu_y j_y = - \frac{1}{m} \frac{\partial}{\partial y} p_p. \tag{34b}$$

Thus the current parallel to $\boldsymbol{E}$ flows now as if there were no intervening magnetic field and another current has made its appearance, flowing in a direction perpendicular to both $\boldsymbol{E}$ and $\boldsymbol{B}$. This is the well known Hall current.

A detailed examination (ref. 10) shows that the conductivity $e^2 m/\nu_z m$ appearing in eq. (34a) is about one third of that of uniform plasma without a magnetic field, owing to the non-uniform spatial distribution of the centra of gyration. This "perpendicular" conductivity is for a hydrogenic plasma

$$\sigma_\perp = 0.5 \times 10^{-4} \frac{T^{3/2}}{\ln \Lambda} \text{ (mho/cm).} \tag{35}$$

There are many plasma geometries in which the drift $\boldsymbol{v}$ can form a closed flow and, therefore, no pressure gradient can be built up. In such a case the magnetic field impairs the conduction of electricity in the direction of $\boldsymbol{E}(\perp \boldsymbol{B})$ and the corresponding conductivity is

$$\sigma'_\perp = \frac{\sigma_0}{1 + \left(\dfrac{\omega_c}{\nu_c} \right)^2}. \tag{35a}$$

However, these are already topics related to diffusion of electric charge in non-uniform plasma and therefore, belonging to the next section.

8.2.2. STOPPING POWER — RELAXATION TO MAXWELLIAN DISTRIBUTION — EQUIPARTITION OF ENERGY

The problem of how long it takes for a test particle to loose either most of its directed velocity or most of its kinetic energy is related to the already analysed problem of the runaway electrons. Let us assume first that the field particles β are infinitely heavy and stationary and, therefore, the test particles α do not change their kinetic energy in collisions with the β's. The motion of the centre of gravity (in the v_a space) of the α's is then described by the equation (20) in which $E = 0$ and z is originally very large. In this case eq. (20) becomes

$$\frac{\mathrm{d}v_a}{\mathrm{d}t} = -\, n_\beta \Gamma_a v_a^{-2} \tag{36}$$

which gives for the time required to reduce the initial speed v_0 to zero

$$t_s = \frac{v_0^3}{3n_\beta \Gamma_a}. \tag{37}$$

The total kinetic energy of the test particles will, however, remain constant. In absence of collisions between the test-particles themselves their individual velocities will remain always equal to v_0, i.e. in the velocity space they will populate a circle (arc A in fig. 133). It is, therefore, obvious that our original assumption of a displaced Maxwellian distribution is no longer valid when $v(t) \lesssim \frac{1}{2}\, v_0$ and the formula for the stopping time t_s is only an approximation.

Eq. (37) can be generalised for relatively light field particles whose mean speed ($\beta_\beta^{-1/2}$) is appreciably smaller than v_{0a}. In that case one has to consider the factor $\dfrac{m_a + m_\beta}{m_\beta}$ appearing in eq. (14b) and in eq. (19):

$$t_s = \frac{m_\beta}{m_\beta + m_a} \, \frac{1}{3\Gamma_a} \, \frac{v_0^3}{n_\beta}. \tag{37a}$$

Let us define an energy- and density-normalized stopping time as

$$\tau_s = t_s \cdot W^{-3/2} \cdot Z_\beta n_\beta = \tau_0 \cdot \frac{m_\beta}{m_\beta + m_a} \sqrt{\frac{m_a}{m}} \frac{1}{Z_a^2 Z_\beta} \tag{38}$$

where $\tau_0 = \dfrac{\sqrt{m}}{3\sqrt{2}\pi e^4 \ln \Lambda}$ which is the normalized stopping time for fast electrons on slow protons.

For fast electrons stopped by slow electrons $\tau_s = \frac{1}{2}\tau_0$. Fast protons on slow protons are stopped after $\tau_s = \frac{1}{2} \sqrt{\dfrac{M}{m}}\, \tau_0$. The shortest stopping time corresponds to heavy multiply ionized atoms stopped by electrons, $\tau_s = \left(\dfrac{M_a}{m}\right)^{-1/2} \cdot \tau_0 Z_a^{-2}$.

A quantity used more often than the stopping time is the range of a fast test particle which is *

$$R \sim \frac{1}{2} v_0 \cdot t_s = \frac{m_\beta}{m_\beta + m_a} \quad \frac{W^2}{24\pi Z_a^2 Z_\beta^2 e^4 \ln \Lambda} \quad \frac{1}{n_\beta}. \tag{39}$$

The largest range corresponds to fast tritons impinging on a hydrogen target.

When t_s refers to particles all of the same kind (e-e, p-p collisions) it can be interpreted as the self-collision time, i.e., the time in which two groups of particles of the same kind, whose original velocity difference is v_0, relax to an almost isotropic distribution characterised by a temperature

$$T = \frac{m_a v_0^2}{3k} \cdot \frac{n_a}{n_a + n_\beta}. \tag{40}$$

The same time is then a characteristic time τ_m for Maxwellisation of such a distribution. When we have $n_a \sim n_\beta$ and using eqs. (37) and (40) there is

$$t_m = \frac{13.5}{Z_a^4 \ln \Lambda} \left(\frac{m_a}{m_p}\right)^{1/2} \frac{T^{3/2}}{n} \text{ (sec)} \tag{41}$$

where m_p is the mass of a proton. Evidently the lighter the particles the shorter is the self-collision and the Maxwellisation time.

Another important situation corresponds to both α and β particles having concentric Maxwellian distributions with temperatures T_a and T_β.

According to thermodynamics, in the final state the two groups will be at the same temperature

* It can be shown (ref. 11) that a better approximation is $R \propto W^{3/2}$, e.g., the range of He^{++} in air at atmospheric pressure is given by $R \simeq 0.32 \cdot W^{3/2}$ (Mev, cm).

$$T = \frac{n_\alpha T_\alpha + n_\beta T_\beta}{n_\alpha + n_\beta}.$$

Let us assume that $n_\alpha = n_\beta = n$ and that during the temperature equilibrium both groups retain their Maxwellian distributions. In order to see how T_α and T_β depend on time we must solve the Fokker-Planck equations for $f_{\alpha,\beta}$.

Let

$$f_\alpha = n(\sqrt{\pi}u_\alpha)^{-3} \exp\left(-\frac{v^2}{u_\alpha^2}\right) \tag{42a}$$

$$u^2_{\alpha.\beta} = \frac{2kT_{\alpha.\beta}}{m_{\alpha.\beta}}. \tag{42b}$$

Having assumed a Maxwellian distribution at all times we may forget the self-collision terms α-α and the differential operators will be only those corresponding to a spherically symmetric solution in the velocity space. Thus eq. (13) becomes

$$\frac{\partial f_\alpha}{\partial t} = -\frac{1}{v^2}\frac{\partial}{\partial v}\left[v^2 f_\alpha \langle \Delta v \rangle_{\alpha\beta}\right] + \frac{1}{v^2}\frac{\partial}{\partial v}\left[v^2 \frac{\partial}{\partial v}\left(f_\alpha \langle \Delta v^2 \rangle_{\alpha\beta}\right)\right]. \tag{43}$$

Multiplying both sides by $\frac{1}{2}m_\alpha v^2$ and integrating over Π_α we get an equation for the change of total energy of the α particles, i.e.,

$$3/2nk\frac{\partial T_\alpha}{\partial t} \equiv \int_{\pi_\alpha} \frac{\partial f_\alpha}{\partial t}\cdot \tfrac{1}{2}m_\alpha v^2 d\Pi_\alpha = -\tfrac{1}{2}m_\alpha \int_0^\infty \frac{\partial}{\partial v}\left[v^2 f_\alpha \langle \Delta v \rangle_{\alpha\beta}\right] 4\pi v^2 dv$$

$$+ \tfrac{1}{2}m_\alpha \int_0^\infty \frac{\partial}{\partial v}\left[v^2 \frac{\partial}{\partial v}\left(f_\alpha \langle \Delta v^2 \rangle_{\alpha\beta}\right)\right]\cdot 4\pi v^2 dv. \tag{44}$$

If $m_\alpha \sim m_\beta$ the problem reduces to that of the Maxwellisation time (eq. (41)). Let us, therefore, study the case when $m_\beta \ll m_\alpha$ and, therefore, in most cases $u_\beta \ll u_\alpha$. In that case using eqs. (14b), (7), (8) and (9) we have *

* For spherical symmetry only the $\frac{m_\alpha}{m_\beta}$ factor has to be considered out of the $\frac{m_\alpha + m_\beta}{m_\beta}$ factor present in eq. (14b). For the same geometrical reason the

$$\frac{\partial}{\partial \varphi^2}\langle \Delta v_\perp^2 \rangle = \frac{\partial}{\partial \theta^2}\langle \Delta v_\perp^2 \rangle = 0.$$

Furthermore $\langle \Delta v^2 \rangle_{\alpha\beta}$ can be ignored owing to the smallness of u_β^2/v^2.

$$\langle \Delta v \rangle_{\alpha\beta} = \frac{m_\alpha}{m_\beta} \Gamma_\alpha \frac{n}{v^2} \tag{45a}$$

$$\langle v^2 \rangle_{\alpha\beta} = \langle v_{\parallel}^2 \rangle_{\alpha\beta} = \Gamma_\alpha \frac{n u_\beta^2}{v^3}. \tag{45b}$$

These simple expressions are not valid in the small region where the α and β distribution overlap and, therefore, the lower limit of the integrals cannot be zero, instead one must choose some $v_{\min}$ where $v^2_{\min} > u_\beta^2$.

Substituting eqs. (45a,b) into eq. (44) we get

$$-\frac{3\sqrt{\pi} k u_\alpha^3}{4 m_\alpha \Gamma_\alpha n} \frac{\partial T_\alpha}{\partial t} = \frac{m_\alpha}{m_\beta} u_\alpha^2 - F(v_{\min}) \cdot u_\beta^2. \tag{46}$$

When the temperatures T_α and T_β of the α and β particles are equal, $\dfrac{\partial T_\alpha}{\partial t} = 0$ and, therefore, we shall require that

$$F(v_{\min}) = 1.$$

Then

$$-\frac{\partial T_\alpha}{\partial t} = \frac{32\sqrt{\pi} e^4 Z_\alpha^2 Z_\beta^2 \cdot n \cdot \ln \Lambda}{3 m_\beta m_\alpha u_\alpha^3} (T_\alpha - T_\beta). \tag{47}$$

Evidently the characteristic time of energy transfer from α to β is

$$t_e = \frac{3\sqrt{2}}{16\sqrt{\pi}} \frac{m_\beta}{m_\alpha} \frac{k^{3/2}}{Z_\alpha^2 Z_\beta^2 e^4 \ln \Lambda} \frac{T_\alpha^{3/2}}{n_\beta}. \tag{48}$$

For the specific case of cooling of hot electrons by cold protons we have (ref. 12):

$$t_e = \frac{244}{\ln \Lambda} \frac{T_e^{3/2}}{n_e} \tag{49}$$

8.3. Diffusion in Configuration Space

Having discussed how, in an infinite and uniform plasma, collisions are responsible for the diffusion of particles in the velocity space we shall now turn our attention to the effect of collisions on a diffusion in a spatially non-uniform plasma *. Such a diffusion may be stimulated by a non-uniformity in the density, in temperature or in the stream velocity of the plasma. Accordingly the response, i.e. the diffusing quantity, may be particles, heat, electric charge or the momentum of the stream. The stimulus is related to the response by a corresponding *diffusion coefficient.* Thus for instance particle flux induced by a density gradient can be expressed by

$$\phi = D \operatorname{grad} n \tag{50}$$

where D is the coefficient of diffusion. These coefficients depend on externally applied constraints impeding the diffusion. The most important constraint influencing the value of all the diffusion coefficients is a magnetic field.

An approximate but very simple method which allows us to estimate the value of diffusion coefficients is based on the theorem of stochastic processes of equal probability such as we have already used in deriving the angle caused by multiple scattering (p. 249).

Let $\Delta\xi$ be the mean step in space that a particle makes between two successive collisions. Provided that such collisions do not alter the probability of the particle making another such step, the distance ξ covered by this particle after N collisions is given by

$$\xi = \Delta\xi \cdot \sqrt{N}. \tag{51}$$

The mean square distance moved per unit time is, therefore

$$D = (\Delta\xi)^2\nu \tag{52}$$

where ν is the collision frequency.

The flux ϕ of particles in the direction parallel to the density gradient in an isothermal plasma can be computed as follows (see eqs. (11) and (12) on p. 252 in the light of footnote on p. 268). This flux is composed of two opposed fluxes ϕ_1 and ϕ_2 where

$$\phi_1 = (n\nu\Delta\xi)_{x = x_0 - \frac{\Delta\xi}{2}} =$$

$$\phi_0 - \frac{\partial\phi_1}{\partial n}\,\frac{\partial n}{\partial x}\,\tfrac{1}{2}\Delta\xi - \frac{\partial\phi_1}{\partial(\nu\Delta\xi)}\,\frac{\partial(\nu\Delta\xi)}{\partial x}\,\tfrac{1}{2}\Delta\xi$$

* There is an essential difference between the diffusion in velocity space and that in configuration space. In the absence of collisions and external forces the points representing particles in the velocity space do not move, whereas those in the configuration space execute rectilinear uniform motion.

The effect of collisions on the movement of representative points in velocity space is, however, analogous to the effect of collisions on the movement of centra of gyration of particles in a magnetic field.

$$\phi_2 = (n\nu\Delta\xi)_{x = x_0 + \frac{\Delta\xi}{2}} =$$

$$\phi_0 + \frac{\partial\phi_1}{\partial n}\frac{\partial n}{\partial x}\,{}^1\!/_2\Delta\xi + \frac{\partial\phi_1}{\partial(\nu\Delta\xi)}\frac{\partial(\nu\Delta\xi)}{\partial x}\,{}^1\!/_2\Delta\xi$$

and

$$\phi = \phi_2 - \phi_1 = \nu\cdot\Delta\xi^2\cdot\frac{\partial n}{\partial x} + n\cdot\Delta\xi\frac{\partial(\nu\Delta\xi)}{\partial x}. \tag{53}$$

It will be shown later that for isothermal plasma $\dfrac{\partial(\nu\Delta\xi)}{\partial x} = 0$ and we obtain in that case

$$\phi = D\frac{\partial n}{\partial x}$$

which is the same as eq. (50). This demonstrates that the coefficient of diffusion can be defined as *the mean square distance a particle covers per unit time.*

If one wishes to find the value of a diffusion coefficient pertaining to the diffusion of a property q of plasma, one must find the flux ψ of this quantity in the same way as was found the flux ϕ.

The corresponding flux equation is then

$$\psi = D_q \text{ grad } q \tag{54}$$

or taking the divergence

$$\frac{\partial q}{\partial t} = \text{div } (D_q \text{ grad } q). \tag{54a}$$

This can be often transformed into the classical diffusion equation

$$\frac{\partial q}{\partial t} = S\Delta q \tag{54b}$$

whose solution is a diffusion wave which progresses with the speed S.

Let us now evaluate, using this approximate method, the diffusion coefficients corresponding to particle-diffusion, to heat-diffusion and to the diffusion of the momentum of plasma flow.

8.3.1. FLUX OF PARTICLES

Magnetic field-free plasma

The mean step executed by a particle between two successive collisions in a field-free isothermal plasma is equal to the mean free path Thus

$$\Delta\xi = \frac{v}{\nu} \tag{55}$$

ν can be defined using the $\langle\Delta v_{\parallel}\rangle$ coefficient by

$$\frac{1}{\nu}\langle\Delta v_{\parallel}\rangle = v$$

or

$$\nu = \frac{4\pi Z^2 e^4 n \ln\Lambda}{m^2 v^3}. \tag{56}$$

The diffusion coefficient for electron follows from eqs. (52), (55) and (56) is

$$D_e = \frac{v^2}{\nu}$$

$$D_e = \frac{m^2 v^5}{4\pi Z^2 e^4 \ln\Lambda}\, n^{-1}. \tag{57}$$

In an isotropic hydrogenic plasma in which drift velocities are less than the mean thermal speed it is clear that the diffusion coefficient for electrons will be v/w, i.e., $\sqrt{M/m}$ times higher than that for the positives.

Unequal diffusion of electrons and positive ions gives rise to an electric field, which in turn slows down the diffusion of electrons and speeds up that of the ions. Thus the effect of the density-gradient in the electron gas, coupled through this electric field to the motion of the positive ion, must be added to the effect of density-gradient in the ion gas and the total diffusion, known as the *ambipolar* diffusion, is characterised by a diffusion coefficient *.

$$\begin{aligned} D &= 2D_+ \\ &= \frac{2\sqrt{2}}{\pi}\,\frac{(kT)^{3/2}}{\sqrt{M}e^4 \ln\Lambda}\, n^{-1}. \end{aligned} \tag{58}$$

Gyrotropic plasma

The effect of a magnetic field on plasma has been discussed in chapter 4. It has been shown that in absence of collisions a density gradient is responsible for the appearance of various drift motions, all of them in the direction perpendicular to both $\boldsymbol{B}$ and grad n. A diffusion parallel to grad n is the result of collisions.

* For non-isothermal plasma and $T_p \neq T_e$ (see ref. 13).

The mean step the centre of gyration of an electron makes as a result of a e-p collision is equal to the radius of gyration ρ. Thus using eq. (52) there is

$$D_e = \left(\frac{mv}{\frac{e}{c}B}\right)^2 \frac{4\pi Z^2 e^4 \ln \Lambda}{m^2 v^3} \cdot n = 4\pi Z e^2 c^2 \ln \Lambda \cdot \frac{n_e}{vB^2}. \quad (59)$$

The mean shift of the centre of gyration of a positive ion as a result of an e-p collision is, according to eq. (2.18)

$$\Delta\xi = \frac{\Delta p}{\frac{e}{c}B} \quad (60)$$

where Δp can be at most equal to the momentum of the electron. The collision frequency is still determined by eq. (56) because $v \gg w$. It thus follows that

$$D_p \approx D_e.$$

In case that $D_p \neq D_e$ a phenomenon similar to ambipolar diffusion will ensure that the electron flux due to diffusion is the same as that of the positive ions.

The speed of diffusion of isothermal plasma across a magnetic field as a result of e-p collisions is from eq. (54)

$$\frac{\Psi}{n} = S \cong 10^{-2} \frac{\operatorname{grad} n_e}{B^2 T^{1/2}}. \quad (61)$$

Let us consider the effect of e-p and p-p collisions on the diffusion process across a magnetic field. The result of a single collision of like particles of equal energy is a shift of the guiding centra of the two interacting particles by equal and opposite amounts (fig. 140). Such collisions will not alter a density distribution, except where such a distribution exhibits a large variation within a radius of gyration ρ, i.e., where

$$\rho \cdot \operatorname{grad} n \not\ll n.$$

An example of such a distribution is one for which the density of the centra of gyration is uniform in a given volume. The particle density falls to zero within a distance ρ from the boundary of such a volume. It is evident that in this case collisions between like-particles will soon blunt the sharp density distribution. The diffusion speed found for this process does not depend linearly on grad n and has been shown to be

$$S = \tfrac{7}{8}\rho^4 \nu_p \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{n}\,\frac{\mathrm{d}^2 n}{\mathrm{d}x^2}\right) \tag{61a}$$

where ν_p is the collision frequency of the positive ions (ref. 14).

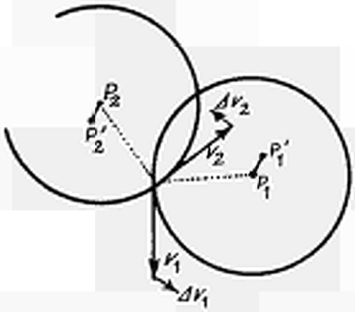


Fig. 140. Shift in the position of centre of gyration in a magnetically immobilized plasma.

In all the calculations of diffusion coefficients the value of the mean step has been defined by the interval between particle collisions. However, the motion of a particle can be considerably perturbed by rapidly oscillating electric fields in the plasma. In thermodynamic equilibrium one would expect that all the possible modes of oscillation of a plasma are excited, each having an energy 1/2 kT (see p. 168). The particle-phonon (or photon) collisions are then of great importance in plasma diffusion across magnetic fields.

Let us consider a particular situation in which the mean step of the centre of gyration between two successive particle-photon collisions is equal to the Larmor radius and the collision-frequency is equal to the gyro-frequency. The corresponding coefficient of diffusion is then

$$D = (\Delta\xi)^2 \cdot \nu = \frac{1}{2\pi}\,\rho v_\perp \tag{62}$$

where

$$\rho = \frac{Mv_\perp}{\dfrac{e}{c}B} \tag{63}$$

and, therefore,

$$D = \frac{1}{2\pi} \frac{Mv_\perp^2}{\frac{e}{c}B} = \frac{kc}{\pi e} \cdot \frac{T}{B} = 0.83 \cdot 10^4 \frac{T}{B} {}^*. \tag{64}$$

Many experiments on plasma diffusion across a magnetic field are in agreement with this diffusion coefficient and the process itself is known as Bohm diffusion (ref. 15). From eqs. (59) and (64) it follows that for a deuterium plasma the classical diffusion is more important than the Bohm one only for (taking $\ln \Lambda \sim 10$)

$$\frac{n}{T^{3/2}} > 2.7 \cdot 10^3 \cdot B. \tag{65}$$

If plasma is confined by the magnetic field then

$$\beta B^2 \geqslant 16\ \pi nkT \tag{66}$$

and, therefore,

$$\frac{\sqrt{n}}{T^2} > 2.2 \cdot 10^{-4}\ \beta^{-1/2}. \tag{67}$$

One sees that (fig. 141) for thermonuclear temperatures ($T > 10^8$ °K) the density of a plasma which would diffuse mainly owing to electron-ion collisions must be greater than $5 \cdot 10^{24}$ ions/cm^2. In order that (65) be true for most plasmas produced in laboratory ($n \sim 10^{15}$) the temperature must be very much lower than $4 \cdot 10^4$ (°K). It follows that, provided Bohm diffusion is always operative, the classical diffusion can be observed only in experiments in low temperature caesium plasmas (ref. 16).

8.3.2. CONDUCTION OF HEAT AND ELECTRICITY

Let us consider a heat conduction in an isotropic and field-free plasma. The density of heat energy is

$$W_e = 3/2 n_e k T_e$$

$$W_p = 3/2 n_p k T_p.$$

Let us assume for the moment that $n_e = n_p$ and $T_e = T_p$. Employing an argument similar to that used in eq. (53) we obtain for the heat-flux Q due to the electrons

* Practically the same for both electrons and ions.

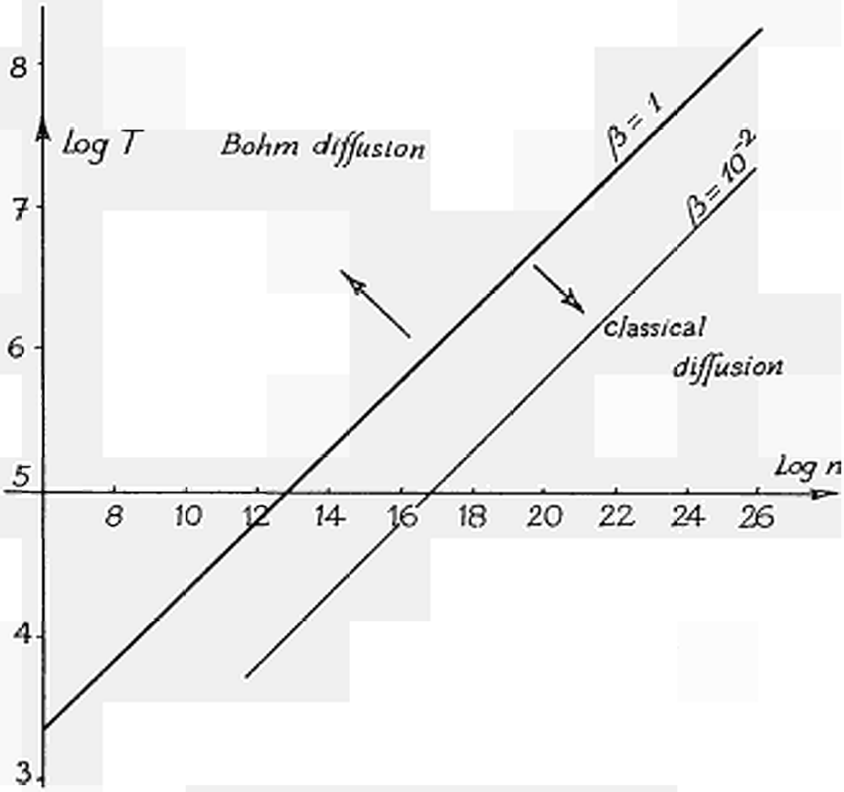


Fig. 141. Graph of eq. (67) for $\beta = 1 \,.\, 10^{-2}$.

$$Q = \frac{\partial}{\partial x}\,(3/2nkT \cdot \nu \cdot \Delta\xi) \cdot \Delta\xi \tag{68}$$

where $\nu \cdot \Delta\xi = v = \sqrt{\dfrac{2kT}{m}}$ and $\Delta\xi = \dfrac{k^2T^2}{\pi e^4 n \ln \Lambda}$ (see eq. (56)).

Let us separate the effect of density variation from that of temperature by assuming that grad n is equal to zero. Then

$$Q = -\frac{3}{\sqrt{2}}\;\frac{nk^{3/2}}{m^{1/2}} \cdot \frac{\partial(T^{3/2})}{\partial x} \cdot \Delta\xi. \tag{69}$$

The heat conductivity is usually defined as

$$\varkappa = \frac{Q}{\text{grad}\ T}\,. \tag{70}$$

and therefore, the heat conductivity of the electron gas is

$$\kappa = \frac{9k^{7/2}}{2\sqrt{2}\pi e^4 m^{1/2}} \frac{T^{5/2}}{\ln \Lambda} = 1.95 \cdot 10^{-5} \frac{T^{5/2}}{n} \text{ (erg/cm}^2 \text{ sec deg).} \tag{71a}$$

In evaluating the total heat flux $Q_e + Q_p$ the concept of ambipolar diffusion cannot be used in the simple form outlined on p. 270.

The value of Q to be taken for heat conduction depends on several assumptions concerning the neutrality of plasma, the current it can carry in the direction of grad T and on the coupling of T_e and T_p through e-p collisions. For a Lorentz gas in which an electric current flows as a result of grad T Spitzer obtains

$$\kappa_L = 4.67 \times 10^{-12} \frac{T^{3/2}}{Z \ln \Lambda} \text{ (cal sec}^{-1} \text{ cm}^{-1} \text{ deg}^{-1}). \tag{71b}$$

The same reasoning can be applied to the conduction of heat in a gyrotropic plasma. According to eqs. (59) and (60)

$$\kappa' = \frac{12\pi}{\sqrt{2}} Ze^2c^2\sqrt{km} \ln \Lambda \frac{n^2}{T^{1/2}B^2} \tag{72}$$

or for a hydrogen plasma

$$\kappa'_H = 2.1 \times 10^{-18} \frac{n^2}{B^2T^{1/2}}. \tag{72a}$$

In the general case of gyrotropic plasma in an electric field the equations describing the electric current and heat flow are ($\|$ and $\perp$ to B)

$$\boldsymbol{i} = \sigma \boldsymbol{E}_\| + \sigma' \boldsymbol{E}_\perp + \sigma'' \frac{\boldsymbol{E} \wedge \boldsymbol{B}}{B} + \sigma_T \frac{c}{e} \text{grad}_\| T + \sigma'_T \frac{\text{grad } T \wedge \boldsymbol{B}}{\frac{e}{c} B} \tag{73}$$

$$Q = \kappa \text{ grad}_\perp T + \kappa' \text{ grad}_\| T + \kappa'' \frac{\text{grad } T \wedge \boldsymbol{B}}{B} + \kappa_e \frac{e}{c} \boldsymbol{E}_\| + \kappa'_e \frac{\boldsymbol{E} \wedge \boldsymbol{B}}{\frac{e}{c} B}. \tag{74}$$

The form of these equations follows from the momentum and energy conservation equations of the two fluid model of plasma, the coefficients can be evaluated on the basis of the approximate diffusion theory outlined here or more rigorously by considering small deviations from a Maxwellian distribution (ref. 17).

The various terms appearing in these equations for $\boldsymbol{i}$ and Q are associated with effects discovered experimentally by physicists investigating electricity and heat conduction in metals.

Thus in the equation for electric current density the first and second terms represent the currents induced by the two components of the applied electric field, the first parallel, the second perpendicular to the magnetic field $\boldsymbol{B}$. The third term represents the well known Hall current, the fourth term describes the thermo-electric (Seebeck) effect and the fifth term corresponds to the Nernst effect.

In the equation of heat-flow the first and second term are the heat flows parallel and perpendicular to $\boldsymbol{B}$ induced by the temperature gradient. The third term represents the Righi-Leduc effect, the fourth term the Peltier effect, whereas the fifth term is associated with the Ettinghousen effect (ref. 18).

8.3.3. DIFFUSION OF MOMENTUM. VISCOSITY

Let us consider two streams of charged particles flowing parallel to each other with different velocities of flow (fig. 142). The particles whose random velocity has a component perpendicular to the velocity of flow will cause a mixing of the two streams. The mixed streams will then exchange momentum through collisions. As a result of this exchange the fast stream will be slowed down, whereas the slow stream will be speeded up and the same time some of the kinetic energy of the flow will be converted into the energy of random motion.

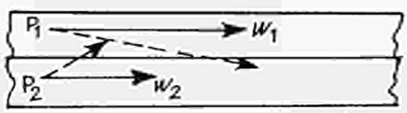


Fig. 142. Exchange of particles between streams of different momentum.

The diffusion of momentum can be calculated by a procedure similar to that used on p. 269 for the diffusion of the fluid property q. Here the flux of momentum is mainly due to the heavy positive ions

$$
\begin{aligned}
P &= M\left(n+\frac{1}{2}\,\frac{\partial n}{\partial x}\,\Delta\xi_{\mathrm{p}}\right)\left(w+\frac{1}{2}\,\frac{\partial w}{\partial x}\,\Delta\xi_{\mathrm{p}}\right)\Delta\xi_{\mathrm{p}}\cdot\nu_{\mathrm{p}} \\
&\qquad - M\left(n-\frac{1}{2}\,\frac{\partial n}{\partial x}\,\Delta\xi_{\mathrm{p}}\right)\left(w-\frac{1}{2}\,\frac{\partial w}{\partial x}\,\Delta\xi_{\mathrm{p}}\right)\Delta\xi_{\mathrm{p}}\cdot\nu_{\mathrm{p}} \\
&= MD\,\frac{\partial}{\partial x}\,(wn). \qquad (75)
\end{aligned}
$$

In a plasma of uniform density

$$P = nMD_{\mathrm{p}} \frac{\partial w}{\partial x}. \tag{75a}$$

This momentum transport is equivalent to a shearing stress exerted parallel to w. This stress is proportional to $\partial w/\partial x$. Thus

$$P = \mu \frac{\partial w}{\partial x}.$$

The constant μ is known as the coefficient of viscosity and is, in our approximation

$$\mu = nMD_{\mathrm{p}}. \tag{76}$$

In a magnetic field-free plasma

$$\mu = \frac{\sqrt{2}}{\pi} \frac{\sqrt{M}(kT_{\mathrm{p}})^{3/2}}{e^4 \ln \Lambda}. \tag{76a}$$

In a gyrotropic plasma the coefficient of viscosity becomes, using the expression for D_{p} derived on p. 271

$$\mu = \frac{4\pi \ln \Lambda}{\sqrt{2}} \frac{Ze^2Mc^2\sqrt{m}}{\sqrt{k}} \frac{n^2}{B^2T^{1/2}}$$

$$= 1.6 \times 10^{-26} AZ \frac{n^2}{B^2T^{1/2}} \tag{76b}$$

where A is the atomic number of the positive ions.

The viscous forces can cause instability (the break-up) of a laminar flow. The criterion for the onset of such an instability is expressed by a quantity known as Reynold's number which is for a cylindrical flow with $v = 0$ at $r = a$

$$R = 2\rho a v \mu^{-1} \tag{77}$$

or more generally

$$R \approx \mu^{-1}\rho \frac{\boldsymbol{v} \wedge \operatorname{curl} \boldsymbol{v}}{\nabla^2 \boldsymbol{v}}.$$

For conditions for which the Reynold's number exceeds a certain value, which depends on the geometry of the flow, the laminar flow produces eddies and becomes turbulent.

REFERENCES TO CHAPTER 8

1) V.I. Veksler, CERN Symposium, p. 80, Geneva (1956).
2) S. Chandrasekhar, *Astroph. J.*, *97*, p. 255 (1943).
3) M. N. Rosenbluth *et al.*, *Phys. Rev.*, *107*, p. 1 (1957).
W. Thompson, An introduction to plasma Physics, p. 193, Pergamon Press (1962).
4) H. Dreicer, Proc. of 2nd Geneva int. conf. on peaceful uses of atomic en. Vol. *31*, pp. 57-64 (1958).
5) S. Chapman and T. G. Cowling, The math. theory of non-uniform gases, pp. 329-337, CUP (1953).
6) J. G. Linhart, CERN-PS report 2 (1957).
7) E. R. Harrison, *Jour. Nucl. En.*, Part C, *1*, p. 105 (1962).
8) E. R. Harrison, *Jour. Nucl. En.*, Part C, *4*, p. 7 (1965).
9) M. D. Kruskal and I. Bernstein, *Phys. Fluids*, *7*, p. 407 (1964).
10) L. Spitzer and R. Härm, *Phys. Rev.*, *89*, p. 977 (1953).
11) Livingstone and Bethe, *Rev. Mod. Phys.*, *9*. p. 265 (1937).
12) L. Spitzer, Physics of fully ionized gases, p. 135, Interscience publ. 1962.
13) S. Chapman and T. G. Cowling, *op. cit.*, p. 338.
14) A. Simon, Phys. Rev. *100*, p. 1557 (1955).
15) D. Bohm, p. 12, in "Characteristics of electrical discharge in magnetic fields" (Guthrie and Wakerling editors) Mc. Graw Hill Co., N.Y. 1949.
J. B. Taylor, *Nucl. Fus.* 1962 Suppl., p. 477 (Proc. Salzburg conf. 1961).
E. Kursunoglu, *Ann. Phys.*, *17*, p. 259 (1962).
F. Boechschoten, *Jour. Nucl. En.*, Part C, *6*, p. 339 (1964).
16) E. Berkl *et al.*, *Phys. Rev. Letters*, *17*, p. 906 (1966).
17) S. Chapman and T. G. Cowling, *op. cit.*, p. 336.
18) A. A. Ware, Proc. Phys. Soc. A, *67*, p. 869 (1954).

List of symbols used in Chapter 8

A	mass number
B, $\mathbf{B}$	magnetic field strength
p	impact parameter
c	velocity of light
d	Debye distance
D	coefficient of diffusion
E, $\mathbf{E}$	electric field strength
e	electronic charge
F, $\mathbf{F}$	force
f	density distribution
G, H	potential functions
g	plasma characteristic
j	current density
k	Boltzmann constant
l	length
m, M	particle mass
n	number density
N	number of collisions
p	momentum or pressure
P	flux of momentum
q	coordinate in configuration space
Q	heat flux
R	Reynold's number
s	velocity
t	time
T	temperature
u, v, w	velocity

V	electrostatic potential
W	energy density
x, y, z	coordinates
Z	atomic number
$\beta =$	v/c normalized speed or $\dfrac{\beta\pi p}{B^2}$
ε	electric charge density
θ, φ, χ	angles
κ	heat conductivity
$\Lambda =$	$\dfrac{p_{\min}}{p_{\max}}$ a ratio of critical impact parameters
ν	collision frequency
μ	coefficient of viscosity
π	probability
ρ	radius of gyration
σ	collision cross-section or electrical conductivity
τ	normalized stopping time flux of particles in configuration space
ψ	flux of particles in velocity space
ξ	distance
ζ	normalized velocity
ω	angular frequency
ω_c	cyclotron frequency

APPLICATIONS

In these two chapters on applications of plasma physics we shall limit ourselves to a brief appraisal of projects that have not yet passed from a research laboratory to a development laboratory. To such belongs the research on controlled thermonuclear reactions to which we shall devote chapter 9. In chapter 10 we shall mention research projects on plasma rocket-motors, the direct conversion of chemical energy into electrical energy, energy storage, plasma-oscillators and plasma-accelerators. In all these cases only basic physical criteria will be given and much of the subject matter will be treated as an exercise in applying the theorems derived in the first eight chapters of this book. It must be appreciated that in these potential applications the rôle played by plasma physics is not always the most important one. Thus surface phenomena, ionization processes, nuclear transformations and purely engineering considerations may often be the source of limitations of the proposed devices. Some of these limitations will be mentioned but not studied, such study being more appropriate to publications on the separate fields of such applications.

CHAPTER 9

RESEARCH ON CONTROLLED FUSION

Introduction

Thermonuclear research is a subject that will remind many physicists of the research on the properties of ordinary chemical flames. This is particularly noticeable when one sub-divides the subject matter into its two component disciplines. The first is concerned with the nuclear reactions and with the manner in which the nuclear energy is released. The second is that of the kinetic theory of plasma, and it is concerned with the heating and confinement of the nuclear fuel.

The first discipline will be the subject of an introductory section, whereas the second one will be discussed mainly in sections 2 and 3. A survey of the various approaches to the problem will be attempted in section 4.

9.1. Sources of Nuclear Energy

9.1.1. ELEMENTARY NUCLEAR CONCEPTS

The atomic nucleus is composed of approximately equal numbers of protons and neutrons. The number of particles within the nucleus belonging to these two groups (nucleons) is equal to the mass number A. If the number of protons is Z, the number of neutrons is $A - Z$. There are certain rules according to which an atomic nucleus is built. These permit reasonably stable nuclear structures to occur only for certain combinations of A and Z. Thus e.g., the population of these stable nuclei is clustered along a curve C in the $A - Z$, Z diagramme (see fig. 143) *.

The reason for the very narrow spread in $A - Z$ (at any particular Z) is to be found in the nature of forces between the nucleons, evidently neutrons and protons prefer to be bound together in pairs.

* This diagram is also known as the "Segrè chart". See e.g., E. Fermi: Nuclear Physics (Univ. of Chicago Press, 1955).

The variation in the degree of nuclear stability is not only noticeable perpendicular to the population curve C but also along it. Thus if two nuclei A′ and A″ collide with each other a nuclear reaction may take place, the products of the reaction being represented by two points B′ and B″ in the Z, $A - Z$ diagramme different from those representing the original nuclei (fig. 144). It is then observed that, if one of the colliding nuclei belongs to a group in the middle of the periodic table (i.e., nuclei centred around $Z = 50$), one of the products of the reaction will be again in this centre group (fig. 145). If, however,

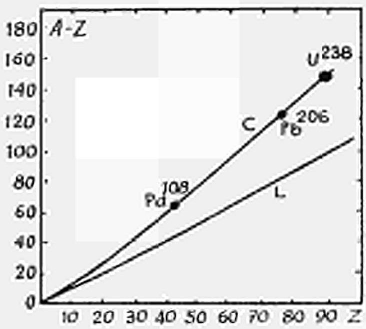


Fig. 143. Line L represents the equation $A - Z = Z$.

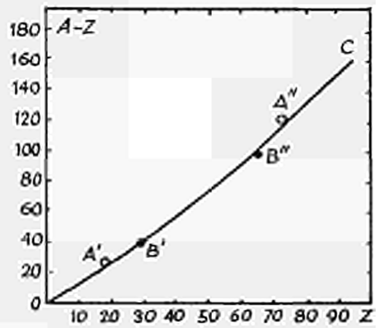


Fig. 144. Reaction A″(A′B′)B″.

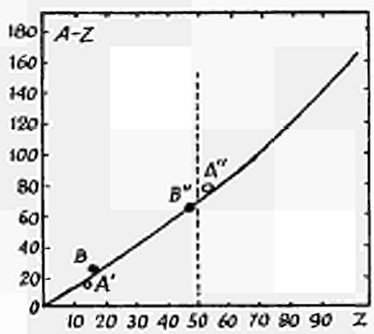


Fig. 145.

one of the colliding partners is a heavy one ($Z \approx 90$) it is possible that both the reaction products will appear in the centre group (fig. 146). Example:

$$U_{92}^{235} + n = Sr_{38}^{94} + Xe_{54}^{140} + 2n. \quad (1)$$

A similar tendency for a shift of Z in the direction of the centre group is noticed in a reaction between two light nuclei. Here one of the reaction products is usually heavier than either of the nuclei entering the reaction (fig. 147). Example,

$$\mathrm{D} + \mathrm{T} \rightarrow \mathrm{He}^4 + \mathrm{n}. \tag{2}$$

This stability in nuclear reactions of the medium mass nuclei can be explained by observing the dependence of the binding energy of nuclei on Z (or A).

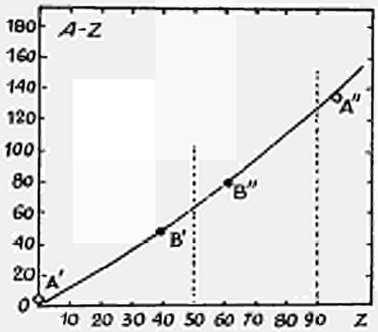


Fig. 146. A typical fission reaction.

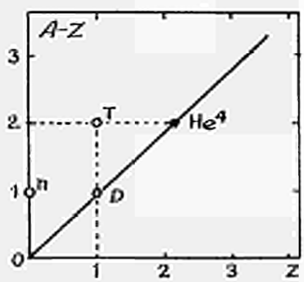


Fig. 147. Reaction D(t, n)He^4.

9.1.2. BINDING ENERGY

The binding energy of a nucleus is the energy difference between a system of A *free* nucleons and a system of these nucleons forming an unexcited nucleus.

The total binding energy W is built up of several contributions.

1. First of all there is the energy due to charge independent forces between adjacent nucleons. These are short range forces similar to the forces between molecules of liquids. A nucleon well inside the nucleus interacts with its immediate neighbours and one may assign to it a binding energy w_1 (which is found to be close to 8.5 MeV). Thus A nucleons in an infinitely large nucleus will have a binding energy $W_1 = Aw_1$.

2. The atomic nuclei are, of course, not infinitely large. There are always nucleons situated on the surface of the nucleus; the value of their binding energy is not w_1 but $w_1 - w_2 < w_1$. As the number of surface nucleons is proportional to the surface $4\pi R^2$ of the nucleus

and $R \cong 1.5 \times 10^{-13} A^{1/3}$ (cm) (owing to the constant density of nuclear matter) we have for the negative component W_2 of the binding energy of the surface nucleons.

$$W_2 = -\alpha w_2 A^{2/3}.$$

3. The other important energy component is due to coulomb repulsion between the protons within the nucleus. This is the potential energy W_3 of a uniformly charged sphere of radius R and containing a total charge Ze. Thus

$$W_3 = -\frac{3}{5}\,\frac{(Ze)^2}{R}$$

as $Z \approx \frac{1}{2}A$ and R is proportional to $A^{5/3}$ one gets

$$W_3 = -\alpha' A^{5/3}.$$

There are other energy components involved in the binding energy W of a nucleus, such as the energy of unpaired nucleons and energy due spin-orbit and spin-spin interactions inside the nucleus. However, for the purpose of our discussion we may ignore these energies. Thus the total binding energy is

$$W = w_1 A - \alpha w_2 A^{2/3} - \alpha' A^{5/3}. \tag{3}$$

A more important expression is the binding energy per nucleon W_n

$$W_n = w_1 - \alpha w_2 A^{-1/3} - \alpha' A^{2/3}. \tag{3a}$$

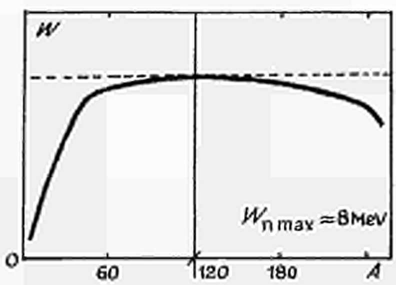
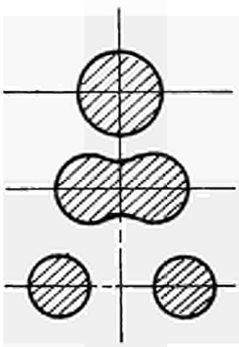


Fig. 148. Binding energy per nucleon.

It can be appreciated that at high A the last term will predominate and the W_n will decrease as A increases (fig. 148). Thus there will be an $A = A_0$ for which W_n becomes a maximum *. Lighter nuclei will then show a tendency to build up into A_0 nuclei and similarly nuclei heavier than A_0 will decay into lighter ones.

In each of these transformations energy will be liberated, the building up of nuclei representing the fusion reactions, the splitting of nuclei the fission reactions.

A nucleus undergoing fission will presumably depart from the spherical shape (fig. 149a) and pass through the shapes of a prolate spheroid and a dumbell into two spherical nuclei of approximately equal size ** (some more recent theories suggest stripping of the outermost shells of the spherical nucleus (fig. 149b).



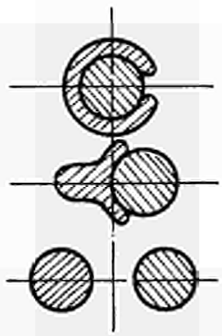


Fig. 149a. Mechanism of a fission event. Fig. 149b.

The changes of surface energy and electrostatic potential energy respectively resulting from the change of shape of the nucleus are obtained as follows:

* The rigorous expression for W_n is more complicater than eq. (3a) and can be written as:

$$W_n = 1.28 \times 10^{-5} - 2.37 \times 10^{-5} A^{-1/3} - 0.09 \times 10^{-5} \frac{Z^2}{A^{4/3}} - 1.195 \times 10^{-4} \left(\frac{1/2\, A - Z}{A}\right)^2 + \delta \text{ (erg)}$$

where $\delta = -5.18 \times 10^{-5} A^{-7/4}$ for A even and Z even,
$\delta = 5.18 \times 10^{-5} A^{-7/4}$ for A even and Z odd,
$\delta = 0$ for all other cases.

This formula does not describe satisfactorily the variation W_n for nuclei whose $A < 12$. However, it does show that there are several values of A for which W_n becomes a maximum (compare Fermi, *loc. cit.*, p. 7 and fig. 1.2).

** See Fermi, *op. cit.*, p. 164.

1. Surface energy is always proportional to the surface area. The surface of the prolate spheroid is

$$S = 4\pi ab \left\{ \frac{b}{a} + \sin^{-1} \sqrt{1 - \left(\frac{b}{a}\right)^2} \right\}$$

where a, b are the major and minor semi-axis respectively.

However, the volume of the nucleus does not change (constant density of nucleons is always preserved) and therefore, if

$$a = R(1 + \varepsilon)$$

the volume Ω is (for small deformations, i.e., for $\varepsilon \ll 1$)

$$\Omega = 4/3\pi R(1 + \varepsilon)b^2 = 4/3\pi R^3 = \text{const.}$$

and therefore

$$b = \frac{R}{\sqrt{1 + \varepsilon}} \cong R(1 - \tfrac{1}{2}\varepsilon + \tfrac{3}{8}\varepsilon^2).$$

The surface becomes

$$S = 4\pi R^2(1 + 2/5\varepsilon^2 + \ldots).$$

and the surface energy is

$$W_2 = 2 \times 10^{-5} A^{2/3}(1 + 2/5\varepsilon^2 + \ldots). \tag{4}$$

2. The electrostatic energy of the prolate spheroid is (ref. 1)

$$W_3 = 1.1 \times 10^{-7} A^{5/3}(1 - 1/5\varepsilon^2). \tag{5}$$

Thus the surface energy increases and the electrostatic decreases with the elliptical deformation. The total change is

$$\Delta W = \varepsilon^2(8A^{2/3} - 0.034A^{5/3}) \times 10^{-6}. \tag{6}$$

When $\Delta W \lessgtr 0$ a spherical nucleus will be unstable. This happens for

$$A \geq \frac{8}{0.034} = 235. \tag{7}$$

Even when $\Delta W > 0$ but very small compared with W_3 a division of the nucleus is possible if such a nucleus is bombarded by particles capable of supplying the energy ΔW. The energy ΔW may be thought of as a triggering energy, which starts the nuclear division. The energy liberated in such a division comes mainly from the Coulomb field of the protons. It may be calculated from the difference between the energy of the orginal spherical nucleus and the energy of the two daughter nuclei. Thus using eqs. (4) and (5)

$$W(AZ) - 2W\left(\frac{A}{2}, \frac{Z}{2}\right)$$
$$= 2 \times 10^{-5} A^{2/3}[1 - 2(\tfrac{1}{2})^{2/3}] + 1.7 \times 10^{-7} A^{5/3}[1 - 2(\tfrac{1}{2})^{5/3}]. \quad (8)$$

Substituting $A = 236$ into this one obtains an approximate value for the fission energy W_{Fi} of uranium

$$W_{\mathrm{Fi}} \cong 2.68 \times 10^{-4} \text{ erg} = 169 \text{ MeV}. \quad (9)$$

The fission energy output per nucleon being $W_{\mathrm{n}} = 0.72$ MeV.

9.1.3. NUCLEAR FUSION

The energy liberated in a fusion of two hydrogen nuclei (the Q-value of that reaction) is mainly the surface tension energy. As an example let us apply eq. (8) to a compound nucleus whose $A = 4$. Then

$$W_{\mathrm{Fusion}} = 2W\left(\frac{A}{2}, \frac{Z}{2}\right) - W(A, Z)$$

$$W_{\mathrm{Fusion}} = -\underbrace{0.62 \times 10^{-7} \times 10.1}_{\Delta W_{\mathrm{elst.}}} + \underbrace{0.52 \times 10^{-5} \times 2.51}_{\Delta W_{\mathrm{surface}}} \quad (8a)$$

it is obvious that $\Delta W_{\mathrm{surface}} \gg \Delta W_{\mathrm{elst.}}$. Thus

$$W_{\mathrm{Fusion}} = 1.22 \times 10^{-5} \text{ erg} = 7.7 \text{ MeV}. \quad (9a)$$

This value represents the order of the Q-values of fusion reactions. Evidently the fusion energy output per nucleon is then $W_{\mathrm{n}} \approx 2$ MeV. This is about 3 times larger than W_{n} encountered in fission. Some typical fusion reactions are described in the following table.

$$\begin{array}{lll}
\mathrm{D + D} & \begin{cases} \rightarrow \mathrm{T + p} + 4.0 \text{ MeV} \\ \rightarrow \mathrm{He^3 + n} + 3.25 \text{ MeV} \end{cases} & \text{equally probable} \\
\mathrm{T + D} & = \mathrm{He^4 + n} + 17.6 \text{ MeV} & \\
\mathrm{T + T} & = \mathrm{He^4 + 2n} + 11.4 \text{ MeV} & \\
\mathrm{He^3 + D} & = \mathrm{He^4 + p} + 18.3 \text{ MeV.} &
\end{array}$$

Not all the nuclear reactions between light elements are fusion reactions; there are some important fission reactions as well. This is due to the dependence of binding energy on the number of unpaired nucleons and some other binding energy terms that we have mentioned in the footnote on p. 285. Owing to these components of the total binding energy the energy per nucleon W_{n} varies rather rapidly and unevenly for $A < 15$ (ref. 1). Thus it is evident from the plot in fig. 150 that there are two regions in which one may expect the occurrence of fission reactions. these regions are

$$6 < A + Z < 11$$
$$12 < A + Z < 16.$$

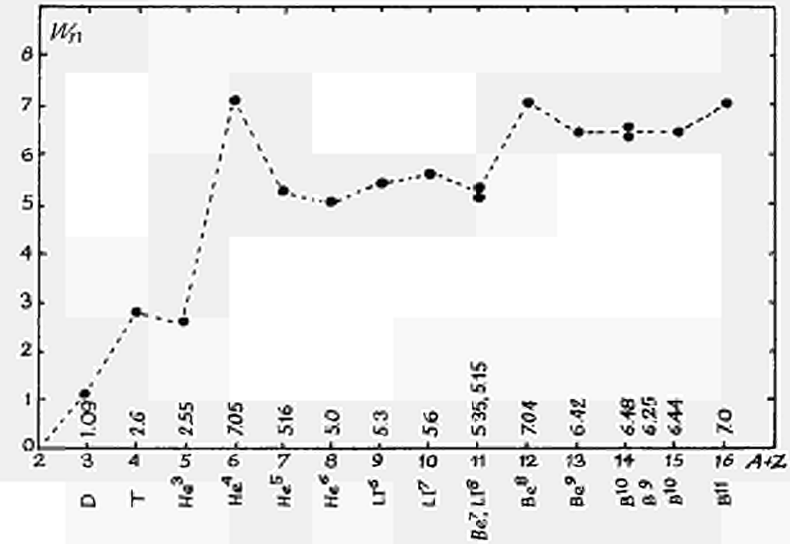


Fig. 150. Binding energy per nucleon of light nuclei.

Typical fission reactions in the first region are:

$$Li^6 + n = He^4 + T + 4.8 \text{ MeV}$$
$$Li^6 + D = 2He^4 + 22.4 \text{ MeV}$$
$$Li^7 + H = 2He^4 + 17.3 \text{ MeV}.$$

The second of these is one of the richest reactions in output energy per nucleons of all fission reactions; it yields $w_n = 2.8$ MeV/nucleon.

The exact calculation of fusion energy output is rather difficult and the accurate Q-values are obtained by experiments. The reaction cross-sections σ are also determined by experiment, except for relatively low kinetic energies of the reacting nuclei, where the cross-sections are computed as follows (ref. 3, 4).

First of all one calculates the probability P per unit time that a collision between two light nuclei will lead to a fusion. This probability is

$$P = P_1 \cdot P_2 \cdot P_3 \qquad (10)$$

where

P_1 is the probability that the nuclei will penetrate each others Coulomb potential barrier,

P_2 is the reaction probability Γ/h, i.e., the probability that the desired reaction will take place within the compound nucleus,

P_3 is the probability that the particle evaporated from the compound nucleus will penetrate the Coulomb potential barrier.

If the evaporated particle is a neutron, then

$$P_3 = 1.$$

As P is also the number of reactions per unit time we have for the reaction cross-section

$$P = nv\sigma \tag{11}$$

where n is the number of bombarding particles per cm^3 and v is their velocity. The general formula for σ as a function of collision energy E is (ref. 5).

$$\sigma(E) = \frac{a}{E} \exp \left(- 0.989 Z_1 Z_2 \sqrt{\frac{A}{E}} \right) \text{ barn} \tag{12}$$

where a is in MeV barns, E in MeV and

$$A = \frac{A_1 A_2}{A_1 + A_2}.$$

The cross-sections for the two most important thermonuclear reactions, i.e., T(D, n) He^4 and D(D, n) He^3 have been measured and are plotted in fig. 151 *.

When fusion reactions proceed in a hot gas (e.g., mixture of T and D) near thermal equilibrium one would be tempted to substitute

$$E = 3/2kT$$

into eq. (12) and calculate the rate of the reaction ν as

$$\nu = \tfrac{1}{2} n \frac{\sqrt{2kT/M}}{\lambda} \text{ (reactions/sec, cm}^3\text{)}$$

where λ is the mean free path between the reactions, i.e.,

$$\lambda = \frac{1}{n\sigma}.$$

* E. J. Stovall, *Phys. Rev.*, ***88*** (1952) 159, Summer Meeting of American Phys. Soc., 1952.

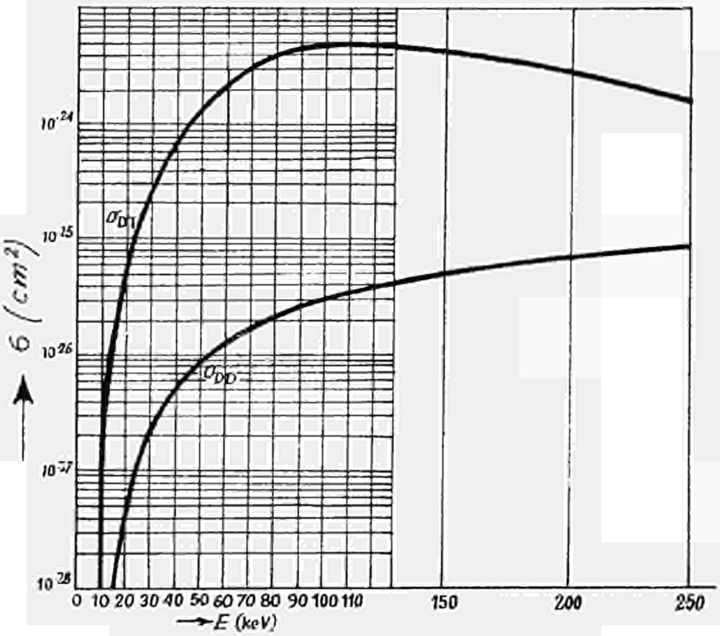


Fig. 151. Collision cross-sections for the two most important fusion reactions.

However, this approach is wrong at temperatures below 10^9 °K because σ is an extremely sharp function of the velocity of the colliding particles. If the velocity distribution of both types of nuclei is Maxwellian then the number of reactive collisions (per cm³, per sec) with collision-energy between E and $E + \mathrm{d}E$ is

$$d\nu = \frac{4c_1c_2n^2}{\sqrt{2\pi}A_1A_2\sqrt{A}} \frac{E}{kT} e^{-E/kT} \frac{\sigma(E)}{\sqrt{kT/M}} \mathrm{d}E \quad (\mathrm{coll^s/cm^3sec}) \quad (13)$$

where $A = A_1A_2/(A_1 + A_2)$, and c_1, c_2 are the respective concentrations of the two types of nuclei.

The function $Ee^{-E/kT}\sigma(E)$, occurring in eq. (13) shows a very sharp maximum at (fig. 152)

$$\frac{E}{kT} = \left(\frac{\pi e^2\sqrt{MA}Z_1Z_2}{\sqrt{2}h}\right)^{2/3} (kT)^{-1/3}. \quad (14)$$

For hydrogen isotopes $Z_1 = Z_2 = 1$, $A \cong 1$. Therefore

$$\frac{E}{3/2kT} \sim \frac{820}{T^{1/3}} \quad (14a)$$

(Example: $T = 10^7$ °K, $2E/3kT = 3.8$.)

This shows that the reaction output is due to a small group of nuclei whose velocity is several times larger than the mean random velocity.

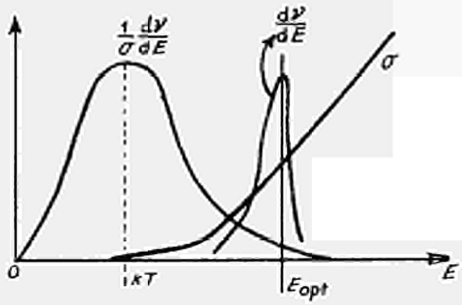


Fig. 152. Most of the fusion events in a plasma possessing a Maxwellian velocity distribution occur between ions having an energy well in excess of the mean thermal energy.

Knowing $\left(\frac{d\nu}{dE}\right)_{max}$ and the effective width ΔE of the $\frac{d\nu}{dE}$ peak, it is now possible to calculate the reaction output W_N (per cm³). Thus (ref. 4)

$$W_N = Q\int_{E=0}^{\infty} \frac{d\nu}{dE}\, dE \approx Q \left(\frac{d\nu}{dE}\right)_{max} \cdot \Delta E. \qquad (15)$$

The complete formula for T(D, n) He reaction is

$$W_N = 0.65 \times 10^{-16} \frac{n^2}{T^{2/3}} \times 10^{-19.7/T^{1/3}} \quad (\text{erg/cm}^3\ \text{sec}).$$

where T is in 10^6 °K.

This formula is valid only as long as the assumption concerning the sharpness of $\frac{d\nu}{dE}$ holds. This is not so for temperatures of several tens of millions of degrees for D,T reactions. In such a case it is necessary to calculate the reaction output using the following formula

$$W_N = n_1 n_2 \langle\sigma v\rangle_{1,2} \cdot W_{\text{Fusion}} \qquad (16)$$

where the mean reaction rate $\langle \sigma v \rangle_{1,2}$ between species 1 and 2 is numerically calculated by integrating eq. (13). It is seen from fig. 153 that the maximum for the $\langle \sigma v \rangle_{D,T}$ is equal to about $0.9 \cdot 10^{-15}$ and occurs at a temperature of about $6 \cdot 10^{8}$ (°K) (ref. 6).

9.1.4. FISSION AND FUSION REACTIONS AS SOURCES OF ENERGY

The energy of radioactive decay is supposed to contribute a certain amount to the heat and magnetic field sources of the planets. There exists also a theory according to which some components of cosmic rays are generated during the disintegrations of heavy nuclei. However, even assuming that these theories are correct, from a cosmic point of view the fission reactions appear as freaks.

The source of energy in our Universe is the energy liberation in fusion reactions. Most stars have been and still are receiving their energy from this type of nuclear reaction. There are, of course, many types of fusion reactions and at present it is difficult to determine which is the most widely used and which stars use which reaction (ref. 2).

However, it appears that the basic energy giving transformation is the formation of an He^4 nucleus out of four protons;

$$4p \rightarrow He_2^4 + 2e^+ + \text{energy}.$$

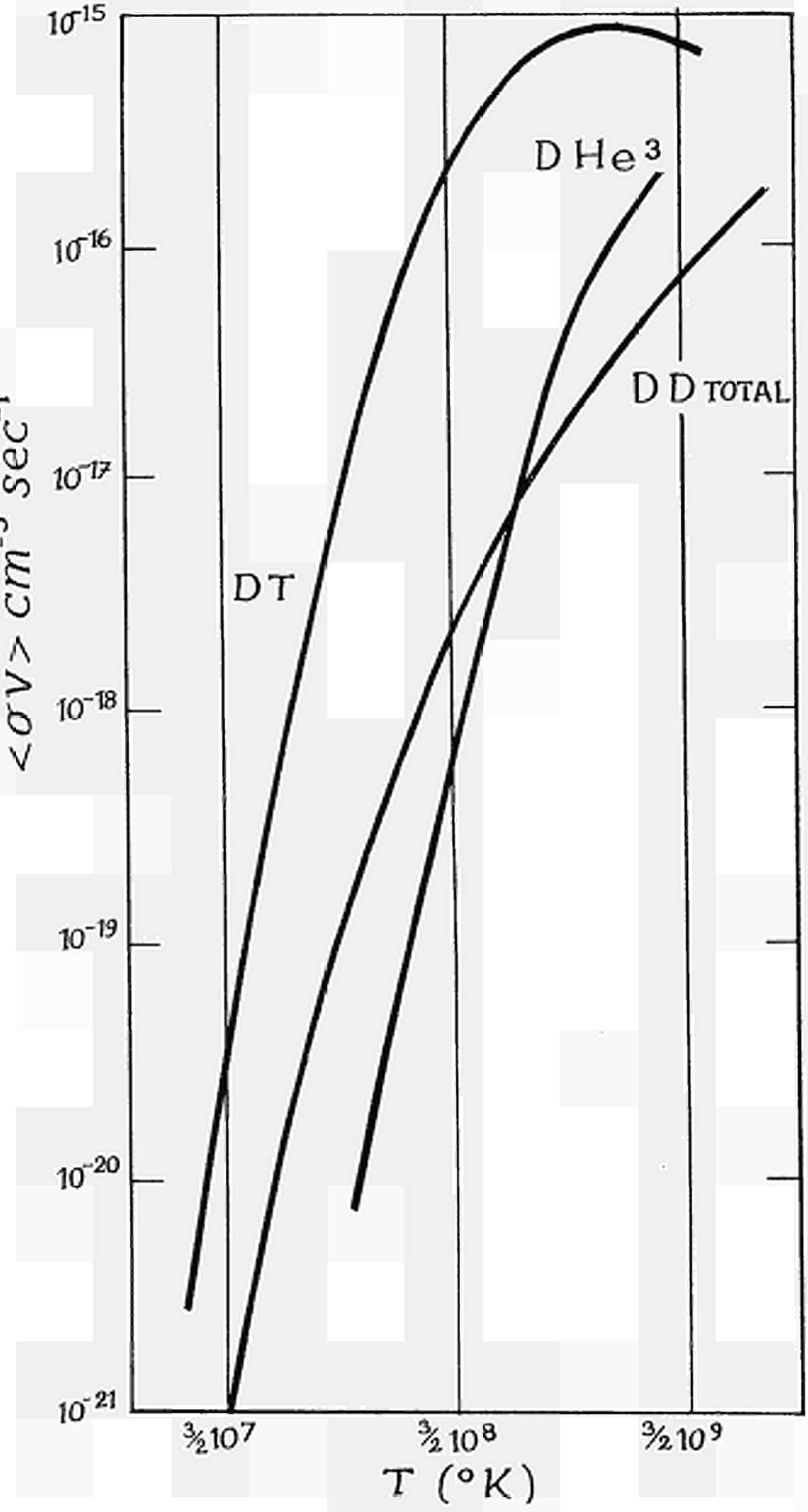
It is thought that stars of the same type as our own sun effect this transformation through the $p(pe^+)D$ reaction (ref. 7).

However, other types of reactions effecting the same transformation may also occur in the stars (ref. 2).

The calculations of energy production show that this process accounts satisfactorily for energy radiated away from our sun. Which reaction cycle is operative in a star is determined by the material, temperature and pressure distribution in that star.

These exothermic reactions can occur in the interior of the stars owing to the high temperature and pressure maintained there and of course, owing to the fact that the material from which our Universe is built is mainly hydrogen.

The reason for the *steady* maintenance of these reactions is the large gravitational pressure of the external layers of the stars. This pressure balances the pressure of the hot plasma in the interior. However, large deviations from this pressure equilibrium may occur. When the gas pressure prevails the star expands rapidly, the process resembling an atomic explosion on a cosmic scale. This expansion causes the nuclear furnace in the interior to cool and the gas pressure drops. The expansion is in many cases eventually arrested by the gravitational pull and the



10^{-15}
10^{-16}
10^{-17}
10^{-18}
10^{-19}
10^{-20}
10^{-21}
$<\sigma v>$ cm^{-3} sec^{-1}
DT
DHe^3
DD TOTAL
3/2 10^7
3/2 10^8
3/2 10^9
T (°K)

Fig. 153.

star begins to collapse towards its centre. This condensation causes the temperature of the star-centre to rise and eventually leads to a new explosion. The stars showing this behaviour are known as pulsating stars (Cepheids) (ref. 2).

Considered as sources of power for mankind the fusion reactions appear at first sight more commendable than the fission reactions. This is due mainly to the following reasons:

1) The fuel for the DD fusion reactions is practically inexhaustible.

2) The final products of fusion reactions are usually not radioactive, or ih they are, theire half-life is short.

The only important radioactive element in all the faster thermonuclear reactions is tritium with a half-life of 12 years. This will occur, however, in very small quantities as an intermediate product. On the other hand the fusion reactions are a source of neutrons and therefore, just as in the case of fission reactions, appropriate screening facilities must be provided.

Recently the situation has, however, changed in favour of fission. Firstly the breeding reactors are now capable of transforming naturally plentiful materials such as uranium 238 and thorium 232 into fissile materials such as plutonium 239 and uranium 233 by neutron-absorption. The reserves of uranium 238 and thorium 232 on earth are such that mankind should not worry about lack of energy for at least another millenium (ref. 8). Secondly the technological developments in dealing with radioactive materials and selecting materials which are resistant to radiation damage has advanced so far that one feels confident that in the future radioactive ashes can be stored or disposed off and reactor accidents rendered extremely improbable. One may even conceive shooting the long lived radioactive products into outer space.

Looking at fusion as an energy source for power plants one may ask. therefore, whether its development is about as untimely as the development of fission reactors would have been at the beginning of the industrial revolution. One may proceed by saying that the problem of fusion reactors should be perhaps reexamined in about 100 years time when it may be resolved faster and better than today or when it may be perhaps recognized as a problem not worth solving. However, this point of view is too conservative. Thus. although fission may meet the power consumption requirements, fusion may do it better — by better is meant more economically, using less complicated technology or having some other substantial advantage.

Man-produced thermonuclear fusion can be divided into two classes — the uncontrolled fusion, i.e., the H bombs and the controlled fusion.

Even the first one is not really uncontrolled, as the maximum final energy released by a bomb is determined by its size.

9.1.5. UNCONTROLLED FUSION REACTIONS

The example of the uncontrolled fusion reactor is the hydrogen bomb. The "modus operandi" of these bombs has not been officially disclosed. However, the guess work of some of the scientists outside this effort provides probably a good picture of how these "fast reactors" work.

It is thought that the bomb mechanism is based on the T(D, n) He^4 and T(T, 2n)He^4 reactions. The fuel could be a mixture of tritium and deuterium but as this would have to be provided in a liquid state it would require a large refrigeration plant. The result would be a very bulky installation hardly suitable to be used as a bomb. The solution to this problem could be found in the use of solid or liquid chemical compounds of T and D. Some physicists suggest that the tritium may be continually produced by another reaction Li^6(n, He^4)T, so that the whole reaction chain is

$$\begin{array}{ccccl} & T + D & \rightarrow & He^4 + n + 17.6\,\mathrm{MeV} & \\ & \uparrow & & \downarrow & \qquad (17) \\ 4.8\,\mathrm{MeV} + & T + He^4 & \leftarrow & Li^6 + n. & \end{array}$$

Thus the most suitable form in which the Li, T and D may be provided is that of lithium hydrides.

It is even possible that some of these fusion reactions work in conjunction with a fission reaction, the fusion reactions delivering fast neutrons for the fission processes, the fission reaction energy is capable of compensating for heat losses and thus sustaining the fusion reaction. In such a case U^{238} could be used (ordinary uranium), which is relatively cheap and does not normally sustain a neutron chain reaction. Thus there would be no difficulty about critical size and the bomb could be made much larger than fission bombs using U^{235} or plutonium.

Whichever fusion reaction one employs one must, in order to start it, produce stellar temperatures in the fusion-capable material. It is usually held that a uranium bomb explosion is necessary for this purpose. It has been suggested that even ordinary chemical explosives may produce spherical shock waves strong enough to heat the fuel for the fusion reaction (at the centre of the sphere occupied by it) to high enough temperature to start the reaction off.

Calculations have been carried out on such a converging shock-wave ignition (ref. 9) which show that for bomb radii of the order of a meter ordinary chemical explosives could generate neither the required energy density w nor the total energy W necessary for triggering off a self-sustained thermonuclear fusion detonation (see also p. 231).

The two quantities w and W can be calculated relatively simply. Let us consider a fusionable medium (e.g, deuterium or a mixture of deuterium and tritium) of uniform ion density n, (or it two species are present n_1 and n_2). At the time $t = 0$ a spherical volume Ω of this medium, whose radius is r_0, is heated instantly to a temperature T whereas the rest of the medium is at a temperature $T_0 \ll T$. The expansion of the hot volume generates a shock wave which heats the cold surrounding medium. Both the expansion and the shock transmission cools the hot plasma contained initially in Ω. When the shock advances to a radius $r = 2r_0$ the temperature, in absence of any energy input, will undoubtedly drop to value much smaller than T. Thus the cooling time is approximately

$$\tau_0 \cong \frac{r_0}{2v_s} \tag{18}$$

where the shock speed $v_s \sim v_t = \sqrt{\frac{2kT}{M}}$ (see 6.4.1.).

If the hot focus were to start a spherical fusion detonation, the fusion energy deposited in Ω during τ would have to at least compensate the cooling, i.e.,

$$\Omega \cdot n_1 n_2 \langle\sigma v\rangle_{1.2}\, Q \cdot \tau_0 > 3(n_1 + n_2)kT \cdot \Omega \tag{19}$$

where Q is the energy (per reaction) of the charged products — which are the only ones which can be hoped to be absorbed localy, i.e., within $r < 2r_0$. This gives

$$r_0 n_1 \frac{\alpha}{1+\alpha} \frac{Q}{3} \sqrt{\frac{M}{2k^3}} > \frac{2\,T^{3/2}}{\langle\sigma v\rangle_{1,2}} \equiv f(T) \tag{20}$$

where $\alpha = \dfrac{n_2}{n_1}$, $M = \dfrac{M_1 + \alpha M_2}{1+\alpha}$.

Graphs of f(T) for DD and DT reactions are plotted in fig. 154, from which it follows that the minima are

for DT : $f_{min} = 0.45 \cdot 10^{28}$ at $T \cong 1.5 \cdot 10^8$ (°K)

for DD : $f_{min} = 0.34 \cdot 10^{30}$ at $T \cong 3.6 \cdot 10^8$ (°K).

If the medium is a 50/50 mixture of D and T we have $\alpha = 1$, $Q = 2.2 \cdot 10^{-6}$ (erg) and criterion (20) becomes

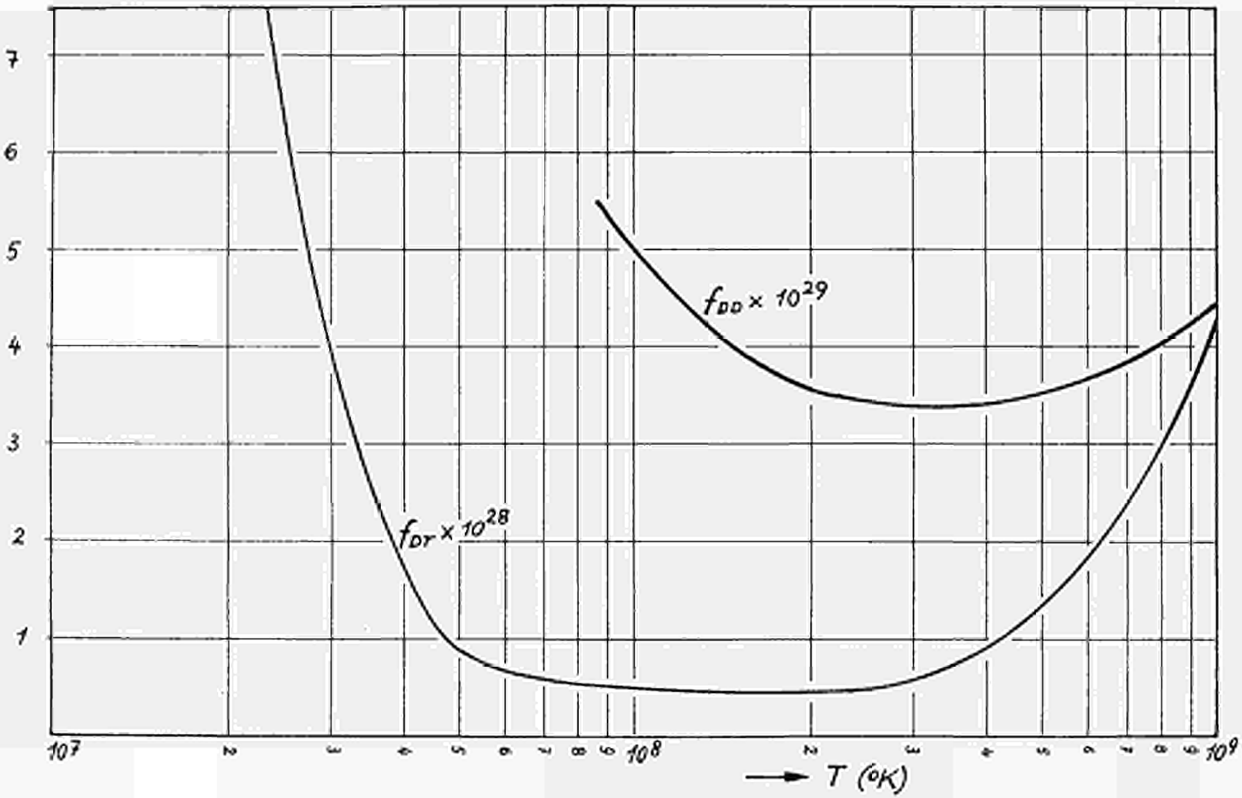


Fig. 154.

$$nr_0 > 2 \cdot 10^{22} \quad (\text{ions/cm}^3, \text{cm}) \tag{20a}$$

which at solid density ($n = 5 \cdot 10^{22}$; D, T ice) corresponds to $r_0 >$ 0.4 cm and to an initial thermal energy

$$W \geqq \frac{4\pi}{3} r_0^3 \cdot 3nkT \cong 65\ (MJ). \tag{21}$$

This is equivalent to an energy liberated by 13 kg of high explosive.

If the medium is pure deuterium then the eq. (19) must be written

$$\tfrac{1}{2}n\langle\sigma v\rangle_{DD}\, Q_{DD} \cdot \tau > 3kT \tag{22}$$

where $Q_{DD} = 0.77 \cdot 10^{-5}$ (erg) and one obtains

$$nr_0 > 3.4 \cdot 10^{23} \tag{22a}$$

which at solid density corresponds to $r_0 > 6.8$ (cm) and to an initial thermal energy W_0 equal to 21 (GJ), equivalent to energy liberated by 4.2 ktons of TNT. The magnitude of this ignition energy excludes all triggers except an A-bomb.

This analysis can be refined in many ways (including radiation loss, an accurate description of the diverging shock, the non-local deposition of the charged nuclear products, etc.), however, even without these improvements of analysis, the order of magnitude of W is correctly predicted by eq. (21).

If one could use the H-bombs as energy sources, the world power-problem could be considered as solved for all foreseeable future. Research on this possibility and also on other uses of H-bombs (canal-excavation, cratering, launching of space vehicles, etc.) is going on in several laboratories — in the USA under the name of Plowshare.

9.2. Controlled Fusion Reactors

Because of the appalling values of temperature and pressure required to sustain a fusion reaction it is natural to enquire whether the ordinary laboratory methods used to investigate fusion reactions may not be amplified so as to generate power. This means that one would use a beam of fast particles (say T nuclei) to bombard a target (say D_2 ice), the beam particles and the target particles being the participants in a fusion reaction.

The most suitable reaction is certainly the T(D, n)He^4 reaction. The cross-section σ becomes maximum for tritons possessing 107 keV energy (fig. 151) where it is

$$\sigma_{\max} = 5 \times 10^{-24} \text{ cm}^2. \tag{23}$$

If each T reacted with one D, the ratio b of output energy over input energy will be

$$b = \frac{17.6 \times 10^6 \text{ (eV)}}{0.107 \times 10^6 \text{ (eV)}} = 165, \tag{24}$$

which is promisingly large.

Unfortunately the bombarding particle will lose its energy very quickly, mainly in collisions with electrons. This loss is given for liquid deuterium by (see also p. 264)

$$\frac{dW}{dx} = \frac{4\pi e^4 n}{mv^2} \ln \Lambda. \tag{25}$$

We must, therefore, calculate the probability P that an incident triton will react with one of the deutons in the target. The fusion output per triton is

$$W_F = P \cdot 17.6 \cdot 10^6 \text{ (eV)} \tag{26}$$

and must be compared with the initial triton energy which is $W = 3/2\ Mv^2$, where M is the mass of a proton. The ratio of these energies is

$$\eta = \frac{W_F}{W} = 1.17 \cdot 10^{19} \cdot \frac{P(v)}{v^2}. \tag{27}$$

The probability P can be calculated as follows. Let us denote the range of the tritons in the target (p. 265) by R, then the probalitity that a fusion reaction occurs along R is

$$P = \frac{R}{\lambda_F} \quad \text{where} \quad \lambda_F = \frac{1}{n\sigma_{DT}}. \tag{28}$$

Using eq. (8.39) we get

$$P = \frac{m}{3M} \cdot \frac{\sigma_{DT} W^2}{24\pi e^4 \ln \Lambda}. \tag{29}$$

and finally (putting $ln\Lambda = 1$)

$$\eta \cong 2 \cdot 10^{15} \cdot W(\text{eV})\ \sigma_{DT}. \tag{30}$$

The maximum of the function $\sigma_{DT} . W$ is at $W = 160$ (kev) and is equal to $6.4 . 10^{-19}$. Therefore

$$\eta \cong 1.28 \cdot 10^{-3}. \tag{30a}$$

It is thus obvious that nuclear energy output is extremely small compared with the beam input energy and consequently this process cannot be exploited for power generating purposes.

If a beam of tritons were shot into a plasma in which the mean energy of the electrons was much smaller than that of the tritons the proportion between energy lost and nuclear energy liberated would be about the same as that found for a target of a liquid deuterium.

However, if the mean random energy of the particles in a deuterium plasma is the same as the kinetic energy of the bombarding tritons the latter will not lose energy on the average. The direction of their motion will be changed in collisions with the deuterons and therefore, the energy of their initially directed motion will be transformed into thermal energy. Thus from the point of view of the bombarding tritons no energy is lost and after a time

$$\tau = \frac{1}{n\langle\sigma v\rangle} \tag{31}$$

60 % of them undergo a nuclear reaction with deutrons. A graph showing the dependence of τ on n for two different energies of the tritons is found in fig. 155. From this graph it is evident that these times are longer than mean reaction times for chemical reactions for the same particle densities.

The relationship between energy lost from the system of tritons — hot plasma cannot be evaluated as simply as that of the system triton beam — cold deuterium target, derived in the preceding pages. The

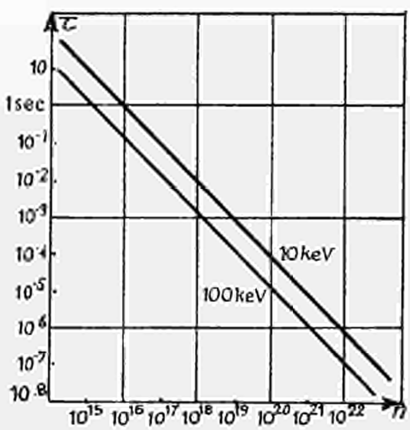


Fig. 155. The mean DT reaction time in a plasma of density n and T = 10, 100 (keV).

energy, no longer lost by the tritons, is lost by the hot plasma target. Thus in order to obtain a net energy gain from the system one must

1) increase the number of bombarding tritons,
2) improve the thermal insulation of the plasma target.

This leads us naturally to a reactor with a plasma mixture of energetic deuterons and tritons confined by some field of force in vacuum. If there is no such confining field an energy gain is still possible in devices resembling the hydrogen bomb.

The criterion of net energy gain known also as the reactor criterion has much the same form as the relationship (19) except that Q is now the total energy yield of one reaction, τ is the cooling (or dispersal) time of the plasma and ε is the efficiency with which the fusion output can be converted into the thermal energy of the plasma. Thus we have

$$n_1 n_2 \langle \sigma v \rangle_{1,2} \, \varepsilon Q \, \tau \geqslant 3(n_1 + n_2) \, kT(1 - \varepsilon) \tag{32}$$

which gives

$$\frac{\varepsilon}{1-\varepsilon} \, n_1 \, \tau \, \frac{\alpha}{1+\alpha} \cdot \frac{Q}{3k} \geqslant \frac{T}{\langle \sigma v \rangle_{1,2}} \equiv f(T). \tag{33}$$

In the case of a 50/50 mixture of deuterium and tritium we have $n_1 = \frac{1}{2} n$, $Q = 2.82 \cdot 10^{-5}$ (erg) and we get

$$n\tau \geqslant 0.58 \cdot 10^{-10} \, \frac{1-\varepsilon}{\varepsilon} \, f(T) \tag{34}$$

known as the Lawson's criterion.

This determines a set of curves in a T, $n\tau$ diagramme (see fig. 156), which are boundaries of a reactor region. The minimum of $f(T)$ occurs at $T \cong 1.7 \, . \, 10^8$ and is equal to $0.34 \, . \, 10^{24}$.

Thus the condition for an optimum zero-energy reactor is (putting $\varepsilon = \frac{1}{3}$)

$$n\tau \geqslant 6 \cdot 10^{13}. \tag{35}$$

Eq. (33) suggests that a reactor could be possible even at comparatively low temperatures provided the $n\tau$ is sufficiently large. This is unfortunately untrue and such a conclusion is due to our neglect of the radiation loss from the plasma. Of the three types of radition (pp. 68-85) the Bremsstrahlung is one that one cannot hope to recuperate (soft and hard X-rays for thermonuclear temperatures) and, therefore, at least this radiation must be considered as a loss. Eq. (32) is then written

$$\varepsilon [n_1 n_2 \langle \sigma v \rangle_{12} \, Q + 2 \cdot 10^{-27} \, (n_1 + n_2)^2 \sqrt{T}] \, \tau \geqslant$$
$$(1-\varepsilon) \cdot 3(n_1 + n_2) \, kT + 2 \cdot 10^{-27} \, (n_1 + n_2)^2 \sqrt{T} \cdot \tau. \tag{36}$$

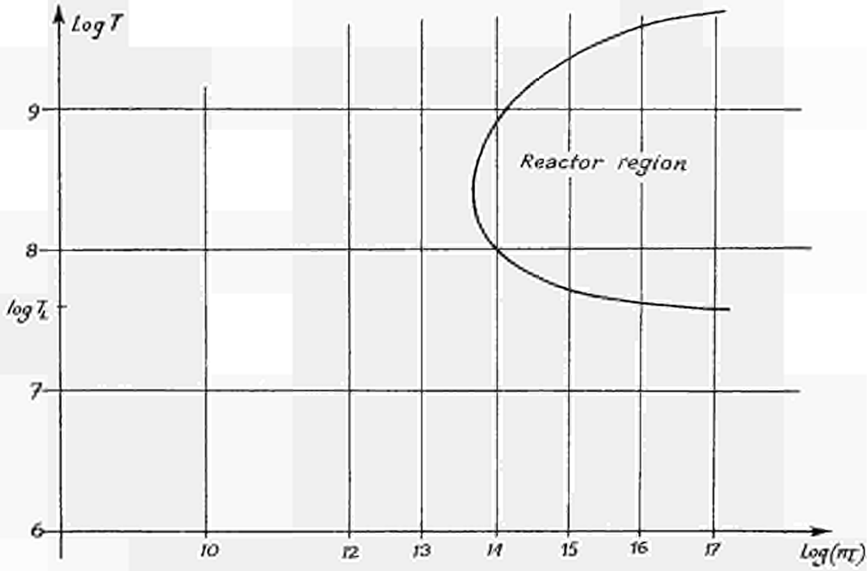


Fig. 156. Simple Lawson's criterion for 50/50 D, T and $\varepsilon = 1/3$.

It is obvious that, even if $\tau \to \infty$, the inequality can be satisfied only when

$$n_1 n_2 \langle \sigma v \rangle_{12}\, Q \geqslant (1 - \varepsilon) \cdot 2 \cdot 10^{-27}\, (n_1 + n_2)^2 \sqrt{T}. \qquad (36a)$$

for 50/50 mixture of D and T and if $\varepsilon = 1/3$ we have

$$\frac{\langle \sigma v \rangle_{DT}}{\sqrt{T}} \geqslant 2 \cdot 10^{-22} \qquad (37)$$

from which the limiting temperature is $T_L \cong 40 \,.\, 10^6$ (°K). Below this temperature the losses due to Bremsstrahlung cannot be compensated by the fusion output. If other radiation losses are present the limiting temperature T_L will be higher than that for Bremsstrahlung alone.

It is clear that the D,T plasma forming the core of any future reactor must have a temperature of the order of 10^8 (°K). It is also equally clear that such a reactor will have to burn a mixture of deuterium and tritium — in which case tritium must be bred using the fast neutrons coming mainly out of the D,T reaction as suggested by equation (17). As it is unlikely that all the neutrons coming from D,T can be utilised, some neutron multiplying reaction must also be used such as

$$Be^9 + n \to 2\ He^4 + 2n. \qquad (38)$$

All this implies a presence of a fairly thick blanket of Li^6, Be^9 around the envelope containing the reacting plasma (ref. 10).

All the proposals for a fusion reactor may be divided in two classes. The first is that of stationary or quasi-stationary machines, the second corresponds to machines in which the plasma pressure is so high that it cannot be balanced statically by magnetic and mechanical forces.

9.2.1. STATIONARY FUSION REACTORS

Almost all the structures proposed for the confinement of plasma can be discussed in terms of a cylindrical system in fig. 157. The central hot plasma-core (radius r) is confined by the magnetic pressure

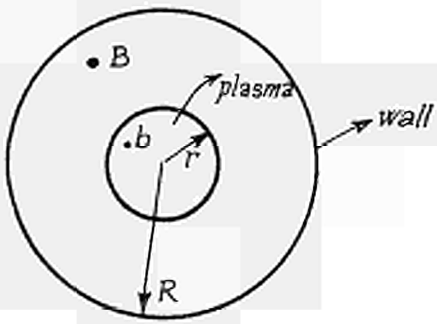


Fig. 157.

$\frac{1}{8\pi}(B^2 - b^2)$ and loses energy mainly due to diffusion of particles to the wall (radius R) *. In absence of all macro-instabilities it has been found experimentally that the diffusion rate is given by the Bohm's formula (see p. 273, ref. (8-15)). The diffusion loss per unit length of the system amounts to

$$\phi = D_B \cdot \frac{\partial n}{\partial r} 2\pi r \sim D_B \cdot \frac{n}{r} \cdot 2\pi r. \tag{39}$$

Using eq. (8.64) we get $\phi = 1.7 \cdot 10^4 \frac{T}{B} \cdot n$ (ions/cm, sec) from which the confinement time (defined by the loss of ½ of all the ions) is

* The radiation losses can be made relatively small in a D, T plasma at a temperature $T \sim 10^8$ (°K).

$$\tau = \frac{N}{2\phi} = 0.9 \cdot 10^{-4} \frac{r^2 B}{T}. \tag{40}$$

Writing Lawson's criterion as $n\tau = \alpha \,.\, 10^{14}$ where $\alpha(\varepsilon) > 1$ and assuming that $T \sim 2 \,.\, 10^8$ we get for the line density

$$N = \pi r^2 n = 10^{14}\, \pi r^2 \, \frac{\alpha}{\tau} \tag{41}$$

$$N \cong \alpha \cdot 10^{27}\, B^{-1} \quad \text{(ions/cm)}. \tag{42}$$

In a system in which plasma is magnetically confined one must satisfy the pressure balance, i.e.,:

$$B^2 - b^2 = 16\pi nkT. \tag{43}$$

Defining $\dfrac{2nkT}{b^2/8\pi + 2nkT} = \beta$ we write the pressure balance as

$$\beta B^2 = 16\pi nkT. \tag{43a}$$

Using eq. (40), the Lawson's criterion and eq. (43a), we get for r

$$r \cong \frac{2 \cdot 10^{10}}{B^{3/2}} \sqrt{\frac{\alpha}{\beta}}. \tag{44}$$

There exist now two different possibilities for the projet of a stationary fusion reactor:

a) The magnetic fields B, b are produced by supraconducting coils — in which case the energy stored in these fields does not enter critically in the energetics of the reactor and, therefore, the radius R can be made much larger than r. The maximum field B_m is then limited by that attainable using hard supraconductors. It does not appear likely that B_m could exceed 250 kG in large coils, whereas β must be sufficiently small in order that the plasma does not perturb the confining magnetic fields.

Considering as a basis for discussion toroidal systems with average minimum B (see p. 214, ref. (21)), we are forced to assume that β will be of the order of 10^{-2}.

We get then

$$r \cong 1.6 \cdot 10^3 \sqrt{\alpha} \quad \text{(cm)}. \tag{45}$$

The plasma density is then of the order of 10^{14} (ions/cm^3). The radius of the wall is determined by the considerations of wall-dissipation. In a steady-state reactor the wall must not be heated to more than about 1000 (°C). This is possible only if the power density does not exceed the order of a few hundred watts/cm^2 (ref. 12). Taking $\dot{W} = 500$ watts/cm^2 we get using eqs. (40), (41) and (45).

$$\dot{W} = \frac{3NkT}{2\pi\tau R} \approx 10^{14}\ R^{-1} \quad (\text{ergs/cm}^2,\ \text{sec}) \tag{46}$$

and

$$R \approx 2 \cdot 10^4 \quad (\text{cm}).$$

It is clear that the volume of such a system will be of the order of a cubic kilometer and, therefore, out of bounds of reality. It is interesting to speculate on how much weaker the diffusion flux ϕ would have to be in order that R would correspond to a practicable apparatus. Let us suppose that the Bohm diffusion is cut by a factor γ. As

$$n \propto \text{const}, \quad \tau \propto \text{const}, \quad N \propto \gamma^{-1}, \quad r \propto \gamma^{-1/2}$$

and

$$R \propto \gamma^{-1}.$$

It seems that a torus whose minor radius is of the order of a few meters could be constructed. In such a case $\gamma = 100$-1000. It appears, therefore, that a stationary fusion reactor using supraconducting coils would be conceivable only if the rate of Bohm's diffusion were cut by at least two orders of magnitude (ref. 13).

b) The second possibility is already a step towards pulsed systems. The magnetic field is not produced by superconductors and the only limitation on its intensity is the mechanical strength of the coil. It is conceivable that conductors with a steel core and jacket of a suitable copper alloy may generate fields up to 1/2 MGauss without suffering meachanical damages. If the plasma is confined in a θ-pinch geometry in which end-losses are made small (ref. 14) we can assume that β is not far from unity and

$$r \sim 10^2\ \sqrt{\alpha}. \tag{47}$$

Since we have to include the energy lost in the coil in the energy-balance equation of the reactor, it is important that R is not much greater than r (for $\frac{R^2}{r^2} \gg 1$ also $\alpha \gg 1$).

This leads to wall dissipations which are very much higher than 500 Watts/cm^2. The only way to absorb such energy bursts is by wall-sweating, i.e., by letting a thin layer of the wall vaporize. In order that the vapours do not polute the plasma during its useful lifetime τ we require that the transit time of these vapours is longer than τ. As the speed v_0 of expansion of wall vapour is of the order of 10^5 cm/sec, we get:

$$\frac{R - r}{v_0} > \tau \tag{48}$$

$$R - r > 9 \frac{r^2 B}{T}. \tag{49}$$

Assuming that $R \gg r$ we get

$$R > 225\ \alpha. \tag{50}$$

If, e.g., $\alpha = 4$ than the radius of the plasma is 2 meters and that of the inner wall 9 meters. The plasma density is of the order of 10^{17} (ions/cm^3). After a time $\tau \sim 6$ (msec) the machine must be emptied of plasma, wall surface reformed before a new pulse can start. Reactor of this type represents a gigantic enterprise — the stored energy W_m in the magnetic field is of the order of 10^4 GJoules — and it lies, therefore, at present in the realm of science fiction. As

$$W_m \propto R^3 \propto \gamma^{-3}$$

it seems that already $\gamma \sim 20$ would bring W_m within engineering concepts.

9.2.2. PULSED FUSION REACTORS

When pressure of plasma or of the magnetic field exceeds the mechanical strength of an envelope or a coil the time of plasma confinement is limited by the characteristic expansion time τ_e of the confining structure. This can be, in many cases, sufficiently long to permit the reactor criterion to be satisfied and we shall call this mechanism — the inertial confinement. Let us choose for the base of our analysis a simple cylindrical model (fig. 158) in which a hot plasma core is compressed or held compressed during τ_e by a relatively heavy cylindrical envelope — the liner *. The thermal insulation of the core from the liner is achieved by means of a magnetic field — either B_θ or B_z. Without confinement the hot core would practically cease to produce fusion after it freeley expanded to twice its original radius. Its lifetime would be in that case $\tau_0 = \dfrac{r}{v_t} = r\sqrt{\dfrac{M}{2kT}}$

* Known also as tamper.

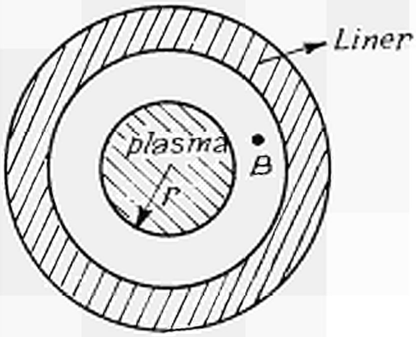


Fig. 158.

Let us now define a coefficient of confinement as

$$\kappa = \frac{\tau}{\tau_0} \tag{51}$$

where τ is the life-time of the core achieved as a result of confinement (or thermal insulation). The Lawson's criterion (eq. (32)) gives then

$$n_1 n_2 \langle \sigma v \rangle_{1,2} \; \varepsilon Q \; \kappa \tau_0 \geqslant 3(n_1 + n_2) kT (1 - \varepsilon). \tag{52}$$

Let ε_0 be the efficiency with which one is able to produce 1 cm^3 of plasma in the hot core using an electromagnetic energy-source and ε_1 the conversion efficiency of heat into electricity. For a D, T plasma we get at $T = 2 \,.\, 10^8$ (°K) and for $\varepsilon = \varepsilon_1 \varepsilon_0 \ll 1$

$$n \kappa r \sim 1/3 \cdot 10^{22} \; \varepsilon_0^{-1} \; \varepsilon_1^{-1} \tag{53}$$

which gives for the minimum plasma energy per unit length of the core

$$W_p \sim 0.87 \cdot 10^8 \frac{r}{\varepsilon_1 \varepsilon_0 \kappa} \quad \text{(J/cm)} \tag{54}$$

and the total energy input per unit length is

$$W = \varepsilon_0^{-1} \, W_p = 87 \frac{r}{\varepsilon_1 \varepsilon_0^2 \kappa} \quad \text{(MJ/cm)}. \tag{55}$$

Take as an example $\varepsilon_1 = 1/3$, $\varepsilon_0 = 1/10$, $\kappa = 1000$ and $r = 0.33$, then $W = 8.7$ (MJ/cm), $n = 3 \,.\, 10^{20}$ (ions/cm^3) and the pressure in the plasma core is $16.5 \,.\, 10^6$ (atm) which corresponds to a magnetic pressure exerted by a field of 20 MG (*). It is evident that the energies involved

* The magnitude of κ assumed here is not in confliot with κ_B derived on the basis of Bohm diffusion. Thus: $\kappa_B = \tau/\tau_* \cong rBT^{-1/2}$. In the above example $\kappa_B = 500$.

are considerable and that this is due mainly to the presence of the factor ε_0^2 in the formula for W.

One way to diminish W is, therefore, to use systems in which the energy in the liner and in the zone B at the moment of maximum compression is not much larger than the plasma energy in the core and consequently ε_0 is relatively large, let us say equal to or larger than 1/3. This implies a use of thin liners and relatively small volumes for the zone B. Experiments have been started to do this using highly compressible plasma liners (focal machines, ref. 15).

A second way to reduce the rôle played by ε_0 is to regard the hot core as a trigger of a vaster fusion reaction. Let us suppose that the region B contains cold D,T plasma whose density is n'. The layer L of this cold plasma adjacent to the hot core can be heated to the same temperature as that of the core if the fusion input (charged particles, i.e., He^4, only) during a time $\kappa\tau_0$ is at least equal to the loss of heat to the layer L. This can be expressed as

$$1/4\ n^2\langle\sigma v\rangle\kappa\tau_0\ Q^+\ \pi r^2 \geqslant 2\pi r \cdot \delta \cdot 3n'kT \tag{56}$$

where $Q^+ = 0.5\ 64 \,.\, 10^{-5}$ (erg). The thickness δ can be determined on the basis of either heat-skin-depth or interdiffusion of the two plasmas. In many cases it is conceptually correct to take $\delta \sim r_0$ in accordance with the deminition of κ (expansion of the hot core to $2r$). In this case

$$\frac{n^2}{n'}\,\kappa r > 4 \cdot 10^{22}. \tag{57}$$

If one is to pay for the energy of the magnetic field in the B-zone by the fusion energy coming from n' plasma in the same zone we have $n' \sim 2/3n$ and

$$n\kappa r > 8/3 \cdot 10^{22} \tag{58}$$

much the same as condition (57), the neutron dissipation outside making up the degradation of energy expressed by ε. The energy needed is, therefore,

$$W = 700\ \frac{r}{\kappa\varepsilon_0}\ \ (\mathrm{MJ/cm}). \tag{59}$$

Taking the same values as in previous example we get $W = 2.3$ (MJ/cm). Projects based on this trigger concept tend to assume a form of small bombs which could be periodically exploded in a suitably lined underground cavity-boiler, whose radius is of the order of ten meters (ref. 16).

9.2.3. EXPERIMENTS IN THE RESEARCH ON CONTROLLED FUSION

Most experiments on controlled fusion aim at finding more about ways to heat plasma and about confinement of hot plasma. There is not an extremely large choice of different methods and consequently it is possible to catalogue most experiments using two labels: heating and confinement. This has been done in table I. Whenever possible the experiments are given general descriptive names rather than local laboratory nick-names. The numbers in brackets correspond to suitable references in literature (found at the end of chapter 9).

It is frequently difficult to assess the importance of these experiments for controlled nuclear fusion — sometimes the results may not be significant for plasma physics; instead they may be indispensable for the solution of technical or technological problems. Perhaps the most relevant criterion for the importance of the experiment is its position in the $n\tau$, T diagramme (fig. 159 in which is drawn the most optimistic boundary of the reactor region given by eq. (36)). For many experiments this boundary should be shifted much further to the right side of the diagramme owing to unavoidable waste of energy associated with the generation and confinement of the hot plasma (e.g., θ-pinches and injection devices).

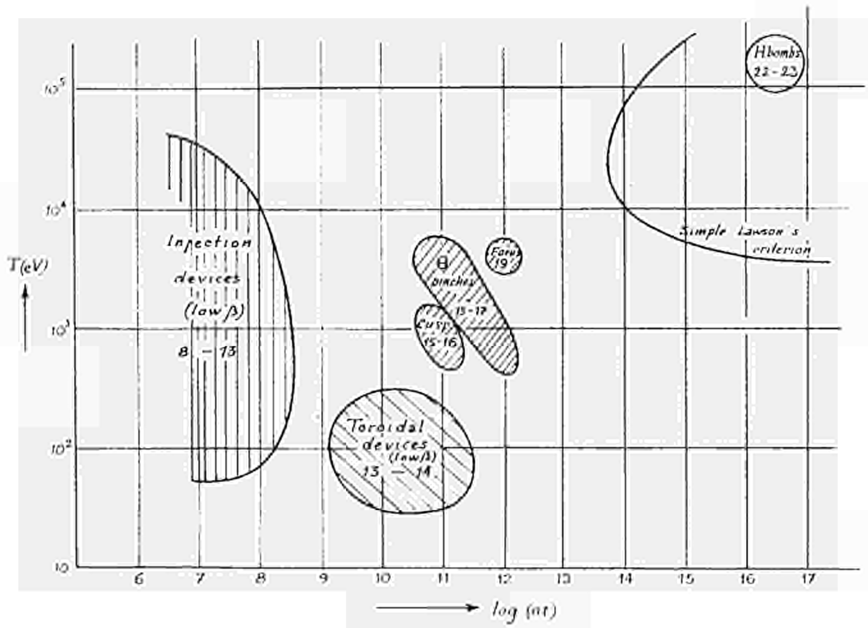


Fig. 159.

SURVEY OF THERMONUCLEAR RESEARCH TRENDS

Heat Supply \ Type of Confinement		Self-magnetic field and image currents: toroidal	Self-magnetic field and image currents: linear	External magnetic field: toroidal	External magnetic field: mirror dipole or linear	External magnetic field: mirror multipole	External field and self-field	External magnetic field and centrifugal forces	Other confinement schemes or no confinement
Heating within the confinement volume	Joule's heating	Zeta (17)	Anomalous resistance Z-pinch (19)	Stellarator preheat (23) Tokamak (24)			Astron (43)		Cold gas insulation schemes (48) and Heavy liner implosion (49)
	Self-field compression or acceleration	Toroidal Z-pinch (18)	Z-pinch (20) Tubular pinch (21) Focal devices (22)				Stabilized Z-pinch (44) Unpinch (45)		
	External field compression or acceleration			Toroidal Θ-pinch (25)	Θ-pinch (29, 30) Multiple plasma compression (31)	Cusp (38) Picket-fence (39)	Θ-pinch with reversed trapped field (46)	Plasma homopolar motor (47)	
	Ion waves or magnetic pumping			Stellarator (26)	Cyclotron resonance in mirrors (32)				
Heating before fuel injection	Injection of fast ions or neutrals			Average minimum B. torus (27)	Ogra (33) DCX (34) Ion magnetron (35)	Cusp (40)	Astron (43)		
	Injection of fast plasma			Average minimum B. torus (28)	Mirror machine (36, 37)	Cusp (41) Ioffe bottle (42)			
Heat supplied from nuclear sources									H-bomb (50)

TABLE I

It is quite evident that generally the higher is the plasma density the more successful is the attempt to get near the reactor boundary. The ion density is indicated in the circles by its decimal logarithm.

REFERENCES TO CHAPTER 9

1) W. R. Smythe, Static and dynamic electricity, p. 111, Mc Graw Hill (1939).
2) G. Gamov and C. L. Critchfield, Theory of atomic nucleus and nuclear energy sources, p. 25 (Oxford UP, 1949).
3) R. Atkinson and P. Houtermans, *Zs. f. Phys.*, ***54***, p. 686 (1928).
4) G. Gamov and E. Teller, *Phys. Rev.*, ***53***, p. 609 (1938).
5) W. A. Fowler, *Phys. Rev.*, ***81***, p. 655 (1951).
6) J. L. Tuck, *Nucl. Fus.*, *1*, p. 201 (1961).
7) Z. Kopal, Brit. *J. Appl. Ph.*, *7*, pp. 119-128 (1956).
8) A. M. Weinberg, *Phys. Today*, ***12***, p. 11, 19-25 (1959).
9) J. P. Somon *et al.*, Nucl. Fus. 1962 Suppl., p. 717 (Proc. Salz. conf. 1961).
10) W. G. Homeyer, Tech. rep. 435 MIT (1965).
A. J. Impink, Tech. rep. 434 MIT (1965).
11) See Ref. 15 Chap. VIII.
12) R. Hancox, AERE-R 2957, Wall materials for thermonuclear devices.
13) R. G. Mills, 4th Symp. on Eng. Probl. in Thermonuclear Res., Frascati (1966).
14) A. C. Kolb, Nucl. fus. 1962 suppl., p. 553, (Proc. Salz. conf. 1961).
15) J. G. Linhart, Ch. Maisonnier, J. P. Somon, LA-3770, Proc. conf. on high den. plas. Los Alamos 1967.
16) J. G. Linhart, *Phys. Today*, ***19***, p. 37 (1966).
17) W. M. Burton, Nucl. Fus. 1962 Suppl., p. 903 (Proc. Salz. Conf. 1961).
18) A. A. Ware, *Phys. Rev.*, ***123***, p. 19 (1961).
19) M. V. Babykin *et al.*, Proc. IAEA, conf. Culham 1965, p. 851, *2*.
20) A. M. Andrianov *et al.*, Proc. Geneva conf. 1958, p. 348, United Nations, N.Y.
21) O. A. Anderson *et al.*, Proc. Geneva conf. 1958, *32*, p. 150, United Nations, N.Y.
22) J. W. Mather, Proc. IAEA conf. Culham 1965, p. 389, *2*.
23) W. Stodiek *et al.*, Proc. IAEA conf. Culham 1965, p. 687, *2*.
24) V. S. Mukhovatov, Proc. IAEA conf. Culham 1965, p. 577, *2*.
25) Semiannual report of Los Alamos Sci. lab., 1965, p. 88 (Vienna 1966).
26) S. Yoshikawa, Proc. IAEA conf. Culham 1965, p. 925, *2*.
27) R. A. Dory *et al.*, *Phys. of Fluids*, *9*, p. 997 (1966).
28) T. Ohkawa *et al.*, Proc. IAEA conf. Culham 1965, p. 531, *2*.
29) W. E. Quin *et al.*, Proc. IAEA conf. Culham 1965, p. 237, *1*.
30) H. A. Bodin *et al.*, Proc. IAEA conf. Culham 1965, p. 193, *1*.
31) F. H. Coensgen *et al.*, Nucl. Fus. 1962 Suppl., p. 125 (Proc. Salz. conf. 1961).
32) M. C. Becker *et al.*, Nucl. Fus. 1962 Suppl., p. 345 (Proc. Salz. conf. 1961).
33) G. F. Bogdanov *et al.*, Nucl. Fus. 1962 Suppl., p. 215 (Proc. Salz. conf. 1961).
34) P. R. Bell *et al.*, Proc. IAEA conf. Culham 1965, p. 77, *2*.
35) J. Kistemaker *et al.*, Nucl. fus. 1962 Suppl., p. 1124 (abstract CN-10/83).
36) G. Francis, Proc. IAEA conf. Culham 1965 p. 53, *1*.
37) A. Biguet *et al.*, Proc. IAEA conf. Culham 1965, p. 69, *1*.
38) T. K. Allen, Proc. IAEA conf. Culham 1965, p. 427, *2*.
39) J. E. Osher, Picket fence, LAMS 2916 (Los Alamos 1963).
40) H. de Kluiver *et al.*, Proc. IAEA conf. Culham 1965, p. 177, *2*.
41) T. K. Allen *et al.*, Nucl. Fus. 1962 Suppl., p. 67 (Proc. Salz. conf. 1961)
42) M. S. Toffe *et al.*, Nucl. Fus. 1962 Suppl., p. 1045 (Proc. Salz. conf. 1961).
43) N. Christofilos *et al.*, Proc. IAEA conf. Culham 1965, p. 211, *2*.
44) K. Aitken *et al.*, Nucl. Fus. 1962 Suppl., p. 979 (Proc. Salz. conf. 1961).
45) J. Adam *et al.*, *Nucl.* Fus. 1962 Suppl., p. 1001 (Proc. Salz. conf. 1961).
46) A. C. Kolb *et al.*, Proc. IAEA conf. Culham 1965, p. 249, *1*.
47) J. Bergström *et al.*, Proc. IAEA conf. Culham 1965, p. 341, *1*.
48) H. Alfven *et al.*, Nucl Fus. 1962 Suppl., p. 33 (Proc. Salz. conf. 1961).
49) J. G. Linhart *et al.*, Nucl. Fus. 1962 Suppl., p. 733 (Proc. Salz. conf. 1961).
50) Proc. 2nd Plowshare Symp., UCRL-5675.

List of symbols used in Chapter 9

A	atomic mass
c	speed of light
$c_1 c_2$	degree of concentration
B, $\mathbf{B}$	magnetic field strength
d	distance
D	diffusion coefficient
e	charge of electron
E	energy
f	frequency
i	current density
I	current
k	Boltzmann's constant
l	length
L	self-inductance
m, M	mass
n	particle density
N	linear density
p, P	probability
Q	quality factor
Q_n	energy output of a nuclear reaction
r, R	radius
S	surface
T	temperature
s, v, w	velocity
V	potential
W	energy or energy output
Z	atomic number
α, α'	nuclear constants
β	ratio of plasma pressure to magnetic pressure
δ	nuclear constant
ε	excentricity
η	efficiency
κ	quality factor of confinement
λ	mean free path
$\Lambda =$	$\dfrac{P_{\min}}{P_{\max}}$ ratio of critical impact parameters
ν	collision frequency or normalized linear density
θ	angle
σ	cross-section
τ	characterisitic time
Ø	diffusion loss
ω	angular frequency
Ω	volume

CHAPTER 10

OTHER APPLICATIONS

10.1. Generation of Electromagnetic Waves

The research on controlled liberation of fusion energy had its stimulus and origin in astrophysics. A similar situation may develop in the research on new sources of h.f. electromagnetic waves where a study of the origin of cosmic radio noise may result in reproducing the cosmic mechanism on a laboratory scale. As the emission of radio-waves from stars and cosmic clouds seems to be related to plasma oscillations it is natural to ask how the properties of a plasma can be employed in the control and generation of h.f. oscillations.

The low inertia of plasma conducotrs has been already used in the tuning of microwave cavities (ref. 1). An extension of the tuning mechanism may be used for parametric generation of micro waves.

Consider a resonator in which a particular resonant mode is excited. This can be, for illustration purposes, a piston-tunable cylindrical cavity (fig. 160). Let us now decrease the volume of this resonator by pushing

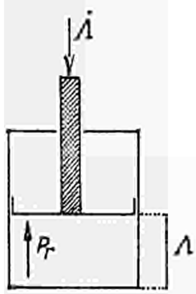


Fig. 160. A cavity analogue of Einstein's pendulum.

in the piston. This increases the resonant frequency of the cavity. If the change of cavity dimensions Λ during one period of oscillation is small compared to the wave length λ of the oscillating electromagnetic field, i.e., if

$$\frac{\frac{d\Lambda}{dt}}{f} \ll \lambda \tag{1}$$

and if one can disregard all energy losses, then the compression can be regarded as adiabatic and one can apply the adiabatic theorem which states that *

$$\frac{\text{energy of the oscillating mode}}{\text{the resonant frequency}} = \text{const.} \tag{2}$$

The compression of the volume of the resonator leads to an increase in the resonant frequency and, therefore, according to the adiabatic theorem, to the increase of the electromagnetic energy stored in the resonator. Thus one achieves not only a frequency multiplication but also a generation of electromagnetic energy.

Let us calculate the rate at which the energy is generated. Consider a cavity filled with a photon gas. If the cavity volume is reduced by $-\Delta V$, the work done on compressing the photon gas is

$$\Delta W = -\Delta V \cdot p_r \tag{3}$$

where p_r is the radiation pressure and $p_r = 2w_r \cos^2 \alpha$.

Here, w_r is the radiation energy density and α is the average angle at which the photons strike the walls of the resonator. If the total energy of the electromagnetic field is W then $w_r = \frac{1}{2}W/V$ and the equation for ΔW becomes

$$\frac{\Delta W}{W} = -\frac{\Delta V}{V} \cos^2 \alpha$$

or

$$\frac{W}{W_1} = \left(\frac{V_1}{V}\right)^{\cos^2 \alpha}. \tag{4}$$

If the volume-compression is three-dimensional

$$\cos^2 \alpha = \tfrac{1}{3}$$

and

$$\frac{W}{W_1} = \left(\frac{V_1}{V}\right)^{1/3} \tag{4a}$$

* This corresponds to conservation of the number n of photons in quantum mechanics. Thus

$$\frac{n \cdot h \cdot \omega}{\omega} = \text{const.} \rightarrow n = \text{const.}$$

as also follows directly from eq. (2) which gives

$$\frac{W}{W_1} = \frac{\omega_1}{\omega} = \left(\frac{V}{V_1}\right)^{1/3}$$

for a uniform decrease in size of a resonator.

Let us now compare the energy input, due to compression, with the losses in the resonator. These losses W' can be expressed by means of the quality factor Q of the resonator. Thus

$$W' = \frac{\omega W}{Q}$$

and from eq. (4a) we have for the input due to uniform compression

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -\frac{1}{3}\frac{\mathrm{d}V}{\mathrm{d}t}\frac{W}{V}. \tag{5}$$

If a characteristic linear dimension of the resonator is denoted by Λ one has

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}\Lambda^3}{\mathrm{d}t} = 3\Lambda^2\frac{\mathrm{d}\Lambda}{\mathrm{d}t}. \tag{6}$$

Substituting this equation into eq. (5) one obtains

$$\dot{W} = -\frac{\dot{\Lambda}}{\Lambda}W. \tag{5a}$$

Let us form a ratio of $\dot{W}$ and W. This is

$$\frac{W}{W'} = -\frac{\dot{\Lambda}}{\Lambda}\frac{Q}{\omega}. \tag{7}$$

If this ratio is larger than unity a generation of electromagnetic energy by radiation-compression becomes possible. Thus the criterion for a generator is

$$\frac{\dot{\Lambda}}{\Lambda}\frac{Q}{\omega} > 1. \tag{7a}$$

However, in most cases

$$\Lambda\omega = \text{invariant} = \Lambda 2\pi c/\lambda$$

and eq. (7a) becomes

$$\dot{\Lambda} > \frac{2\pi c}{Q}\frac{\Lambda}{\lambda}. \tag{7b}$$

For a resonator oscillating in the principal mode

$$\frac{\Lambda}{\lambda} \approx \frac{1}{2} \frac{1}{\sqrt{\varepsilon\mu}}$$

and one has

$$\dot{\Lambda} > c \frac{\pi}{Q\sqrt{\varepsilon\mu}} . \tag{7c}$$

The Q-factor of an orthodox cavity resonator is rarely larger than 3×10^4. This implies compression speeds

$$\dot{\Lambda} > 3 \times 10^6 \text{ (cm/sec)} \tag{8}$$

which is about 3 times larger than the escape speed from earth's gravitational field.

It is clear that the compression speeds required for an efficient radiation compression (eq. (7c)) are not obtainable by ordinary mechanical tuning of cavity resonators. However, such speeds are often attained in fast pinches of plasma cylinders (ref. 2) or in magnetically driven plasmas. Also the r.f. losses in a plasma can be smaller than those in metals.

Experiments have been carried out on this principle (ref. 3, 4). Although the effect has been observed, sofar the technical difficulties are such as to exclude its practical exploitation.

10.2. Direct Conversion of Chemical Energy into Electrical Energy

We have shown in chapter 8, eq. (8.73) that a temperature gradient in a plasma gives rise to a thermoelectric force

$$E_{\mathrm{T}} = \frac{c}{e} \int_{l_1}^{l_2} \operatorname{grad} T \cdot \mathrm{d}l. \tag{9}$$

A drop of temperature across a certain length of plasma makes it into a thermocouple whose output current is

$$I = l^{-1} S \sigma_{\mathrm{T}} E_{\mathrm{T}} \tag{10}$$

where S is the cross-section and l is the length of the plasma conductor. However, as a result of the temperature gradient a heat-flow Q will also appear (eq. (8.79)). The efficiency of the conversion of heat into electricity taken per unit volume is, therefore,

$$\eta_0 = \frac{i\dfrac{c}{e}}{i\dfrac{c}{e} + \kappa + \kappa_e}$$

$$= \frac{1}{1 + \dfrac{\kappa + \kappa_e}{\sigma_T \left(\dfrac{c}{e}\right)^2 \text{grad } T}}. \tag{11}$$

Averageing over the whole volume of such a plasma thermocouple one obtains the total efficiency, which in most cases is higher than that of any bimetal thermocouple. However, it seems that efficiencies larger than 15 % cannot be obtained in this way (ref. 5).

A completely different method of converting heat energy into electrical energy is based on the following principle. Plasma is produced by some source of heat, e.g., an explosion of a combustible mixture. The hot plasma expands into a volume occupied by a magnetic field and if it is conducting enough it will act as a piston and convert some of its kinetic energy into electrical energy.

In order to understand the relation of this process to the orthodox conversion by mechanical and electrical machinery, let us consider a resonant circuit (L_0C_0) coupled to an inductive load (L_1). Let us assume that due to the action of external mechanical forces the value of the inductance L_1 can be varied at a frequency corresponding to the resonant frequency of the resonant circuit L_0C_0 (fig. 161). The voltage induced across the inductor L_1 is then

$$V = \frac{\partial}{\partial t}(L_1 I) = L_1 \frac{\partial I}{\partial t} + I \frac{\partial L_1}{\partial t}. \tag{12}$$

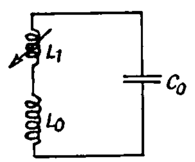


Fig. 161.

The second term corresponds to the voltage induced by the change of the inductance L_1. If the phase of the variation of L_1 with respect to the phase of the oscillations in the circuit L_0C_0 is suitably chosen, the induced voltage $I(\partial L_1/\partial t)$ will increase the stored energy in the system. This increase in energy can be traced back to the work of the external mechanical forces performed in varying the value of the inductance L_1.

In all the methods of conversion of mechanical energy into electrical energy the variation in L_1 is performed by mechanical motors, such as combustion engines, turbines, etc. The motion of these motors is transmitted to electrical generators.

For the purpose of illustrating this let us consider the inductor L_1 to be composed of two separate coils X_1 and X_2 connected in series (fig. 162). These are fixed to the two piston rods of a Diesel engine D.

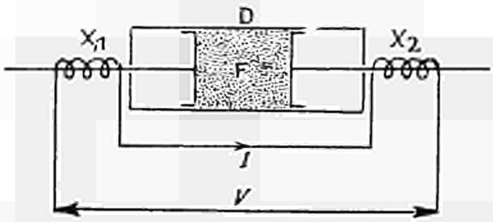


Fig. 162. Mechanical and electrical elements involved in the conversion of heat energy into electrical energy.

The combustion of the injected fuel F forces the pistons apart; the distance between the coils X_1 and X_2 grows and if the electric current flowing through the coils is I then the voltage across the two coils X_1 and X_2 due to the motion is

$$V = I \frac{\delta L}{\delta t} = I \frac{\delta M}{\delta t} \tag{12a}$$

where M is the mutual inductance between the two coils.

This is a typical arrangement and its drawbacks are mainly the following:

a) A large proportion of the heat of combustion is wasted in heating up the walls of the combustion engine and the pistons.

b) Mechanical devices for transmitting the motion of the pistons to the coils X_1 and X_2 must be employed, such as piston rods, bearings, flywheels. The introduction of moving metal components results in large inertia and therefore limits the speed of repetition of the cycle and the speeds of all the moving components.

It has been suggested that one could employ plasma in place of both the piston and the inductor L_1 the simplest form of such a moving plasma member being a hollow cylindrical arc carrying a heavy current I between two plane electrodes (see fig. 163). Such a current flow produces a magnetic field H_φ which at the outer surface of the discharge column reaches a maximum value

$$H_\varphi = \frac{0.2I}{(r_0 + \frac{1}{2}d)} \quad \text{(gauss, amps).} \tag{13}$$

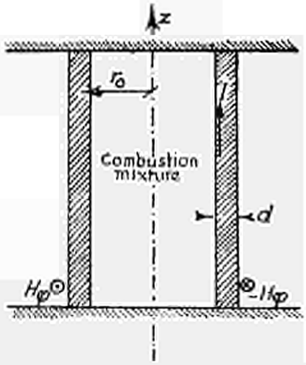


Fig. 163. A plasma shell as a piston in the scheme for direct conversion of chemical energy into electrical energy.

At the inner surface of the current column one has $H_\varphi = 0$. The mean value of this field inside the current shell is

$$H_\varphi \cong \frac{0.1I}{r_0} \quad \text{(gauss, amps).}$$

The action of the field H_φ on the current I produces a force

$$P = 1/10\ i_z H_\varphi \quad \text{(dynes/cm}^2\text{, amp, gauss),} \tag{14}$$

where i_z is the surface current density (on the outer suface)

$$i_z = \frac{I}{2\pi r_0} \quad \text{(amps/cm).}$$

Therefore

$$P = 10^{-2} \frac{I^2}{2\pi r_0^2} \quad \text{(dyne/cm}^2\text{, amps).} \tag{14a}$$

This force is directed radially inwards and tends to constrict (pinch) the current cylinder (p. 238). The constricting current channel compresses the gases that are inside the cylindrical hollow. If these gases form a combustible mixture and their temperature reaches the ignition temperature an explosion will take place.

The gas pressure of the gases inside the current cylinder will rise sharply and cause an expansion of the plasma cylinder *. This expansion is analogous to the separation of the coils X_1 and X_2 in fig. 162. The voltage induced across the discharge is

$$V = I \frac{\partial L_1}{\partial t}$$

where L_1 is the inductance of the current channel.

The expansion can be conveniently continued until the current channel is expelled and thrown on the walls of the confining vessel which represents, at the time of maximum expansion, a small loss of energy. After this the discharge may be struck again and the cycle of compression and expansion can be repeated. The fuel injection must be coordinated with the initiation of a new cycle. One can represent the work done in a cycle by a pressure-volume diagramme (or as the expansion is a radial one by the P, r_0^2 diagramme (fig. 164)). The

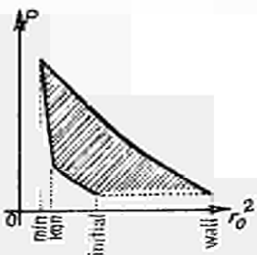


Fig. 164.

discharge is started always at $r_0 = r_{0\ \mathrm{initial}}$. The ignition of the compressed fuel occurs at $r_0 = r_{\mathrm{ignition}}$ after which the pressure P rises rapidly and the compression is finally arrested at some $r_{\min} < r_{\mathrm{ignition}}$. After this the column expands and the pressure decreases. The work done by the plasma piston is proportional to the area enclosed by the P, r^2 curve.

* The density of plasma in the current channel must be large enough not to let the gas molecules leak out radially.

It is most unlikely that the large efficiencies suggested by the large ratio of $(t_{wall}/r_{min})^2$ can be realised. This is connected with the conductivity of plasma at explosion temperatures being much smaller than that of copper. This would be evident to any electric engineer who would have to use a high resistance alloy for conductors in a dynamo. This could be done, provided the frequency of revolution of the dynamo and the frequency of the generated magnetic field is higher than a certain critical value. The criterion, corresponding to the critical value of the speed of the lossy conductor, is similar to that derived for the plasma piston in a tunable cavity on p. 315.

A similar mechanism could be also used in the conversion of energy of fission reactions into electrical energy (ref. 6) although here the long mean free path of neutrons in appropriate fuel gases, such as UF_6, would require high gas pressure in a large combustion chamber.

Some of the propositions of direct conversion of fusion energy into electrical energy are based on the mechanism discussed in chapter 9. However, this depends on a fusion reactor being available in the first instance.

A somewhat different mechanism of generation of electricity is based on the induction of electromotive force in a plasma stream moving across a magnetic field. This effect is described by eqs. (3.62) and (3.63). Neglecting plasma pressure, these can be written as

$$\rho \frac{\partial \boldsymbol{V}}{\partial t} = \boldsymbol{j} \wedge \boldsymbol{B}$$

$$\boldsymbol{E} = -\frac{m}{e^2 nc} \boldsymbol{V} \wedge \boldsymbol{B} + \frac{1}{enc} \boldsymbol{j} \wedge \boldsymbol{B}.$$

If the velocity of plasma flow is perpendicular to the magnetic field an electromotive force is induced. If this e.m.f. can force a current through some external impedance, electric energy is generated at the expense of the kinetic energy of plasma. This electromagnetic reaction on the flow is represented by the last term in the first equation.

The mechanism resembles that of a homopolar generator, the rôle of the rotor being played by a plasma jet. This can be accomplished in two ways.

In the first, known as the open cycle MHD generator, the jet is formed by combustion products similar to the exhaust jet of jet-engines. As the temperature is relatively low, the degree of ionization is often too low for an efficient momentum transfer.

A better method, known as the closed cycle MHD generator, is based on the recycling of the same fluid, usually an inert gas (e.g., He) to which a small amount of material having a low ionization potential has been added (e.g., Cs). This "seeded" substance provides a relatively high degree of ionization, which means a good conductivity and a good efficiency of the "plasma rotor" (ref. 7).

10.3. Applications to Particle-Accelerators

An accelerator consists of four basic components:

1) the injection system,
2) the mechanism for guiding and focusing of the accelerated particles,
3) the accelerating mechanism,
4) the ejector.

It appears that some of the properties of plasmas may be exploited in the construction of the first of these components.

10.3.1. PLASMA BETATRON

The injection of electrons into a betatron has been for a long time an imperfectly understood subject. Only recently it has been shown (ref. 8) that some of the electrons injected into the toroidal betatron chamber from an electron gun are retained in the chamber during the initial phase of acceleration owing to the electric field of the space charge built up by the injected current. The percentage of the captured electrons is generally very low, usually of the order of 1 % of the injected charge.

Let us now consider a plasma of low and uniform density located in the betatron chamber. It has been shown (chapter 8, p. 259) that a strong electric field applied to a uniform plasma induces a run-away current of electrons. In order that the collisional friction on these electrons could be neglected $E \gg 2 \times 10^{-8} n/T_e$ (volt/cm) (eq. (8.27)).

Furthermore if such a field is applied parallel to a plasma cylinder, the skin depth δ in the plasma must be very much larger than the radius of the cylinder. Thus from eq. (5.35a)

$$\frac{1}{2}\,\frac{10^6}{\sqrt{n}} > r_0.$$

Both these conditions must be satisfied in order that the plasma in the betatron chamber could become a source of electrons for the betatron.

The last condition puts an upper limit on the linear density * of such plasmas. Thus

$$\pi n r_0^2 = N < \tfrac{1}{4}\pi \times 10^{12} \tag{15}$$

or putting $e^2/mc^2\, N = \nu$ (see eq. (4.54)) one has

$$\nu < 0.23. \tag{16}$$

If all the electrons were accelerated by the betatron mechanism to relativistic energies the circulating current corresponding to $\nu = 0.23$ would be 3750 (amps).

Apart from the centrifugal force F_c of the accelerated electrons there is also the force F_M of the self-magnetic field of the electron current, both of which have to be compensated by the centripetal forces of the betatron field. The ratio F_M/F_c has been shown to be (p. 140, eq. (4.86)).

$$\frac{F_M}{F_e} = \frac{\nu \ln \dfrac{8R}{r_0}}{\gamma}.$$

It follows that if the magnetic field of an orthodox betatron is to keep the intense electron current in an orbit near the equilibrium orbit then

$$\frac{F_M}{F_c} \ll 1.$$

This gives us a condition for γ (or a condition for ν if $\gamma \sim 1$).
Thus

$$\gamma \gg \nu \ln \frac{8R}{r_0}. \tag{17}$$

For the ν_{max} given by inequality (16) and for $\ln 8R/r_0 \approx 10$ one has

$$\gamma \gg 2.3. \tag{17a}$$

A further condition, binding the minor and major radii of the electron beam, γ and ν follows from the considerations of hydrodynamic stability discussed on p. 213. There, we have shown that according to the two-string model of a neutralized electron beam a kink instability occurs for wave lengths

$$\lambda > \sqrt{2}\pi r_0 \beta \sqrt{\frac{\gamma}{\nu}}.$$

* This limitation is not very stringent owing to the form of the electron trajectory

In order that the electron beam in a plasma betatron be stable the periphery $2\pi R$ of this beam must be shorter than 2λ *. Thus

$$R < \sqrt{2} r_0 B \sqrt{\frac{\gamma}{\nu}} . \tag{18}$$

Substituting for ν and γ values consistent with the previous criteria, e.g.,

$$\nu = 0.1, \gamma = 10$$

one obtains

$$\frac{R}{r_0} < 14.$$

On the basis of these considerations it appears that a plasma betatron with a circulating current of the order of 100 amps is feasible. This is two orders of magnitude higher than the circulating current in orthodox betatrons and with such a current some new experiments in low energy nuclear physics could be contemplated.

The experimental work on these plasma rings has been up to now marred by other instabilities (ref. 8a) some of which are related to the two-stream instability (see p. 185).

10.3.2. COLLECTIVE ION-ACCELERATION

In 1956, Veksler proposed a new acceleration mechanism, now known as the "collective acceleration", for the acceleration of ions.

The idea can be described as follows:

Let us consider a bunch of N electrons and N_p ions ($N_p \ll N$), and let us suppose that this bunch remains confined during the acceleration. If, owing to external fields, the electrons are accelerated, the ions may become trapped in bunch owing to the action of an ambipolar electric field created inside the bunch and thus accelerated together with the electrons up to a common final velocity. Each ion is dragged by $\dfrac{N}{N_p}$ electrons so that, within certain limits, the length of the accelerator can be reduced by the same factor $\dfrac{N}{N_p}$ with respect to a machine accelerating ions only. Experiments have been carried out (ref. 8b) on a prototype of such an accelerator. If this principle should prove to be

* If $2\pi R = \lambda$, the kink instability transforms the originally circular beam into a circle again. This is stabilized by the betatron field and does not lead to instability.

applicable, one could hope of obtaining 10^{12} protons at 1.000 Bev with a machine of only 1.500 m long.

A particularly interesting version of a collective accelerator is known as the electron-ring accelerator (ERA) or "smokotron", in which the electron bunch has the form of a self-focusing electron ring of relativistic electrons. ERA should function as follows:

The generation of the ring

An intense beam of electrons is fired into an "injection box". A magnetic field B_0 applied across the box turns the electrons around the axis of symmetry so that they form a ring, initially with a radius r_0. The magnetic field is increased rapidly, and the ring shrinks down to a radius $r = \left(\frac{B_0}{B}\right)^{1/2} r_0$. This increases the transverse energy of the electrons (see pp. 40-57). After this, hydrogen gas is fed in and is ionized by the fast moving electrons.

The positive ions are attracted into the deep potential well which the intense ring of negative electrons sets up and the slow electrons join the electrons in the ring.

Acceleration

In accelerating the rings, it is important not to pull so hard that the stability is destroyed — i.e., the electrons are pulled away from the protons.

Two methods of acceleration are possible. The first has been called "expansion acceleration". It involves setting up a magnetic field which is progressively weaker along the accelerator tube. In travelling through such a field, the radius of the ring grows; the tranverse energy of the electrons falls and reappears as increased longitudinal energy — increased energy of the ring as a whole travelling down the tube. The energy gain is inversely proportional to the square root of the strength of the magnetic field. Thus if the field decreases by a factor of four over some distance the energy of the protons would be doubled as the ring travels that distance (see pp. 41-42).

An electron ring accelerator (ERA) using only expansion acceleration might be suitable for protons up to energies around 1 GeV. Such an ERA could be a fairly compact machine giving this order of energy over a length of about 10 m. For higher energies, "electric acceleration" is needed, using, for example, r.f. cavities. Here electric fields would accelerate the rings, while an axial magnetic field is continuously applied to keep the ring radius small.

10.4. Rocket Propulsion

In order that a certain load whose mass is M can be lifted against the gravitational field by a rocket motor, the thrust P_0 of the motor must be larger than the weight of the load. Thus

$$P_0 = m_0 v_j > Mg \tag{19}$$

where m_0 is the mass of the jet expelled per second
v_j is the jet-velocity
g is the acceleration in a gravitational field.

Apart from this consideration of momentum-balance one must also take into account the power consumed by the jet which is

$$W > \tfrac{1}{2} m_0 v_j^2. \tag{20}$$

Assuming that m_0, v_j and g are constant, the equation of motion of the rocket is

$$\frac{\mathrm{d}}{\mathrm{d}t}(MV) = - m_0(v_j - V) - Mg \tag{21}$$

where V is the speed of the rocket.

However, M is variable as it includes the jet mass also. Thus

$$M = M_0 + \alpha m_0 - m_0 t \tag{22}$$

where M_0 is the payload.

Obviously, after a time $t = \alpha$ all the jet mass is exhausted and the thrust of the rocket vanishes. We shall distinguish, therefore, two phases of the flight — a powered flight and a free flight.

For the first phase the equation (21) can be written as

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{- m_0 v_j}{M_0 + m(\alpha - t)} - g \tag{23}$$

whose solution is

$$V(t) = |v_j| \ln \frac{M_0 + m_0\alpha}{M_0 + m_0(\alpha - t)} - gt. \tag{24}$$

If one aims at escaping from the gravitational field then at the end of the powered flight the speed $V(\alpha)$ must be at least equal to the escape velocity V_0 from that field. Owing to inequality (19) the term $-gt$ can be neglected at $t = \alpha$ and one gets

$$\exp\left(\frac{V_0}{|v_j|}\right) = 1 + \frac{m_0\alpha}{M_0}. \tag{24a}$$

It follows that if V_0/v_j is a large quantity, the ratio of weight of the jet-fuel to payload must be also very large. However, in the opposite case, i.e. for $v_j/V_0 \gg 1$, one has

$$\frac{M_0}{m_0\alpha} \approx \frac{|v_j|}{V_0} \tag{24b}$$

and the mass of the payload can be many times the mass of the jet-fuel

This desirable result is unobtainable with chemical fuels, as can be seen from the following simplified consideration. Let all the heat energy liberated in a chemical reaction be converted into the kinetic energy of a unidirectional flow. Let us take for this the $2H_2 + O_2$ reaction which is one of the more favourable reactions for this purpose. The energy liberated per H_2O is approximately 1.6×10^{-12} erg. The velocity which this amount of energy imparts to one molecule of H_2O is $1/3 \times 10^6$ cm/sec. This is the maximum velocity that a burning mixture of oxygen and hydrogen could aquire. In practice the jet velocity would be lower by about 50 % because of energy losses, etc Somewhat higher jet velocity would be obtained from the combustion of hydrofluoride fuels. Thus, as the escape velocity from earth's gravitational field is about 10^6 cm/sec the payload to total load ratio is in the region of 0.2 to 0.1.

We have seen in the section on magnetically driven plasmas that a plasma gun can accelerate a bunch of ionized gas to a speed of 10^7 cm/sec without employing any special sources of stored electrical energy. Such a jet speed would, according to eq. (24b), reduce the mass of the jet fuel to a small fraction of the payload. However, there are two major problems connected with a plasma jet:

a) the development of a plasma gun having an appreciable thrust,

b) the energy supply for such a gun.

The first problem is a technological one and can probably be solved by engineering development (ref. 9).

The second problem forces us to look for energy sources other than chemical ones. This can be shown by calculating the power of the jet as a function of jet velocity, keeping the thrust constant. Thus if

$$m_0 v_j = P$$

one has for the power consumed by the jet

$$W = \tfrac{1}{2}Pv_j. \tag{25}$$

This is such a high value, counted per particle, that only nuclear sources can provide the answer. Even if nuclear sources of energy are admitted, further problems must be solved such as the transfer of power from the nuclear reactor to the plasma gun.

In some astronautical applications a large thrust may not be necessary, e.g., in the case of interplanetary probes, where the probe may first be placed into a suitable orbit around the Earth and only subsequently spiral out using a small plasma engine. Encouraging results have been obtained on such small continuously operated plasma guns (ref. 10).

10.5. Energy Storage

The art of storing energy in such a way that it can be rapidly released has its beginning in the invention of bows, battering rams, hammers and other devices. In present day engineering energy is stored mainly in four different forms:

1. chemical energy of fuels and nuclear energy,
2. kinetic energy of flywheels,
3. magnetic energy of currents flowing through some large inductance or energy of atomic currents in permanent magnets,
4. electric energy in condensers.

The maximum energy density W_0 has been obtained so far with chemical explosives in which $W_0 \approx 10^4$ J/cm^3.

In order to obtain the same in magnetic storage systems, the magnetic field-strength would have to be 1.6×10^6 gauss.

Assuming that the rim of a flywheel has the speed of sound and is made of steel one finds that the kinetic energy density at the rim is 500 joule/cm^3. This is also a value dictated by the mechanical strength of flywheels.

The density of electrical energy stored in ordinary condensers is about 10^{-1} J/cm^3 and is, therefore, far below the values of W_0 achieved by the methods 1 — 3. Recent development in titanium oxides having a dielectric constant of the order of 1000 raises hopes that an electric energy density of over 1 J/cm^3 can be obtained with condensers (ref. 11). This is still rather low for some purposes, such as the dynamic pinch and others.

Some properties of rotating plasma discs suggest that a method of storing electric energy can be developed in which W_0 will be comparable to that achieved by flywheels.

Consider a sheet of plasma confined by a magnetic field B between two conducting planes $\sigma_{1,2}$ (fig. 166). Let us apply an electric voltage between the planes. We have shown in chapter 5, p. 177 that plasma behaves as a dielectric whose dielectric constant is

$$\varepsilon = 1 + \frac{4\pi n M c^2}{B^2}, \tag{26}$$

provided the frequency of variation of the electric field is much lower than the ion-cyclotron frequency. At the moment of application of the voltage V a current i_c will start to flow which will transport a total charge q per cm^2

$$q = CV \tag{27}$$

where C is the effective capacity per cm^2. This current is the polarization current in the plasma and is due to the shift of the guiding centra of positive ions and electrons* (fig. 167). As it flows at right

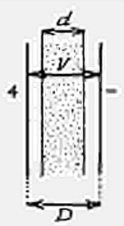


Fig. 165.

Fig. 166.

angles to the magnetic field it gives rise to a force $(1/c)i_c B$ which sets the plasma layer into motion. If, for the moment, we assume that $D \approx d$, i.e., the plasma fills completely the space between the planes then

$$C \cong \frac{nMc^2}{B^2 d}\ (\mathrm{pF})$$

and the total momentum accumulated by the action of the force $(1/c)i_c B$ is

* See also footnotes, pp. 54 and 58.

$$P = \int_0^\infty \frac{1}{c} i_c B \, dt = \frac{nMc}{Bd} V. \tag{28}$$

This follows directly from eq. (2.79a), where it was shown that charged particles in crossed electric and magnetic fields acquire a drift velocity $v_d = c(E/B)$. In our case $E = V/d$ and therefore,

$$P = cnM \frac{V}{Bd}.$$

The electric energy $1/2qV$ supplied to the plasma is, therefore, stored mostly as the kinetic energy of the plasma flow. This kinetic energy can be converted back into electrical energy and the plasma flow will stop. Such a "discharge" of the plasma capacitor can be effected within a very short time. The shortest attainable discharge time is of a few periods of the ion-cyclotron motion.

For the same reasons as those which apply to any flywheel storage, our plasma layer must be wrapped into a ring (radius R). A rotating ring of plasma is subjected to centrifugal forces which must be balanced. For our magnetically confined plasma such a balance will result from the magnetic field B on the outside of the plasma ring being compressed until

$$\frac{B^2}{8\pi} \approx \frac{ndMv_d^2}{R}. \tag{29}$$

From this simple formula follows at once that the ratio of stored kinetic energy to the stored magnetic energy is of the order of $R/4d$.

In experiments on a plasma homopolar devices, energy densities of 1 J/cm^3 were achieved (ref. 12). With large dimensions (R of the order of 100 cm) and with insulation between the electrodes σ_1 and σ_2 capable of withstanding 100 kV, energy densities up to 100 J/cm^3 seem to be attainable. This implies a total stored energy of the order of 10^7 J which would be of considerable interest in research on fast pinches and would be especially useful for the dynamic pinches. However, in spite of many attempts to improve the insulation between the electrodes, no encouraging results have been so far obtained. A short-circuit has been always observed even when the insulating wall has been removed relatively far away from the rotating plasma (ref. 13).

REFERENCES TO CHAPTER 10

1) F. R. Arams and H. K. Jenny, Proc. IRE, p. 1102 (1955).
2) A. M. Andrianov *et al.*, Plasma phys. and thermonucl. res. *1*, p. 251, Pergamon Press 1959.
3) J. G. Linhart and L. Th. Ornstein, Proc. 4th conf. ioniz. plasma in gases, Uppsala, p. 774 (1959).
4) O. G. Zagorodnov *et al.*, *Jour. Tech. Phys.*, *31*, p. 212 (1961).
5) G. M. Grover, *Nucleonics*, *17*, p. 54 (1959).
6) S. A. Colgate and R. R. L. Aamodt, *Nucleonics*, *15*, p. 50-55 (1957).
7) Electricity from MHD, Symp. IAEA and ENEA, Salzburg 1966.
8) A. N. Matveev, Transl. JETP *35*, p. 372 (1959).
8a) J. G. Linhart *et al.*, Proc. High-Energy conf., p. 139, CERN 1959.
8b) Symp. on high en. accelerators, Cambridge (1967).
9) G. M. Giannini, *Sci. American*, *197*, p. 80-88 (1957).
A. von Engel, *Nature*, *183*, pp. 573-74 (1959).
10) C. L. Longmire, Plasma acceleration, p. 3, Stanford Univ. Press 1960.
L. E. Kalikhman, Elements of magneto gas dynamics, p. 333, Saunders Co. 1967.
11) D. Lupfer, Proc. of 2nd Geneva int. conf. on peaceful uses of atomic en. Vol. 32, p. 457 (1958).
12) O. Anderson *et al.*, *Jour. Appl. Phys.*, *30*, p. 188 (1959).
13) See ref. 47, chapter 9, p. 311.

List of symbols used in Chapter 10

a	amplitude	S	surface
B	magnetic field strength	t	time
c	velocity of light	v	velocity
C	capacity	V	volume or voltage
e	charge of electron	w	energy density
E	electric field	W	energy
f	frequency	x	coordinate
F	force	α	angle
g	gravitational acceleration	$\beta =$	v/c
i	current density	$\gamma =$	$(1 - v^2/c^2)^{-1/2}$
I	current	ε	dielectric constant
k	Boltzmann's constant	η	efficiency
l	length	κ	coefficient of thermal conductivity
L	self-inductance		
m, M	mass	λ	wave length
n	particle density or index of the betatron field	Λ	characteristic linear dimension
p	pressure	μ	magnetic permeability
P	momentum	ν	normalized linear density
Q	quality factor	ρ	resistivity
q	electric charge	σ	electric conductivity
r, R	radius	ω	angular frequency

GENERAL LITERATURE

1) H. Alfvén and C. G. Fälthammer, Cosmical electrodynamics, Oxford U.P. (1963).
2) L. A. Artsimovich, Controlled thermonuclear reactions, Oliver & Boyd Ltd. (1964).
3) W. G. Chace and H. K. Moore editors, Exploding wires, Plenum Press (1964).
4) S. Chapman and T. G. Cowling, The mathematical theory of non-uniform gases, CUP (1952).
5) J. L. Delcroix, Physique des Plasmas, Dunod Paris (1966).
6) V. G. A. Ferraro and C. Plumpton, Magnetofluid mechanics, Oxford U.P. (1961).
7) E. Fünfer and G. Lehner, Ergebnisse der exakten Naturwissenschaften, Springer Verlag 1962, vol. 34.
8) B. Lehnert, Dynamics of charged particles, North Holland publ. Co. (1964).
9) C. L. Longmire, Elementary plasma physics, Interscience (1963).
10) A. G. Sitenko, Electromagnetic fluctuations in plasma, Gorki Univ. press (1965).
11) W.B. Thompson, An introduction to plasma physics, Pergamon Press (1962).
12) L. Spitzer, Physics of fully ionized gases, Interscience (1962).
13) Van Kampen, Elementary plasma physics, North Holland publ. Co. (1967).
14) L. E. Kalikhman, Elements of magneto gas dynamics, Saunders Co., Philadelphia and London 1967.

BOOKS ON SPECIAL TOPICS

1) K. G. Budden, Lectures on magnetoionic theory, Gordon and Breach (New York 1964).
2) J. F. Denisse and J. L. Delcroix, Plasma waves, Interscience (New York 1963).
3) V. L. Ginzburg, Propagation of electromagnetic waves in plasma, Gordon and Breach (New York 1961).
4) T. H. Stix, Theory of plasma waves, McGraw-Hill (New York 1962).
5) J. E. Anderson, Magnetohydrodynamic shock waves, MIT (Cambridge Mass. 1963).
6) G. Bekefi, Radiation processes in plasmas, Wiley (New York 1966).
7) M. A. Held and C. B. Wharton, Plasma diagnostics with microwaves, Wiley (New York 1965).
8) H. R. Griem, Plasma Spectroscopy, McGraw-Hill (New York 1964).

NOTICE TO THE READER

All Euratom reports are announced, as and when they are issued, in the monthly periodical **EURO-ABSTRACTS**, edited by the Centre for Information and Documentation (CID). For subscription (1 year: US$ 16.40, £ 6.17) or free specimen copies please write to:

Handelsblatt GmbH
« Euro-Abstracts »
Postfach 1102
D-4 Düsseldorf (Germany)

or

Office de vente des publications officielles des Communautés européennes
37, rue Glesener
Luxembourg