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EUROPEAN ATOMIC ENERGY COMMUNITY - EURATOM

LECTURES ON "RIESZ SPACES"

given by Prof. A. C. ZAANEN

Editor : R. F. GLODEN

1966

Joint Nuclear Research Center
Ispra Establishment - Italy

Scientific Information Processing Center - CETIS

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CONTENTS

SUMMARY

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1st Lecture

Definition of an ordered vector space

L is called : ordered vector space, if

(i) L is a real linear vector space,

(ii) L is a partially ordered set, (i.e. for elements f, g, h,... and $f \leq g$, we have

$f \leq g$ and $g \leq h \Rightarrow f \leq h$

$f \leq f$ for all f

$f \leq g$ and $g \leq f \Rightarrow f = g$).

These two structures are compatible, L is such that

a) $f \leq g \Rightarrow f+h \leq g+h$ for all $h \in L$,

b) $f \leq g \Rightarrow af \leq ag$ for all real $a \geq 0$.

Let us recall the following definitions:

If L is partially ordered and A is a subset of L, then the element $g \in L$ is called an upper bound of A if $f \leq g$ for all $f \in A$. If, in addition, $g \leq g_1$ for every other upper bound g_1 of A, then g is called the least upper bound of A, or the supremum of A.

Notation : $g = \sup A$ or $g = \sup \{ f : f \in A \}$

Similarly, if $h \leq f$ for all $f \in A$, then h is a lower bound of A. If $h \geq h_1$ for any other lower bound h_1 of A, then h is the greatest lower bound of A, or the infimum of A.

Notation : $h = \inf A$ or $h = \inf \{ f : f \in A \}$

Manuscript received on July 29, 1966.

<u>Definition of a RIESZ space</u>

The ordered vector space (o.v. sp.) L is called a RIESZ space if for every pair $f, g \in L$, there exist $\sup(f,g)$ and $\inf(f,g)$.

<u>Examples</u>

(i) The real numbers, i.e. R^1.

(ii) The plan R^2. Partial order defined by

$$f \leq g \Leftrightarrow \begin{array}{l} f_1 \leq g_1 \\ f_2 \leq g_2 \end{array}$$

fig. I, 1

(iii) The space $C([0,1])$, being the set of all real continuous functions on $[0,1]$.

Definition of $f \leq g$: $f(x) \leq g(x)$ for $x \in [0,1]$

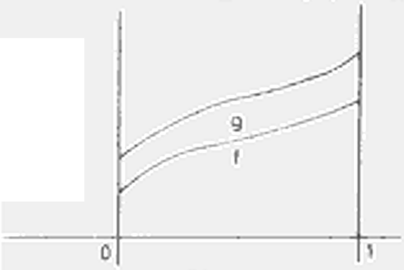


fig. I,2

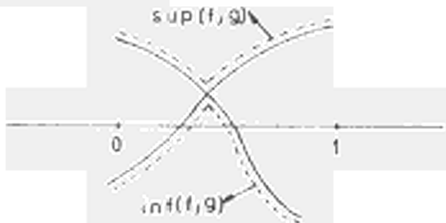


fig. I, 3

<u>Positive cone of L</u>

<u>The positive cone of L, called L^+</u>, is by definition $\left\{f;\ f \geq 0\right\}$, 0 being the null element of L.

<u>Graphic representation of L^+ for the above examples</u>

(i)

fig. I, 4

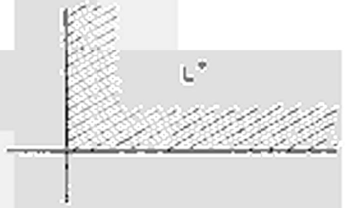
(ii) 

Fig. I, 5

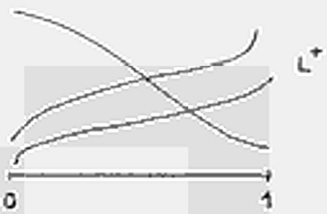
(iii) 

Fig. I, 6

Cone properties

$f, g \in L^+ \Longrightarrow f+g \in L^+$

$f \geq 0 \Longrightarrow af \geq 0$ for real $a \geq 0$

$f, -f \in L^+ \Longrightarrow f = 0$

$f \geq 0 \Longrightarrow f+g \geq g \geq 0 \Longrightarrow f+g \geq 0$

$$\left.\begin{array}{l} f \geq 0 \\ f \leq 0 \end{array}\right\} \Longrightarrow f = 0$$

Simple properties

(i) $f \geq g \Longleftrightarrow f-g \in L^+$

(ii) $f \geq g \Longleftrightarrow f = \sup(f,g)$ and $g = \inf(f,g)$

(iii) $f \geq g \begin{array}{l} \nearrow\!\!\!\swarrow \; af \geq ag \quad \text{for} \quad a > 0 \\ \searrow\!\!\!\nwarrow \; af \leq ag \quad \text{for} \quad a < 0 \end{array}$

Indeed $f \geq g \Longleftrightarrow f-g \geq 0 \Longleftrightarrow -f \leq -g$

(iv) sup (f,g) and inf (f,g) :

$\sup(f,g) = -\inf(-f,-g)$

indeed, if h : upper bound of f,g

then -h: lower bound of -f, -g.

(v) $\begin{matrix}\sup\\ \inf\end{matrix}\,(f+h,\ g+h) = \begin{matrix}\sup\\ \inf\end{matrix}\,(f,g) + h$

We have to prove that $\sup(f+h,\ g+h) = \sup(f,g)+h$.

Since $\sup(f,g) \geqslant \begin{matrix} f\\ g\end{matrix}$, we have $\sup(f,g) + h \geqslant \begin{matrix} f+h\\ g+h\end{matrix}$, and so

$$\sup(f,g) + h \geqslant \sup(f+h,\ g+h) \qquad (1)$$

For the converse, write $k = \sup(f+h,\ g+h)$, only for abbreviation. Then $k \geqslant \begin{matrix} f+h\\ g+h\end{matrix} \Longrightarrow k-h \geqslant \begin{matrix} f\\ g\end{matrix} \Longrightarrow k-h \geqslant \sup(f,g) \Longrightarrow k \geqslant \sup(f,g)+h$, so

$$\sup(f+h,\ g+h) \geqslant \sup(f,g)+h \qquad (2)$$

The desired result follows from (1) and (2).

(vi) $\sup(af,\ ag) = \begin{cases} a\sup(f,g) & \text{for } a \geqslant 0\\ a\inf(f,g) & \text{for } a \leqslant 0\end{cases}$

(vii) $\sup\left\{\sup(f,g),h\right\} = \sup\left\{\sup(f,h),\ \sup(g,h)\right\} = \sup(f,g,h)$

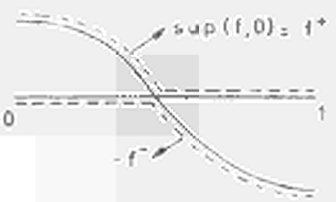


fig. I,7

<u>Notational</u> : $\sup(f,0) = f^+$

$$-\inf(f,0) = \sup(-f,0) = f^-$$

$$\sup(f,-f) = |f|$$

<u>Properties of f^+, f^- and $|f|$</u>

(i) $f^+, f^- \in L^+$; $f^+ = (-f)^-$ and $f^- = (-f)^+$,

because $(-f)^- = \sup(f,0) = f^+$

$$|f| = |-f|$$

(ii) $f = f^+ - f^-$ and $\inf(f^+, f^-) = 0$

Indeed $f^+ - f = \sup(f,0) - f = \sup(0, -f) = f^-$

and $\inf(f^+, f^-) = \inf(f+f^-, f^-)$

$= \inf(f,0) + f^- = -f^- + f^- = 0$

$$|f| = f^+ + f^-,$$

because $|f| = \sup(f, -f) = \sup(2f, 0) - f$

$= 2\sup(f,0) - f = 2f^+ - (f^+ - f^-) = f^+ + f^-$

2^{nd} Lecture

From the formula that $|f| = f^+ + f^-$ we deduce

$$0 \leq f^+ \leq |f|,$$
$$0 \leq f^- \leq |f|.$$

The formula that $f = f^+ - f^-$ yields

$$-f^- \leq f \leq f^+.$$

We prove now the following proposition:

$$f \leq g \iff \begin{array}{l} f^+ \leq g^+ \\ f^- \geq g^- \end{array}$$

If $f \leq g$, then $\sup(f,0) \leq \sup(g,0)$; indeed, let us denote $\sup(g,0)$ by h:

$$\left.\begin{array}{l} h \geq g \geq f \\ h \geq 0 \end{array}\right\} \Rightarrow h \geq \sup(f,0) = f^+, \text{ i.e., } g^+ \geq f^+.$$

$f \leq g \Rightarrow -f \geq -g \Rightarrow \sup(-f,0) \geq \sup(-g,0)$, i.e. $f^- \geq g^-$.
Conversely, if $f^+ \leq g^+$ and $f^- \geq g^-$, then

$$f = f^+ - f^- \leq g^+ - g^- = g.$$

<u>List of formulas for sup(f,g) and inf(f,g)</u>

(i) $\sup(f,g) = (f-g)^+ + g = f + (g-f)^+$

(ii) $\inf(f,g) = f - (f-g)^+ = g - (g-f)^+$

(iii) $\sup(f,g) + \inf(f,g) = f+g$

(iv) $\sup(f,g) - \inf(f,g) = |f-g|$

(v) $\sup(f,g) = \frac{1}{2}\left\{f+g+|f-g|\right\}$

(vi) $\inf(f,g) = \frac{1}{2}\left\{f+g - |f-g|\right\}$

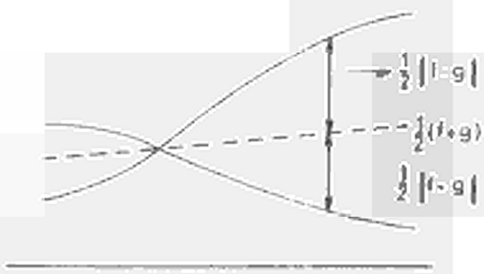


fig. II, 1

Proof of these formulas

$$\text{(i)} \quad \sup(f,g) = \sup(f-g,o) + g = (f-g)^{+}+g$$
$$\text{(ii)} \quad \inf(f,g) = \inf(f-g,o) + g = -\sup(g-f,o) + g = g-(g-f)^{+}$$

To prove (iii) we have to add the left hand sides to the right hand sides of (i) and (ii); to prove (iv) we have to subtract them:

$$\text{(iv)} \quad \sup(f,g) - \inf(f,g) = (f-g)^{+}+(g-f)^{+} = (f-g)^{+}+(f-g)^{-} = |f-g|$$
because $f^{+} = (-f)^{-}$

To prove (v), we add the left hand sides to the right hand sides of (iii) and (iv), then we divide by 2. To prove VI, we repeat these operations, subtracting instead of adding.

Other formulas

$$\sup(|f|,|g|) = \frac{1}{2}\,(|f+g| + |f-g|\,)$$
$$\inf(|f|,|g|) = \frac{1}{2}\,\big||f+g| - |f-g|\big|$$

Proof. $\sup(|f|,|g|) = \sup(f,-f,g,-g) = \sup(\sup(f,-g),\ \sup(g,-f))$
$= \sup(\frac{1}{2}\,(f-g) + \frac{1}{2}|f+g|,\ \frac{1}{2}(g-f)+ \frac{1}{2}\,|f+g|) = \frac{1}{2}(|f+g|+|f-g|)$

by virtue of (v). The first formula is proved.

To prove the second formula, we set $f = p+q$ and $g = p-q$, so $f+g = 2p$ and $f-g = 2q$.

Let us apply formula (iii); we obtain

$$\begin{aligned}\inf(|f|,|g|) &= |f|+|g| - \sup(|f|,|g|)\\ &= |f|+|g| - \tfrac{1}{2}(|f+g|+|f-g|)\\ &= |p+q|+|p-q| - |p| - |q|\\ &= 2\sup(|p|,|q|) - |p| - |q|\end{aligned}$$

by virtue of the first formula.

The formula (v) yields

$$
\begin{aligned}
& 2\sup(|p|,|q|)-|p|-|q| \\
= & \{|p|+|q|+||p|-|q||\} -|p|-|q| \\
= & ||p|-|q|| \\
= & \frac{1}{2}\big|\,|f+g| - |f-g|\,\big|
\end{aligned}
$$

q.e.d.

Standard decomposition of f.

$f = f^+ - f^-$ with $\inf(f^+,f^-) = o$: standard decomposition of f.

The standard decomposition is the "minimal" decomposition.

Then (i) If $f = u-v$; $u,v \in L^+$, then $f^+ \leq u$
$f^- \leq v$

(ii) If $f = u-v$; $u,v \in L^+$ and $\inf(u,v) = o$,
then $u = f^+$ and $v = f^-$.

Proof.(i) $\left.\begin{array}{l} f \leq u \\ o \leq u \end{array}\right\}$ $\sup(f,o) \leq u$, (u is an upper bound, greater or equal the least upper bound),

i. e. $f^+ \leq u$.

Furthermore, $f^- = f^+ - f \leq u - f = v$.

So (i) is proved.

(ii) By virtue of the formula for infimum

$o = \inf(u,v) = u-(u-v)^+ = u-f^+ \Rightarrow u=f^+$,

so, since $f = u-v$ and $f = f^+-f^-$, we obtain

$$v = f^-$$

Triangle inequality

$$\big||f| - |g|\big| \leqslant |f+g| \leqslant |f| + |g|$$

Proof. We prove first the second inequality.

$$\left.\begin{array}{r} f+g \leqslant f^+ + g^+ \\ o \leqslant f^+ + g^+ \end{array}\right\} \quad (f+g)^+ \leqslant f^+ + g^+$$

$$(f+g)^- = (-f-g)^+ \leqslant (-f)^+ + (-g)^+ = f^- + g^-$$

Let us add the left hand sides and the right hand sides:

$$|f+g| \leqslant |f| + |g|$$

To prove the first inequality, we substitute f-g for f in the second inequality

$$|f| \leqslant |f-g| + |g|$$

or $\quad |f| - |g| \leqslant |f-g|$
and $\quad |g| - |f| \leqslant |f-g|$
since $\quad |f| = |-f| = \sup(f,-f)$ by definition.

Let us now introduce the supremum of the left hand sides:

$$\big||f| - |g|\big| \leqslant |f-g|$$

Now we change g in -g, this gives a difference on the right, but not on the left; and we find the desired result.

Theorem . If $\{f_\tau^*: \tau \in \{\tau\}\}$ is a subset of L, such that $f = \sup f_\tau$ exists, then

$$\sup(f,g) = \sup_\tau \left[\sup(f_\tau , g)\right]$$

---

* For both countable and uncountable sets, we use the notation f_τ , where τ runs through an appropriate index set.

<u>Proof.</u> $f \geqslant f_\tau$ for all τ, $\Rightarrow \sup(f,g) \geqslant \sup(f_\tau, g)$ for all τ, so $\sup(f,g)$ is an upper bound of the set of all elements $\sup(f_\tau, g)$.
Let h be also an upper bound of this set, so $h \geqslant \sup(f_\tau, g)$ for all τ,

$$\left.\begin{array}{c} \Rightarrow h \geqslant f_\tau \text{ for all } \tau \Rightarrow h \geqslant f \\ \text{(f being the least upper bound} \\ \text{of the } f_\tau \text{)} \\ \text{and, since } \quad h \geqslant g \end{array}\right\} \Longrightarrow h \geqslant \sup(f,g),$$

so $\sup(f,g)$ is the least upper bound of the $\sup(f_\tau, g)$.

Still assuming that $f = \sup f_\tau$, the equality

$$\inf(f,g) = \sup_\tau (\inf(f_\tau, g))$$

is also true. It is the distributive law. This law is of the same kind as the distribution law of the set of real numbers.

<u>Proof.</u> $f \geqslant f_\tau$ for all $\tau \Rightarrow \inf(f,g) \geqslant \inf(f_\tau, g)$ for all τ. Hence $\inf(f,g)$ is an upper bound of the set of all $\inf(f_\tau, g)$. We still have to prove that $\inf(f,g)$ is the least upper bound of the $\inf(f_\tau, g)$.
Let h be another upper bound of this set, so

$$h \geqslant \inf(f_\tau, g) \text{ for all } \tau.$$

Since the infimum of two elements equals the sum of these two elements minus their supremum,

$$h \geqslant f_\tau + g - \sup(f_\tau, g) \Rightarrow$$
$$h - g + \sup(f_\tau, g) \geqslant f_\tau \text{ for all } \tau.$$

We make the left hand side still larger, without changing the right hand side,

$$h - g + \sup(f,g) \geqslant f_\tau \text{ for all } \tau$$

At the left hand side we have now a fixed element, hence an upper bound of the f_τ; f being the least upper bound of the f_τ, we obtain

$$h - g + \sup(f,g) \geqslant f \Rightarrow$$
$$h \geqslant f+g - \sup(f,g) = \inf(f,g).$$

So we have proved that $\inf(f,g)$ is the least upper bound.

3rd Lecture

We have proved that if $f = \sup\limits_{\tau} f_{\tau}$, $\sup\,(f,g) = \sup\limits_{\tau}\,(\sup(f_{\tau},g))$.

In particular, if $g = 0$, then $f^{+} = \sup\limits_{\tau} f^{+}_{\tau}$.

From the distribution law we deduce for $g = 0$:

$$\inf\,(f,0) = \sup_{\tau}\,(\inf\,(f_{\tau},\ 0))$$

$$-f^{-} = \sup\,(-f^{-}_{\tau})$$

$$f^{-} = \inf_{\tau} f^{-}_{\tau}\ ;$$

hence

$$f = \sup f_{\tau} \begin{array}{l} \nearrow\!\!\!\!\nearrow\ f^{+} = \sup f^{+}_{\tau} \\ \\ \searrow\!\!\!\!\searrow\ f^{-} = \inf f^{-}_{\tau} \end{array}$$

Similarly, if $f = \inf f_{\tau}$, then $\sup(f,g) = \inf\limits_{\tau}\,(\sup(f_{\tau},\ g))$.

Remark : $\sup(f,g)$ is written sometimes as $f \vee g$,

$\inf(f,g)$ " " " " $f \wedge g$.

("cup" and "cap")

Particular case : case of two elements.

If $f = f_1 \vee f_2$, then $f \wedge g = (f_1 \wedge g) \vee (f_2 \wedge g)$, or

$$(f_1 \vee f_2) \wedge g = (f_1 \wedge g) \vee (f_2 \wedge g);$$

we have also

$$(f_1 \wedge f_2) \vee g = (f_1 \vee g) \wedge (f_2 \vee g).$$

The first formula remains correct if we interchange the symbols $\vee$ and $\wedge$.

That is the distributive law.

On the other hand, if f_1, f_2 and g are real numbers, we have the law

$$(f_1 + f_2) \cdot g = (f_1 \cdot g) + (f_2 \cdot g),$$

in this case the symbols + and · cannot be interchanged.

Theorem. If $f,g,h \in L$, then

$$\underbrace{|\sup(f,h) - \sup(g,h)|}_{\mathrm{I}} + \underbrace{|\inf(f,h) - \inf(g,h)|}_{\mathrm{II}} = |f-g|$$

Corollaries

$$|\sup(f,h) - \sup(g,h)| \leq |f-g|$$

$$|\inf(f,h) - \inf(g,h)| \leq |f-g|$$

In particular, if h is the null element, we have

$$|f^+ - g^+| \leq |f - g|$$

$$|f^- - g^-| \leq |f - g|$$

We first illustrate the theorem by a graph.

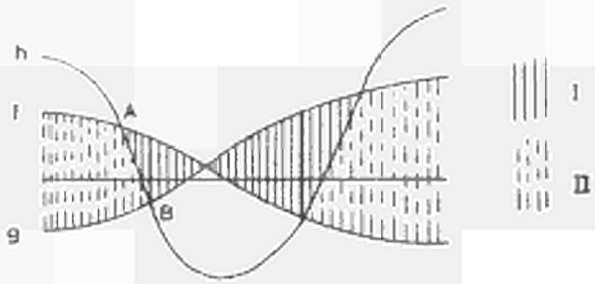


fig. III, 1

Let us consider different regions, first the region at the left of the intersection point A of the curves h and f; the supremum of f and h and the supremum of g and h are h; the expression I represents nothing in the figure.

Now we pass A and go up to B, intersection point of the curves h and g; the supremum of f and h is f, the supremum of g and h is h; we represent the expression I in the figure by a continuous line.

We go on thus up to the end of the chosen interval.

We proceed similarly for the expression II and use in that case dotted lines.

In this figure, therefore, I + II = $|f-g|$ for all possible choices of the point x on the horizontal axis. Note : this is no proof, only an illustration for the case that L is the space of continuous functions on an interval.

Proof. We first prove the equality, from which the inequalities can easily be derived.

Using the relation

$$|p-q| = \sup(p,q) - \inf(p,q)$$

we obtain

$$\mathrm{I}+\mathrm{II} = f \vee h \vee g \vee h - (f \vee h) \wedge (g \vee h)^{*} + (f \wedge h) \vee (g \wedge h) - f \wedge h \wedge g \wedge h$$

$$= f \vee g \vee h - (f \wedge g) \vee h + (f \vee g) \wedge h - f \wedge g \wedge h$$

By arranging the terms of the last expression in pairs, we find

$$\mathrm{I}+\mathrm{II} = \left\{(f \vee g) \vee h + (f \vee g) \wedge h\right\} - \left\{(f \wedge g) \vee h + (f \wedge g) \wedge h\right\}$$

$$= \left\{(f \vee g) + h\right\} - \left\{(f \wedge g) + h\right\}$$

since the sum of the supremum and of the infimum of two terms equals the sum of the two terms. We arrive at the result

---

* For this demonstration we use cup and cap notation.

$$I+II = f \vee g - f \wedge g$$

$$= |f-g|$$

Remark. Sometimes the inequalities are called:

G. BIRKHOFF's inequalities.

He proved first the inequalities. But it is more striking to begin with the equality.

Definition. f and g are called disjoint, when $\inf(|f|, |g|) = 0$

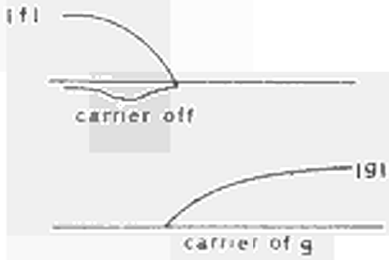


fig. III, 2

Notation : $f \perp g$

(the same as for orthogonal)

The carriers of f and g have no point in common (English: carrier , French support).

Theorem

(i) If $f \perp g$, and $|h| \leqslant |f|$, then $h \perp g$.

(ii) If $f \perp g$, then $af \perp g$ for every real a.

(iii) If $f_1, f_2 \perp g$, then $f_1 + f_2 \perp g$.

(In other words, the set of all elements, which are orthogonal to a fixed element, is a linear subspace)

(iv) $f \perp g \Leftrightarrow f^+ \perp g$ and $f^- \perp g$.

(v) If $f = \sup f_\tau$ and all $f_\tau \perp g$, then $f \perp g$.

<u>Proof</u>. (i) The proof of the first proposition is trivial:

$$0 \leqslant \inf(|h|, |g|) \leqslant \inf(|f|, |g|) = 0 .$$

(ii) $f \perp g \Rightarrow \inf(|f|, |g|) = 0 \Rightarrow \inf(\alpha|f|, \alpha|g|) = 0$

for $\alpha = |a|+1$, i.e. $\inf(|\alpha f|, |\alpha g|) = 0$,

$$\text{i.e.} \quad |\alpha f| \perp |\alpha g|$$

Now $\quad |af| \leqslant |\alpha f|$, so by (i) $\quad af \perp |\alpha g|$,

finally $\quad |g| \leqslant |\alpha g|$, so by (i) $\quad af \perp g$,

q.e.d.

(iii) *) We prove first that $(|f_1| \vee |f_2|) \perp g$, or that $(|f_1| \vee |f_2|) \wedge |g|$ is the null element.

By virtue of the distribution law,

$$(|f_1| \vee |f_2|) \wedge g = (|f_1| \wedge |g|) \vee (|f_2| \wedge |g|) = 0 \vee 0 = 0 \ .$$

Since each of two elements is smaller than or equal to its supremum,

$$|f_1| + |f_2| \leqslant 2(|f_1| \vee |f_2|),$$

so

$$(|f_1| + |f_2|) \perp g;$$

the inequality

$$|f_1 + f_2| \leqslant |f_1| + |f_2|$$

shows that $\quad |f_1 + f_2| \perp g$, i.e. $\quad (f_1 + f_2) \perp g$.

(iv) If $f \perp g$, then $|f| \perp g$;

$0 \leqslant f^+ \leqslant |f|$; so $f^+ \perp g$;

$0 \leqslant f^- \leqslant |f|$; so $f^- \perp g$.

----

*) For $f_1 \perp g$ and $f_2 \perp g$ can be said: f_1 and f_2 are disjoint to g.

Conversely, if $f^+, f^- \perp g$, then

$$f = (f^+ - f^-) \perp g,$$ q.e.d.

(v) We have $f = \sup f_\tau$, and all $f_\tau \perp g$.

We have proved earlier that

$$f^+ = \sup f_\tau^+$$

$$f^- = \inf f_\tau^-$$

and $$f_\tau^+ \perp g$$

was proved sub (iv).

So

$$\inf(f_\tau^+, |g|) = 0$$

Or, by virtue of the distribution law

$$\inf(f^+, |g|) = \sup_\tau(\inf(f_\tau^+, |g|) = \sup_\tau 0 = 0;$$

since the supremum of a set of elements, which are the null element, is the null element. Thus

$$f^+ \perp g$$

Then we have $f_\tau^- \perp g$, so $\inf(f_\tau^-, |g|) = 0$.

Or, $0 \leq f^- \leq f_\tau^-$ for all τ,

thus $0 \leq \inf(f^-, |g|) \leq \inf(f_\tau^-, |g|) = 0$;

and so $f^- \perp g$.

Finally $$\left.\begin{array}{l} f^+ \perp g \\ f^- \perp g \end{array}\right\} \Rightarrow f \perp g,$$

which completes the proof.

Theorem

$$f \perp g \Rightarrow |f+g| = |f-g| = |f|+|g| = \big||f|-|g|\big| = \sup(|f|,\ |g|)$$

Proof

We use the formula

$$\inf(p,q) = \frac{1}{2}\left\{p+q - |p-q|\right\}$$

thus, $0 = 2\inf(|f|,\ |g|) = |f|+|g| - \big||f|-|g|\big| \Rightarrow |f|+|g| = \big||f|-|g|\big|$

But we have proved

$$\big||f|-|g|\big| \leqslant \begin{matrix}|f+g| \\ \\ |f-g|\end{matrix} \leqslant |f|+|g|$$

Since the outer expressions are equal, they are also equal to the expressions in the middle.

Finally, we observe that

$$\sup(|f|,\ |g|) = |f|+|g| - \inf(|f|,\ |g|)$$

The infimum being zero, the last inequality is proved.

4^{th} Lecture

Theorem
$$\begin{cases} \text{If } |f+g| = |f-g|, \text{ then } f \perp g. \\ \text{If } |f+g| = \sup(|f|,|g|), \text{ then } f \perp g. \\ \text{If } |f-g| = \sup(|f|,|g|), \text{ then } f \perp g. \end{cases}$$

Proof. We have proved

$2\inf(|f|,|g|) = \big||f+g| - |f-g|\big|$, so if $|f+g| = |f-g|$, then

the hypothesis implies $\inf(|f|,|g|) = o$;

and if $|f+g| = \sup(|f|,|g|)$, it follows from

$$2\sup(|f|,|g|) = |f+g| + |f-g| \quad \text{and}$$
$$\sup(|f|,|g|) = |f+g|$$

by subtraction that

$$\sup(|f|,|g|) = |f-g| \Longrightarrow |f+g| = |f-g| \Longrightarrow f \perp g.$$

For the third formula we replace g by $-g$,

q.e.d.

Notation

Given the sequence $f_1 \leqslant f_2 \leqslant f_3 \ldots$, we write $f_n \uparrow$, i.e. we say that the sequence f_n is monotonically increasing.

If, in addition, $f = \sup f_n$ exists, we write $f_n \uparrow f$.

Similarly if $f_1 \geqslant f_2 \geqslant \ldots$, we write $f_n \downarrow$.

If, in addition, $f = \inf f_n$ exists, we write $f_n \downarrow f$.

Theorem

(i) If $f_n \uparrow f$ and $g_n \uparrow g$, then $f_n+g_n \uparrow f+g$.

(ii) If $f_n \uparrow f$, $g_n \uparrow$ and if $f_n+g_n \uparrow f+g$, then $g_n \uparrow g$.

(iii) If $f_n \uparrow f$, $g_n \uparrow g$, then $\sup_{\inf}(f_n, g_n) \uparrow \sup_{\inf}(f, g)$.

Proof. (i) Let $n \geqslant m$: $f_m+g_m \leqslant f_m+g_n \leqslant f_n+g_n$

and $f_n+g_n \leqslant f+g$, this shows that

$f+g$ is also an upper bound of the set $f_m+g_n (m,n=1,2,\ldots)$

Let h be another upper bound of the set f_n+g_n. Hence h is also an upper bound of f_m+g_n, that means that for all m,n such that $n \geqslant m$, $f_m+g_n \leqslant h$.

Keep m fixed, then $g_n \leqslant h-f_m$, but $g_n \uparrow$,

hence $g = \sup g_n \leqslant h-f_m \Rightarrow f_m \leqslant h-g \Rightarrow$

$f \leqslant h-g \Rightarrow f+g \leqslant h \Rightarrow f+g$ is the least upper bound of f_n+g_n,

q.e.d.

(ii) and (iii) are proved in the same manner.

Definition of Convergence. (Side Remarks)

$$\text{If} \quad \left. \begin{array}{l} g_n \leqslant f_n \leqslant h_n \\ g_n \uparrow f, \; h_n \downarrow f \end{array} \right\} \quad \text{then we say that } f_n \text{ "order" converges to } f.$$

Definition of Series: If $u_n \in L^+$ for $n=1,2,\ldots$, if

$S_n = u_1+u_2+ \ldots + u_n$, and if $S_n \uparrow S$, then we write

$$\sum_{n=1}^{\infty} u_n = S.$$

Theorem (Dominated decomposition theorem)

(i) If $u, v, z \in L^+$, $u+v = z$ and $o \leqslant z_n \uparrow z$, then there exist sequences $o \leqslant u_n \uparrow u$ and $o \leqslant v_n \uparrow v$ such that $u_n + v_n = z_n$.

(ii) If $u, v, z \in L^+$, $u+v = z$ and $z = \sum_{n=1}^{\infty} z_n'$ where all $z_n' \in L^+$, then there exist series $\sum u_n' = u$ and $\sum v_n' = v$ (all $u_n', v_n' \in L^+$) such that $u_n' + v_n' = z_n'$.

(ii) is the same as (i), so we prove only (i).

(iii) If $o \leqslant u \leqslant z_1' + \dots + z_p'$, all $z_i' \in L^+$ (p finite), then there exist $u_1', \dots, u_p' \in L^+$ such that
$u_1' + \dots + u_p' = u$ and $o \leqslant u_i' \leqslant z_i'$ $(i=1, \dots, p)$.

Therefore the name: dominated decomposition theorem.

It is a particular case of (ii), so we prove only (i).

Proof of (i)

We make a guess.

The guess is: Let us define $u_n = \inf(u, z_n)$.

Then $o \leqslant u_n \uparrow \inf(u,z) = u$, become

$$f_n \uparrow f, \quad g_n \uparrow g \Rightarrow \inf(f_n, g_n) \uparrow \inf(f,g).$$

We define $v_n = z_n - u_n$. Then

$$o \leqslant v_n \quad \text{and} \quad u_n + v_n = z_n.$$

We still have to prove that $v_n \uparrow$ and that $v_n \uparrow v$.

We have

$$o \leqslant u_{n+1} - u_n = \inf(u, z_{n+1}) - \inf(u, z_n) =$$
$$|\inf(u, z_{n+1}) - \inf(u, z_n)| \leqslant |z_{n+1} - z_n| = z_{n+1} - z_n,$$

(BIRKHOFF's inequality)

$$\Rightarrow z_n - u_n \leqslant z_{n+1} - u_{n+1}, \text{ i.e. } v_n \leqslant v_{n+1}.$$

We have already proved that $o \leqslant u_n \uparrow u$, $o \leqslant v_n \uparrow$,
$o \leqslant u_n + v_n \uparrow u + v \Rightarrow v_n \uparrow v$ by (ii) of convergence statements.

Generalization of the notion of monotone convergence

Definition

The set $\{ f_\tau \,;\, \tau \in \{\tau\} \}$ is called directed upwards, if to any pair τ_1, τ_2, there exists τ_3 such that

$$f_{\tau_3} \geqslant \begin{matrix} f_{\tau_1} \\ f_{\tau_2} \end{matrix}$$

We define similarly: directed downwards.

Notation: $f_\tau \uparrow$ or $f_\tau \downarrow$ (directed upwards, downwards)

If $f_\tau \uparrow$ and $f = \sup f_\tau$ exists, then $f_\tau \uparrow f$.

Theorem. If $u, v, z \in L^+$, $u+v = z$ and $o \leqslant z_\tau \uparrow z$, then there exist an upwards directed set $o \leqslant u_\tau \uparrow u$ and $o \leqslant v_\tau \uparrow v$ such that $u_\tau + v_\tau = z_\tau$ for each τ.

The proof is almost the same as for preceding theorems.

Theorem. Let there be given an arbitrary subset of L, say $\{ f_\sigma ; \sigma \in \{\sigma\}\}$. Then there exists an upwards directed set $\{g_\tau\} \supset \{f_\sigma\}$ having the same upper bounds as $\{f_\sigma\}$.

Proof. Let $\tau_{\sigma_1, \dots, \sigma_n} = (\sigma_1, \dots, \sigma_n)$ and then define (n finite but variable) :

$g_{\tau_{\sigma_1 \dots \sigma_n}} = \sup(f_{\sigma_1}, \dots, f_{\sigma_n})$ and take the set of all these g's.

Hence take the set of all suprema of finite subsets of $\{f_\sigma\}$.
Of course $\{g_\tau\} \supset \{f_\sigma\}$.

$\langle g_\tau \rangle$ is directed upwards. In fact

$$\sup(g_{\tau_{\sigma'_1,\ldots,\sigma'_m}}, g_{\tau_{\sigma_1,\ldots,\sigma_n}}) \in \{g_\tau\}$$

Any upper bound of g_τ is of course an upper bound of f_σ.

Conversely if $k \geqslant f_\sigma$, for all σ, then $k \geqslant g_\tau$ for all τ.

q.e.d.

5^{th} Lecture

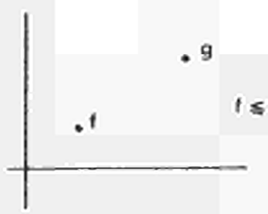
Examples

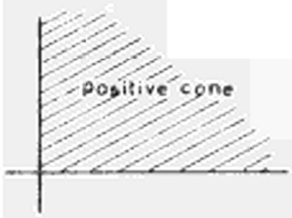
1) The n-dimensional real space R^n, with elements $f = (f_1, \dots f_n)$, is a real linear vector space;

$$f \leqslant g \quad \text{iff} \quad f_i \leqslant g_i \quad \text{for } i = 1, \dots, n.$$

(f is smaller than g if every coordinate of f is smaller than the corresponding coordinate of g).

Graphic representations for R^2:





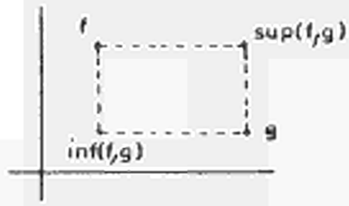


fig. V, 1 fig. V, 2 fig. V, 3

2) R^n, with a different ordering: lexicographical ordering : for R^2 : $f \leqslant g$ iff either $f_1 \leqslant g_1$ or $f_1 = g_1$, $f_2 \leqslant g_2$.

fig. V, 4

This is a linear ordering, i.e. $(\forall f)\ (\forall g)$: $f \leqslant g$ or $g \leqslant f$.

In this case two elements are always comparable, whereas in the former case it can occur that two elements are not comparable.

3) An arbitrary non-empty point set X and the set L of all real (finite valued) functions on X; the algebraic operations are as usual, i.e. L is a real vector space.

$f \leqslant g$ iff $f(x) \leqslant g(x)$ for all $x \in X$, L is a RIESZ space. All pairs of functions have a least upper bound; for $f(x)$, $g(x)$,

$$(\sup (f,g))\ (x) = \max\ (f(x),\ g(x))$$

and we have an analogous relation for the infimum.

Example 1) is a particular case of 3), $X = \{1, 2, \ldots, n\}$

$$\begin{array}{cccccc} f_1 & f_2 & f_3 & & & f_n \\ \bullet & \bullet & \bullet & --- & & \circ \\ 1 & 2 & 3 & & & n \end{array}$$

Every point in R^n is a function on this point set; $f_1 = f(1)$, etc.

4) A topological space X, and L the set of all real continuous functions on X; the ordering is as in 3). Then L is a RIESZ space. The subspace of bounded continuous functions is also a RIESZ space.

5) X: non empty point set; μ : countably additive* non negative measure in X; M:set of all real, finite-valued, μ-measurable functions on X.

In M we identify functions which are μ-almost equal; (Officially we have $L = M/N$, N: set of all μ-null functions); we call it L now; then L is a real linear vector space;

$$f \leqslant g \text{ in } L \text{ iff } f(x) \leqslant g(x) \quad \mu\text{-almost everywhere.}$$

L is then a RIESZ space. $(\sup (f,g))(x) = \max (f(x), g(x))$ a.e.$[\mu]$)

6) Many subsets of the above L are also RIESZ spaces, e.g. L_p $(1 \leqslant p \leqslant \infty)$; in particular ℓ_p (the point set X is countable).

$$X = \{1, 2, \ldots \qquad \}$$
$$\mu(n) = 1 \quad \text{for every } n.$$

7) A non-empty point set X; Γ field (algebra) of subsets of X. If Γ is a collection of subsets of X, Γ is called a field iff

$$\begin{cases} \text{(i)} & X \in \Gamma \\ \text{(ii)} & A, B \in \Gamma \Rightarrow A \cup B \in \Gamma \\ & \qquad\qquad\quad A - B \in \Gamma \end{cases}$$

Let μ be a finitely additive signed measure defined on Γ ; i.e. to each $A \in \Gamma$ is assigned a finite real number μ (A), such that

$$\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$$

----

*one says also: σ-additive or completely additive .

if A_1, A_2 are disjoint. (finitely additive: concerns the above condition; signed measure: means μ is not necessarily positive). μ is sometimes called a <u>charge</u>.

Assume also that $\sup(|\mu(A)| : A \in \Gamma)$ is finite.
Let L be the set of all such charges.

Addition: $\begin{cases} (\mu_1+\mu_2)(A) = \mu_1(A)+\mu_2(A) \\ (a\,\mu)(A) = a\,\mu(A) \end{cases}$

L becomes so a linear vector space, since the charges behave like a vector:

$$\mu_1 \leqslant \mu_2 \text{ iff } \mu_1(A) \leqslant \mu_2(A), \text{ all } A \in \Gamma.$$

Hence the positive cone L^+ is the subset of all non negative charges.

We show now that L is a RIESZ space.
Let the charges μ_1, μ_2 be given.

<u>Remark</u>. We can't write : $\mu_3(A) = \max(\mu_1(A), \mu_2(A))$, μ_3 isn't additive. We have to find a supremum that is additive.

For any set $A \in \Gamma$, we write

$$\nu(A) = \sup(\mu_1(B)+\mu_2(A-B) : B \subset A,\ B \in \Gamma)$$

First of all $\nu(A) < \infty$, for

$$|\mu_1(B)+\mu_2(A-B)| \leqslant |\mu_1(B)| + |\mu_2(A-B)|$$

$$\leqslant \sup(|\mu_1(B)| : B \in \Gamma) + \sup(|\mu_2(C)| : C \in \Gamma) < \infty$$

Let A_1, A_2 be disjoint; we shall prove that

$$\nu(A_1 \cup A_2) = \nu(A_1) + \nu(A_2)$$

(From now on B is always in the collection Γ).

If $B \subset A_1 \cup A_2$, then $B = B' \cup B''$, where $B' = B \cap A_1$ and $B'' = B \cap A_2$. B' and B'' are disjoint, so (calling $A = A_1 \cup A_2$)

$$\mu_1(B) + \mu_2(A-B) =$$
$$\left\{\mu_1(B') + \mu_2(A_1-B')\right\} + \left\{\mu_1(B'') + \mu_2(A_2-B'')\right\}$$
$$\leq \qquad \nu(A_1) \qquad + \qquad \nu(A_2)$$

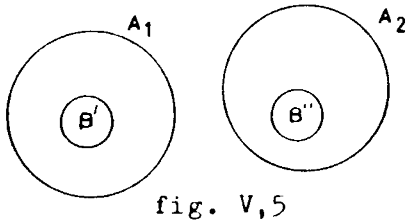


fig. V,5

Hence $(\forall B\ ,\ B \in \Gamma,\ B \subset A_1 \cup A_2)\ \mu_1(B) + \mu_2(A-B) \leq \nu(A_1) + \nu(A_2)$

$$\sup\left(\mu_1(B) + \mu_2(A-B)\ :\ B \in A\ ,\ B \in \Gamma\right) \leq \nu(A_1) + \nu(A_2)$$

i.e. $\qquad \nu(A) \leq \nu(A_1) + \nu(A_2)$

On the other hand take $\varepsilon > 0$, and let

$B_1 \subset A_1$ such that $\mu_1(B_1) + \mu_2(A_1-B_1) > \nu(A_1) - \varepsilon$

$B_2 \subset A_2$ " " $\mu_1(B_2) + \mu_2(A_2-B_2) > \nu(A_2) - \varepsilon$

Call $B_1 \cup B_2 = B$,

by addition we have $\mu_1(B) + \mu_2(A-B) > \nu(A_1) + \nu(A_2) - 2\varepsilon$

$$\sup\left(\mu_1(B) + \mu_2(A-B)\right) > \nu(A_1) + \nu(A_2) - 2\varepsilon$$
$$\nu(A) > \nu(A_1) + \nu(A_2) - 2\varepsilon$$
$$\nu(A) \geq \nu(A_1) + \nu(A_2)$$

Hence ν is additive.

We now prove that $\sup(|\nu(A)|\ :\ A \in \Gamma)$ is finite.
We already proved that for any $A \in \Gamma$ and any $B \subset A$, $B \in \Gamma$:

$$|\mu_1(B) + \mu_2(A-B)| \leq \sup(|\mu_1(B)|\ :\ B \in \Gamma) + \sup(|\mu_2(C)|\ :\ C \in \Gamma)$$
$$= C_1 + C_2 < \infty,$$

C_1 and C_2 being fixed constants. Hence

$$\sup(|\mu_1(B) + \mu_2(A-B)|\ ;\ B \subset A\ ;\ A,\ B \in \Gamma) \leq C_1 + C_2$$
$$(\forall A \in \Gamma)\ |\nu(A)| \leq C_1 + C_2;$$

this for every A, so

$$\sup(|\nu(A)|\ :\ A \in \Gamma) \text{ is finite;}$$

hence ν is a charge.

But $\nu(A) = \sup(\mu_1(B) + \mu_2(A-B))$

For the particular choice $B = \emptyset$: $A-B = A$,

and it results : $\nu(A) \geqslant \mu_2(A)$.

If we take $B = A$, then $A-B = \emptyset$,

and $\nu(A) \geqslant \mu_1(A)$.

So ν is an upper bound of μ_1 and μ_2.

To prove that ν is the least upper bound:

Let ν' be another upper bound of μ_1 and μ_2.

$\nu'(A) = \nu'(B) + \nu'(A-B)$ for any $B \subset A$,

so $\nu'(A) \geqslant \mu_1(B) + \mu_2(A-B)$

$\Rightarrow \nu'(A) \geqslant \sup(\mu_1(B) + \mu_2(A-B); B \subset A) = \nu(A)$,

thus any upper bound $\nu'(A)$ is greater or equal $\nu(A)$ and ν is the least upper bound. Hence L is a RIESZ space.

One can easily prove that

$$\lambda(A) = \inf(\mu_1(B) + \mu_2(A-B) : B \subset A,\ B \in \Gamma)$$

is a charge, such that λ is the greatest lower bound of μ_1 and μ_2, i.e., $\lambda = \inf(\mu_1, \mu_2)$.

6^{th} Lecture

Remark.

We have required that $\sup(|\mu(A)| : A \in \Gamma)$ would be finite. If Γ is a σ-algebra and μ σ-additive, this condition is automatically verified.

8°) A HILBERT space H (on the complex numbers), the elements being x, y, z.
The bounded linear transformation A (of H into H) is HERMITIAN (self-adjoint) if

$$(Ax, y) = (x, Ay) \quad \text{for all} \quad x,y.$$

$$(Ax, x) = (x, Ax) = (\overline{Ax, x}) \Longrightarrow (Ax, x) \quad \text{real for all } x.$$

The set of all HERMITIAN transformations * is a real linear vector space.
(If A, A_1, A_2 are HERMITIAN,

$$\left.\begin{matrix} A_1 + A_2 \\ 5A \end{matrix}\right\} \text{ are HERMITIAN, but iA is not HERMITIAN).}$$

$A \leqslant B$ iff $(Ax, x) \leqslant (Bx, x)$ for all $x \in H$.

Positive cone: all $A \geqslant \Theta$, Θ being the null transformation,
so $(Ax, x) \geqslant 0$ for all x.

Given A, B, HERMITIAN $\to$ C, C being an upper bound of A and B. This set is not a RIESZ space (KADISON; 1951), but many subspaces are RIESZ spaces.

Example of a subspace containing an element A and being a RIESZ space:
2^{nd} commutant of A, C"(A).

----

* A bounded transformation is also called operator.

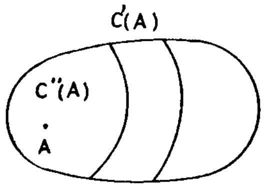


fig. VI, 1

Commutant of A: set of all B (HERMITIAN) such that BA = AB : commutant C'(A).

2^{nd} commutant of A: set of all C (HERM.) " " CB = BC for all B ϵ C'(A): it is called C"(A).

We have : A ϵ C"(A) $\subset$ C'(A).

The proof of this assertion is not trivial. We don't give it now.

Subspaces

Let L be a RIESZ space (linear lattice).

Definitions

(i) RIESZ subspace K (linear sublattice; sous-espace propre)
- 1^o K : linear subspace of L
- 2^o If f,g ϵ K, then sup (f,g) ϵ K, (and hence inf (f,g) ϵ K).

(ii) Ideal A (solid subspace, sous-espace épais, U.R.S.S. : semi-normal subspace)
- 1^o A : linear subspace
- 2^o If f ϵ A and $|g| \leq |f|$, then g ϵ A.*

(Ex. : $g = |f|$)

(iii) Band A (normal subspace; bande; U.R.S.S. : component)
- 1^o A : ideal
- 2^o If any subset of A has a sup in L, then this sup is already in A.

----

* Any subset, which verifies (ii) (2), is a solid subset.

<u>Theorem</u> Band $\Longrightarrow$ Ideal $\Longrightarrow$ RIESZ subspace

The first proposition can be derived from the definition (iii).

<u>Proof</u> of the second proposition.

Let A be an ideal; from the definition of the ideal it follows:

If $f \in A$, then $|f| \in A$.

$\Longrightarrow$ If $f,g \in A$, then $|f-g| \in A$.

$\Longrightarrow$ If $f,g \in A$, then $\frac{1}{2}\left\{f+g + |f-g|\right\} \in A$,

i.e. $\sup(f,g) \in A$.

Thus A is a RIESZ space.

<u>Examples</u>

1° In $C[0,1]$: the set of all polynomials is a linear subspace, but not a RIESZ subspace.

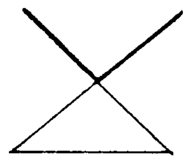


fig. VI, 2

Graph for two linear functions: the supremum is not a polynomial.

2° The set of all real constants is a RIESZ subspace, but not an ideal.

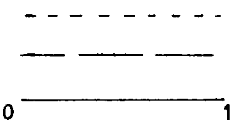


fig. VI, 3

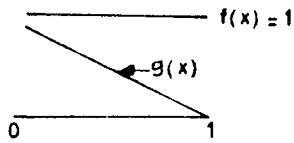


fig. VI, 3 bis

fig. VI, 3 bis : The function $g(x)$ verifies $|g| \leq |f|$, but is not in the subspace.

3° The set of all $f \in C[0,1]$ satisfying $f(0) = 0$ is an ideal.

The set of these functions is a linear subspace;

$$f(0) = 0, \quad |g(x)| \leq |f(x)| \Longrightarrow g(0) = 0 \ ;$$

hence it is an ideal.

It is not a band. Indeed, let us consider the sequence of functions f_n (see fig. VI, 4), having always steeper slopes, when n increases.

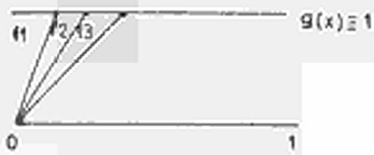


fig. VI, 4.

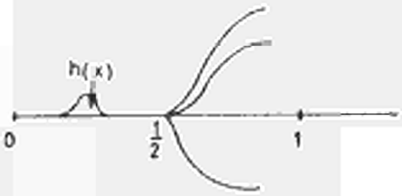
$$f_1 \leqslant f_2 \leqslant f_3 \leqslant \qquad \uparrow g$$

In the big space, g is the least upper bound of the sequence. In this space the function null at the origin and one everywhere else doesn't exist.

Since $g \notin A$, the considered set is not a band.

4^o Example of a band.

The set of all f such that $f(x) = 0$ for $0 \leqslant x \leqslant \frac{1}{2}$ is a band.



(h(x) is no supremum, a supremum is identically zero for $0 \leqslant x \leqslant \frac{1}{2}$)

fig. VI, 5

5^o We consider now other spaces.

a) $X = \left\{1,2,3,\dots\right\}$, (s) RIESZ space of all real functions on X.

$$f = (f(1),\ f(2),\ f(3),\dots)$$

$$f = (f_1,\ f_2,\ f_3,\dots)$$

(s) set of all real sequences.

Properties of (s) :

$$f \leqslant g \Longrightarrow f_k \leqslant g_k \quad \text{for} \quad k = 1,2,\dots$$

Let $g = (g_1,\ g_2,\dots)$,

then $\sup(f,g) = (\max(f_1,g_1),\ \max(f_2,\ g_2),\dots)$

$$f^+ = (f_1^+,\ f_2^+,\dots)\ ;\ |f| = (|f_1|,\ |f_2|,\dots)$$

b) ℓ_∞ : subspace of all <u>bounded</u> real sequences,

ℓ_∞ : ideal in (s)

$$\sup_{n=1,2,..} |f_n| \text{ is finite, } |g| \leq |f|$$

then $\sup_{n=1,2,..} |g_n|$ is also finite;

ℓ_∞ by itself is again a RIESZ space.

c) (c_o) : subspace of all sequences $f = (f_1, f_2, \ldots)$ such that $f_n \to 0$.

(c_o) : subspace of ℓ_∞ : since $f_n \to 0$, the sequence f must be bounded.

(c_o) : ideal in ℓ_∞ : if $|g| \leq |f|$, g is also a null-sequence .

d) ℓ_1 : subspace of all sequences $f = (f_1, f_2, \ldots)$ such that $\sum_{n=1}^{\infty} |f_n|$ converges.

ℓ_1 : ideal in (c_o) : if $|g| \leq |f|$, the condition : $\sum_{n=1}^{\infty} |g_n|$ converges is satisfied.

Ideal

$$(s) \supset \ell_\infty \supset (c_o) \supset \ell_1$$

None of these ideals is a band.

From definitions it results:

	RIESZ subspaces		RIESZ subspace
An arbitrary intersection of	ideals	is again a	ideal
	bands		band.

We will prove this for the intersection of two ideals. Let A_1, A_2 be ideals.

Then $A_1 \cap A_2$ is a linear subspace, since the intersection of two subspaces is a linear subspace.

We prove : If $f \in A_1 \cap A_2$ and if $|g| \leq |f|$, then $g \in A_1 \cap A_2$.

$$f \in A_1 \cap A_2 \quad \begin{matrix} \nearrow \\ \\ \searrow \end{matrix} \quad \left.\begin{matrix} f \in A_1;\ |g| \leq |f| \Rightarrow g \in A_1 \\ \\ f \in A_2;\ |g| \leq |f| \Rightarrow g \in A_2 \end{matrix}\right\} \ g \in A_1 \cap A_2 ,$$

and the proposition is proved.

$\cap A\tau$ is also an ideal; indeed:

If $f \in \cap A\tau$ and if $|g| \leq |f|$, then $g \in \cap A\tau$, since

$$f \in \cap A\tau \Rightarrow \underset{\text{for all } \tau}{f \in A\tau}\ ;\ |g| \leq |f| \Rightarrow \underset{\text{for all } \tau}{g \in A\tau} \Rightarrow g \in \cap A\tau .$$

If D is an arbitrary non empty subset of L, the intersection of

all $\left\{\begin{matrix}\text{RIESZ subspaces}\\ \text{ideals}\\ \text{bands}\end{matrix}\right.$ containing D is, therefore, a $\left\{\begin{matrix}\text{RIESZ subspace}\\ \text{ideal}\\ \text{band.}\end{matrix}\right.$

This $\left\{\begin{matrix}\text{RIESZ subspace}\\ \text{ideal}\\ \text{band}\end{matrix}\right.$ is called the $\left\{\begin{matrix}\text{RIESZ subspace}\\ \text{ideal}\\ \text{band}\end{matrix}\right.$ generated by D.

If D consists of one element f_o, then the $\left\{\begin{matrix}\text{ideal}\\ \text{band}\end{matrix}\right.$ generated by f_o

is called a principal $\left\{\begin{matrix}\text{ideal}\\ \text{band.}\end{matrix}\right.$

The present notion of ideal is analogous to the notion of ideal in the algebraic sense:

Let us consider a ring R of elements x,y,...

$$\text{I ideal} \left\{\begin{array}{l} \text{I : subring} \\ \text{If } x \in I \text{ and } y \in R \\ \text{then } xy \in I . \end{array}\right.$$

In an analogous manner we have:

If $f \in A$ and $g \in L$, then $\inf(|f|, |g|) \in A$; the infimum is analogous to the product in the ideal.

7^{th} Lecture

Let V be a linear vectorspace with elements f,g,h; A,B : linear subspaces of V. Then

$$A \dot{+} B = \left\{ f : f = f_1 + f_2,\ f_1 \in A,\ f_2 \in B \right\}$$

is a linear subspace, called the algebraic sum of A and B.

If $A \cap B = \left\{ 0 \right\}$, then the representation $f = f_1 + f_2$ ($f_1 \in A$, $f_2 \in B$) is unique. Indeed,

if also $f = f_1' + f_2'$ $\quad (f_1' \in A,\ f_2' \in B)$,

then $\quad f_1 - f_1' = f_2 - f_2' = 0,$

since $\quad f_1 - f_1' \in A$ and $f_2 - f_2' \in B$.

We write, instead of $A \dot{+} B$, now $A \oplus B$: direct sum.

We consider now a RIESZ space L.

Theorem. Let A, B be ideals in L.

(i) $A \dot{+} B$ ideal

(ii) $A \perp B$ (i.e. $f \perp g$ for every $f \in A$ and every $g \in B$),
iff $A \cap B = \left\{ 0 \right\}$; hence, in this case, $A \dot{+} B$ is a direct sum.

(iii) If $A \perp B$, and if $f, g \in A \oplus B$, and $f = f_1 + f_2$, $g = g_1 + g_2$
(with $f_1, g_1 \in A$, $f_2, g_2 \in B$); if $f \leq g$, then $f_1 \leq g_1$ and $f_2 \leq g_2$.

Proof.

(i) We know already that $A \dot{+} B$ is a linear subspace.
We have to prove only that if $f \in A \dot{+} B$, and $|g| \leq |f|$,

then $g \in A \dot{+} B$.

We leave the details of the proof of (i) to the reader.

(ii) Let $A \perp B$, and assume $f \in A \cap B$. Then $f \perp f$, i.e.

$$\inf(|f|, |f|) = 0 \Rightarrow |f| = 0 \Rightarrow f = 0.$$

$$\Rightarrow A \cap B = \{0\}$$

Conversely, let $A \cap B = \{0\}$. Take $f_1 \in A$, $f_2 \in B$.

$f_1 \in A$, and $|f_1|$ is also in A.

Then $\inf(|f_1|, |f_2|) \leq \begin{cases} |f_1| \Rightarrow \inf(|f_1|, |f_2|) \in A \\ |f_2| \Rightarrow \inf(|f_1|, |f_2|) \in B \end{cases}$

$$\inf(|f_1|, |f_2|) = 0.$$

$A \dotplus B$ is a direct sum : We have $f_1, g_1 \in A$, $f_2, g_2 \in B$, the decomposition of f and g is uniquely determined.

But now we have to show that if $f \leq g$, the components f_1, g_1, f_2, g_2 satisfy $f_1 \leq g_1$ and $f_2 \leq g_2$. For that purpose, consider g-f.

$$g - f = g_1 - f_1 + g_2 - f_2$$

$$g - f \in A \oplus B \qquad g_1 - f_1 \in A \qquad g_2 - f_2 \in B$$

$f \leq g \Rightarrow g-f \geq 0$; it will be sufficient to prove that $g_1 - f_1 \geq 0$ and $g_2 - f_2 \geq 0$.

For this aim:

(iii) Assume that $u \geq 0$ is an element of $A \oplus B$ and $u = u_1 + u_2$ is the decomposition, so $u_1 \in A$, $u_2 \in B$.

We split up u_1 and u_2 in a positive and a negative part:

$$u = (u_1^+ + u_2^+) - (u_1^- + u_2^-)$$

(We recall that

$$f = f^+ - f^-,\ \inf(f^+, f^-) = 0, \text{ i.e. } f^+ \perp f^-, \text{ for every element } f).$$

$u_1 \in A \Longrightarrow |u_1| \in A$, thus $u_1^+ \in A$

$u_1^- \in A$

$$\left.\begin{array}{l} u_1^+ \perp u_1^- \\ u_1^+ \perp u_2^- \quad (u_1^+ \in A \text{ and } u_2^- \in B) \end{array}\right\} \Rightarrow u_1^+ \perp (u_1^- + u_2^-)$$

In the same manner we find: $u_2^+ \perp (u_1^- + u_2^-)$

$$\left.\begin{array}{l} u_1^+ \perp (u_1^- + u_2^-) \\ u_2^+ \perp (u_1^- + u_2^-) \end{array}\right\} (u_1^+ + u_2^+) \perp (u_1^- + u_2^-)$$

Thus $u_1^+ + u_2^+$, $u_1^- + u_2^-$ are disjoint.

But for $f = u-v$; $u,v \in L^+$, $\inf(u,v) = 0$, it results that $u = f^+$, $v = f^-$.

Here we have:

$$u = (u_1^+ + u_2^+) - (u_1^- + u_2^-),$$

the two terms being disjoint; we obtain

$u_1^+ + u_2^+ = u^+$, u^+ being the plus part of u ,

$u_1^- + u_2^- = u^-$, u^- being the minus part of u ,

Now $u^- = 0 \Longrightarrow u_1^- = u_2^- = 0$.

Then $u_1 \geq 0$ and $u_2 \geq 0$.

This can be said: The greater the elements, the greater the components.

Disjoint complements

Definition : If D is an arbitrary non empty subset of L, then the set $D^p = \{f : f \perp g \text{ for every } g \in D\}$ is called the disjoint complement of D.

(p means perpendicular) *

We have $D^{pp} = (D^p)^p$

----

* Sometimes one uses the notation D^d, d : disjoint.

<u>Theorem</u>. (i) D^p is a band.

(ii) $D \subset D^{pp}$; $D^p = D^{ppp}$; $D^p \cap D^{pp} = \{0\}$, so $D^p \dotplus D^{pp}$ is a direct sum : $D^p \oplus D^{pp}$.

<u>Proof</u>. (i) Let g denote an arbitrary element od D.

We have proved that if f_1 and f_2 are disjoint to D, their sum is also disjoint to D:

$$\left.\begin{array}{l} \left.\begin{array}{l} f_1 \perp g \\ f_2 \perp g \end{array}\right\} (f_1 + f_2) \perp g \\ \\ \text{and} \quad f \perp g \Rightarrow af \perp g \quad \text{for every real } a \end{array}\right\} \begin{array}{l} \Rightarrow D^p \ \underline{\text{is a linear subspace}}, \\ \\ f_1 \perp g \text{ and } |f_2| \leq |f_1| \Rightarrow f_2 \perp g. \end{array}$$

From these two results it follows that <u>D^p is an ideal</u>.

Finally we have to prove that, if

$f_\sigma \perp g$ for all $\sigma \in \{\sigma\}$, $f = \sup f_\sigma \Rightarrow f \perp g$.

But, according to a theorem, if a subset contains elements $f_\sigma \perp g$, and this subset has a supremum f, then $f \perp g$.

Hence <u>D^p is a band</u>.

(ii) D^{pp} : all elements disjoint to D^p.

Every element of D satisfies this condition

$D \subset D^{pp}$

D^{pp} is generally a band, but D is generally no band.

D and D^{pp} can be equal, but are not always equal.

An interesting question is : When does D^{pp} equal D?

Continuation of the <u>proof</u> :

If $D_1 \subset D_2$, thus $D_1^p \supset D_2^p$.

(The smaller the set, the larger the complement)

Indeed: if $f \in D_2^p$, thus $f \perp D_2 \Rightarrow f \perp D_1 \Rightarrow f \in D_1^p$.

By virtue of the relation

$$(D^p)^{pp} = (D^{pp})^p$$

we obtain

$$\begin{tikzcd} & D^p \supset D^{ppp} \\ D \subset D^{pp} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow, "\text{replacing } D \text{ by } D^p"'] & \\ & D^p \subset D^{ppp} \end{tikzcd}$$

We can write the last inequality, since a proposition true for every subset D is true for D^p.

Hence $D^p = D^{ppp}$.

From this it follows that

$$D^{(n+2)p} = D^{np}$$

$$n = 1,2,\ldots$$

As soon as two ideals are disjoint, their intersection is the null element.

$D^p \perp D^{pp}$; are ideals $\Rightarrow D^p \cap D^{pp} = \{0\}$

Remarks on bands

Is every band a disjoint complement of a subset? No.

Example

L : lexicographically ordered plane

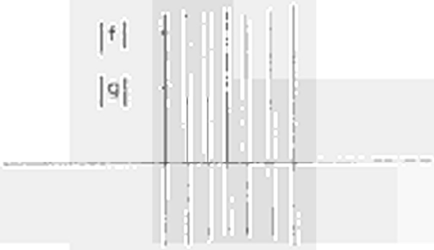


fig. VII, 1

$$f = (f_1, f_2) \qquad g = (g_1, g_2) \qquad f \leq g \text{ iff } \begin{cases} f_1 < g_1 \\ \text{or} \\ f_1 = g_1,\ f_2 \leq g_2 \end{cases}$$

It is a linear ordering.

All $f = (0, f_2)$ form a band D.

It is a linear subspace.

It is an ideal, if $|g| \leq |f| \in D$, then $g \in D$, because $|g|$ must be on the positive vertical axis.

It is a band. If we consider a subset, lying on the vertical axis and having a supremum, this supremum lies on the vertical axis.

(Subset with elements $f_\sigma = (0, f_{2\sigma})$;

$f = \sup f_\sigma$; then $f = (0, f_2)$ with $f_2 = \sup f_{2,\sigma}$)

D^p : all element $\{g : g \perp f \text{ for all } f \in D\}$

$= \{g : \inf(|g|, |f|) = 0 \text{ for all } f \in D\}$:

$|g|$ and $|f|$ are comparable.

The only element satisfying the condition is the zero element .

$\Longrightarrow D^p = \{0\}$ and $D^{pp} = L$.

Hence : D is a proper subspace of the second disjoint complement D^{pp}.

The set of all bands which are disjoint complements is not always the set of all bands.

Any set D^{pp} is a disjoint complement ,

$D^{pp} = (D^p)^p$

Any set D^p is of the form A^{pp} for some A,

$D^p = D^{ppp} = (D^p)^{pp}$

Any band which is a first disjoint complement, is also a second disjoint complement. But D in the above example is not a first disjoint complement. In L there are three bands : the null element, the vertical axis and the whole space. The horizontal axis is not a band, not even an ideal. Indeed,

in the figure we have $|g| \leq |f|$, and f is on the horizontal axis, but g is not.

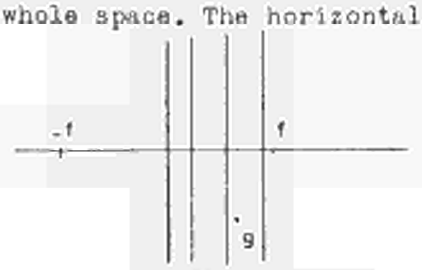


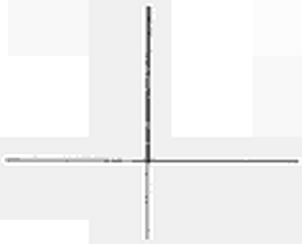
fig. VII, 2

8^{th} Lecture

D : arbitrary subset of L $\Longrightarrow D^p$ band, say A, so $A = D^p \Longrightarrow A^{pp} = D^{ppp} = D^p = A$.

Conversely, if A is a band such that $A^{pp} = A$, thus A is the disjoint complement of A^p.

Example : The lexicographically ordered plane:



A: positive vertical axis $\qquad A^{pp} \neq A \Longrightarrow A$ not a disjoint complement.

$A^{pp} = L$

fig. VIII, 1

$\left.\begin{array}{l}\mathcal{A}: \text{set of all bands} \\ \mathcal{B}: \text{set of all disjoint complements}\end{array}\right\} \mathcal{B} \subset \mathcal{A}$, in general $\mathcal{B}$ is a proper subset.

Let now :

$\mathcal{A}$: partially ordered by inclusion;

$\mathcal{B}$: also partially ordered by inclusion; $\{0\}$ is the "smallest" element in $\mathcal{B}$ and L the "largest" element in $\mathcal{B}$.

<u>Theorem</u> : If $A_1, A_2 \in \mathcal{B}^*$, then $A_1 \cap A_2 \in \mathcal{B}$.

<u>Proof</u> : By virtue of

$$D_1 \subset D_2 \Longrightarrow D_1^p \supset D_2^p \Longrightarrow D_1^{pp} \subset D_2^{pp}$$

we have

$$(A_1 \cap A_2)^{pp} \subset \begin{array}{l} A_1^{pp} = A_1 \\ \\ A_2^{pp} = A_2 \end{array} \Rightarrow (A_1 \cap A_2)^{pp} \subset (A_1 \cap A_2)$$

---

* Script B

The converse

$$A_1 \cap A_2 \subset (A_1 \cap A_2)^{pp}$$

has been already proved.

Hence

$$(A_1 \cap A_2)^{pp} = A_1 \cap A_2 \Longrightarrow A_1 \cap A_2 \in \mathfrak{B}.$$

We have

$$A_1 \cap A_2 = \inf (A_1, A_2).$$

Indeed, $A_1 \cap A_2$ is, in $\mathfrak{B}$, a lower bound of A_1 and A_2. Any other lower bound must be included in A_1 and A_2, so in $A_1 \cap A_2$. Hence $A_1 \cap A_2$ is the greatest lower bound.

<u>Theorem</u> : If $A_1, A_2 \in \mathfrak{B}$, then $(A_1^p \cap A_2^p)^p \in \mathfrak{B}$ and

$$(A_1^p \cap A_2^p)^p = \sup (A_1, A_2) \text{ in } \mathfrak{B}.$$

<u>Proof</u> : $(A_1^p \cap A_2^p)^p$ is a disjoint complement, hence it is in the set $\mathfrak{B}$.

Then

$$\left.\begin{array}{l} (A_1^p \cap A_2^p)^p \supset A_1^{pp} = A_1 \\ (A_1^p \cap A_2^p)^p \supset A_2^{pp} = A_2 \end{array}\right\} \Longrightarrow \text{so } (A_1^p \cap A_2^p) \text{ is an upper bound of } A_1 \text{ and } A_2.$$

Let $B \in \mathfrak{B}$ such that B is also an upper bound of A_1 and A_2.

$$B \supset \begin{matrix} A_1 \\ A_2 \end{matrix} \Longrightarrow B^p \subset A_1^p \cap A_2^p \Longrightarrow$$

$$B = B^{pp} \supset (A_1^p \cap A_2^p)^p, \quad B^{pp} \in \mathfrak{B}.$$

The set B is larger than $(A_1^p \cap A_2^p)^p$, hence this is the least upper bound.

$\mathfrak{B}$ has a smallest and a largest element, $\mathfrak{B}$ is a <u>lattice</u>.

Given $A \in \mathfrak{B}$, there exists a "complementary element", namely A^p, such that

$$\sup (A, A^p) = L,$$

$$\inf (A, A^p) = \{0\}.$$

We apply now the relation:

$$\sup(A_1, A_2) = (A_1^p \cap A_2^p)^p;$$

thus indeed

$$\sup(A, A^p) = (A^p \cap A^{pp})^p = \{0\}^p = L,$$

$$\inf(A, A^p) = A \cap A^p = \{0\}.$$

$\mathfrak{B}$ is a <u>Boolean algebra</u> : it is a lattice with a largest element and smallest element, and every element has a complementary element.

In general $\mathcal{A}$ is not a Boolean algebra.

An element has not always a complementary element.

In the lexicographically ordered plane the set of all bands is not a Boolean algebra.

There are three bands : $\{0\}$, A, L.

The smallest is the complement of the largest, but A has no complementary element.

fig. VIII, 2

<u>Archimedean RIESZ spaces</u>

Axiom of ARCHIMEDES (axiom of EUDOXOS) :

Let us have line segments of lengths u,v.

$\Longrightarrow$ There exists a natural number n, such that $nu \geq v$.

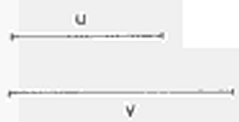


fig. VIII, 3

For functions in C ([0,1]) we have

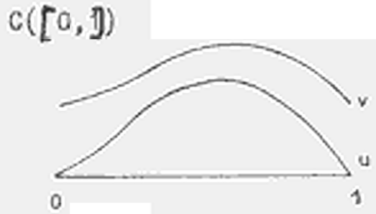


fig. VIII, 3 bis

Since u = 0 for 0 and 1, these is no n such that $nu \geq v$.

For real numbers, one can equivalently say : If $u \geq 0$, $v \geq 0$, and $nu \leq v$ for all $n = 1,2,\dots$, then $u = 0$.

That relation is true for functions in $C[0,1]$. It is true for some (but not all) RIESZ spaces.

<u>Definition</u> : If a RIESZ space L has the property that, for any $u, v \in L^+$ satisfying $nu \leq v$ for $n = 1,2,\dots$, we have $u = 0$, then L is called an Archimedean RIESZ space.

The lexicographically ordered plane is not Archimedean.

$u \not\leq 0$

$u \leq v$

$2u \leq v$

$3u \leq v$

$\vdots$

fig. VIII, 4

But C([0,1]) is Archimedean and the space of real numbers is Archimedean. Most spaces are Archimedean.

In C([0,1]) $nu \leq v$, $u \not\leq 0$, for all n, is impossible.

fig. VIII, 5

<u>Lemma</u> : If A is an ideal, and $0 < u \in A^{pp}$, then there exists an element $z \in A$, such that $0 < z \leq u$.

<u>Remark</u> : A^{pp} is often much larger than A.

Proof : We assume that no such z exists. Then $\inf(u, |f|) = 0$ for every $f \in A$; (because, if, for some $f \in A$, $z = \inf(u, |f|) > 0$, thus $z \in A$ and $0 < z \leq u$), so $u \perp f$ for every $f \in A$,

$\Rightarrow u \in A^p$; $u \in A^{pp} \Rightarrow u \in A^p \cap A^{pp} = \{0\}$;

$\Rightarrow u = 0$. This is a contradiction.

Hence there exists such a z.

Definition : The band generated by A is the smallest band containing A or the intersection of all bands containing A.

Theorem : L Archimedean; A ideal in L $\Rightarrow$ The band generated by A is A^{pp}.

We can prove, but it is more difficult, that if L is a RIESZ space, such that for every ideal A the band generated by A is A^{pp}, then L is Archimedean.

Proof : We denote the band generated by A by $\{A\}$. Then $\{A\} \subset A^{pp}$. To prove is that $\{A\} = A^{pp}$.

We take $0 < u \in A^{pp}$.

If every positive element in A^{pp} is in $\{A\}$, we can prove that every element of A^{pp} is in $\{A\}$; (we can split it up in u^+ and u^-).

We consider the set $A_u = \{v : v \in A, 0 \leq v \leq u\}$;

i.e. the elements of the ideal A, which are between 0 and u.

We will prove that $u = \sup A_u$;

then $u = \sup$ (subset of A),

hence $u = \sup$ (subset of $\{A\}$) ; and so $u \in \{A\}$ by the definition of a band.

We prove the assumption by contradiction.

Assume that it is not true (i.e., although u is an upper bound A_u, u is not the **least** upper bound).

$\Rightarrow$ There is another upper bound $\dot{\dot{\dot{u}}}$, such that $u \leq u_1$ is not true.

Let us take $w = \inf(u, u_1) \Longrightarrow w$ is an upper bound of A_u, and $w < u$; (for, if $w = u$, $u_1 \geq u$, this is exactly not the case).

But, if we make an element smaller, without making it negative, we remain in A^{pp}. Thus

$$0 < u-w \in A^{pp} \xrightarrow{\text{using the lemma}} \text{There exists } 0 < z \in A \text{ such that } 0 < z \leq u-w.$$

We take now $v \in A_u$.

Then $v+z \in A$, and $0 \leq v+z \leq w+z$; but $w+z \leq u$, so $0 \leq v+z \leq u$,

$\Longrightarrow v+z \in A_u$.

Hence, if we take any element in A_u and add the element $z \in A$, we remain in A_u.

$\Longrightarrow v+nz \in A_u$ for $n = 1,2,\ldots$

In particular $nz \in A_u$ for $n = 1,2,\ldots$

i.e. $nz \leq u$ for $n = 1,2,\ldots$

Since L is Archimedean, $\Longrightarrow z = 0$.

We have obtained a contradiction, because $0 < z \leq u$.

Hence the assumption made is not true and $u = \sup A_u$.

9th Lecture

The set of all real numbers has the property that every subset bounded above has a least upper bound. (DEDEKIND $\simeq$ 1870)

Definition. Any RIESZ space L with the property that every subset of L which is bounded above has a least upper bound, is called a DEDEKIND (or conditionally) complete RIESZ space.

Any RIESZ space L with the property that every countable subset of L which is bounded above has a least upper bound, is called a DEDEKIND σ-complete RIESZ space.

Theorem. Any RIESZ space L:

DEDEKIND complete $\Longrightarrow$ DEDEKIND σ-complete $\Longrightarrow$ Archimedean.

Proof. Assume L DEDEKIND σ-complete, and let $nu \leqslant v$ for $u, v \in L^+$ and $n = 1,2,\ldots$

Let us take the set of elements $\{nu : n = 1,2,\ldots\}$ bounded above by v,

$$\overset{\sigma D.C.}{\Longrightarrow} \sup_{n=1,2,} nu = u_o \text{ exists. Then,}$$

(since $\sup f_\sigma = f \Longrightarrow \sup af_\sigma = af,\ a > o$),

we have

$$2u_o = \sup_{n=1,2,} 2nu = \sup_{n=1,2,} nu = u_o \Longrightarrow u_o = o \Longrightarrow u = o$$

$$\Longrightarrow \text{L is Archimedean.}$$

Any space, that is not Archimedean, is not DEDEKIND σ-complete, any space that is not DEDEKIND σ-complete, is not DEDEKIND complete.

Example of a RIESZ space, that is not DEDEKIND σ-complete : the space $C([o,1])$.

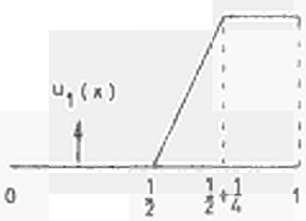


fig. IX,1

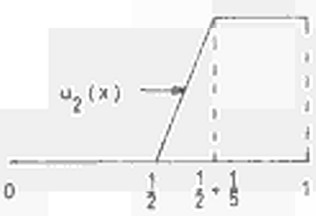


fig. IX,2

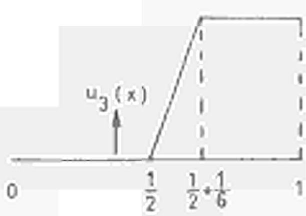


fig. IX,3

and so on.

$$o \leqslant u_1(x) \leqslant u_2(x) \leqslant u_3(x) \leqslant \ \ldots \qquad \leqslant 1$$

This sequence has no supremum in C $([o,1])$. There exists a least upper bound in the space of all real functions:

fig. IX,4

$u(x) = o$, $o \leqslant x \leqslant \frac{1}{2}$

$u(x) = 1$, $\frac{1}{2} < x \leqslant 1$.

This function is not in the space C $([o,1])$

Thus C $([o,1])$ is not DEDEKIND σ-complete.
and hence not DEDEKIND complete.

<u>Theorem</u>. Let L be DEDEKIND complete.

(i) If A and B are bands such that $A \cap B = \{o\}$, then $A \oplus B$ is also a band.

(ii) For any band A , $A \oplus A^{p} = L$.

<u>Remark</u>. If L is not DEDEKIND complete, then (ii) is not necessarily true. If, in the space C $([o,1])$, we have

$$A = \left\{ f : f(x) = o \quad \text{on } o \leqslant x \leqslant \tfrac{1}{2} \right\} ,$$

$$B = \left\{ f : f(x) = o \quad \text{on } \tfrac{1}{2} \leqslant x \leqslant 1 \right\} ,$$

then $A \oplus B = \left\{ f : f(\tfrac{1}{2}) = o \right\}$, so $A \oplus B \neq L$.

<u>Proof</u>. (i) We assume that $f_\sigma \in A \oplus B$, such that $f = \sup f_\sigma$ exists.
Since $A \oplus B$ is an ideal, f_σ^{+} , $f_\sigma^{-} \in A \oplus B$, for all σ , and
$f^{+} = \sup f_\sigma^{+}$ and $f^{-} = \inf f_\sigma^{-}$.
To prove is that $f^{+}, f^{-} \in A \oplus B$.

For f^{-} it is very easy.
We take one $f^{-}_{\sigma_0}$; $o \leqslant f^{-} \leqslant f^{-}_{\sigma_0} \in A \oplus B \Longrightarrow f^{-} \in A \oplus B$.
For f^{+} we have:
$f_\sigma^{+} \in A \oplus B \Longrightarrow f'^{+}_\sigma = u'_\sigma + u''_\sigma$, where $o \leqslant u'_\sigma \in A$ and $o \leqslant u'_\sigma \in B$.
Then $o \leqslant u'_\sigma \leqslant f^{+}$ for all σ .

L being DEDEKIND complete

$$\Longrightarrow u' = \sup u'_\sigma \text{ exists in } L \Longrightarrow u' \in A;$$

according to the definition of a band, a supremum is not only in L, but also in A.

Similarly

$$u'' = \sup u''_\sigma \text{ exists, and } u'' \in B.$$

$$f^+_\sigma = u'_\sigma + u''_\sigma \leqslant u'+u'' \in A \oplus B \quad \text{for all } \sigma .$$

f^+_σ being less than a fixed element, the least upper bound of f^+_σ is smaller than this fixed element. Hence

$$f^+ \leqslant u'+u'' \in A \oplus B$$

$$\Longrightarrow A \oplus B \text{ is an ideal, } f^+ \in A \oplus B.$$

Hence $A \oplus B$ is a band.

Proof of (ii). By (i), $A \oplus A^p$ is a band, since A and A^p are two disjoint bands.

$$\text{Since } A \oplus A^p \supset \begin{matrix} A \\ A^p \end{matrix}, \quad (A \oplus A^p) \subset \begin{matrix} A^p \\ A^{pp} \end{matrix} \Longrightarrow (A \oplus A^p)^p = \{0\} \Longrightarrow (A \oplus A^p)^{pp} = L.$$

Since L is ARCHIMEDEAN, the band generated by $A \oplus A^p$, is equal to $(A \oplus A^p)^{pp}$, i.e. $A \oplus A^p = L$;
since a band generated by a band, is the band itself.

Question. If L is an arbitrary RIESZ space, and A, B are disjoint ideals such that $A \oplus B = L$, what can we prove about A and B?

Theorem. If A, B are ideals, such that $A \oplus B = L$, thus $B = A^p$ and $A = B^p$, so $A = A^{pp}$ and $B = B^{pp}$.

Remark. Hence, A and B are bands determining each other uniquely.

<u>Proof.</u> $L = A \oplus B \Longrightarrow A \perp B \Longrightarrow B \subset A^p$.

Let us take an arbitrary positive element in A^p,

$$o \leqslant u \in A^p;$$

by hypothesis we can decompose u :

$$u = u_1 + u_2 \quad \text{with} \quad o \leqslant u_1 \in A \quad \text{and} \quad o \leqslant u_2 \in B;$$

$$\left.u_1 \begin{matrix} \in A \\ \in A^p \end{matrix} \quad (\text{because } o \leqslant u_1 \leqslant u \in A^p)\right\} \Longrightarrow u_1 = o, \quad u = u_2 \in B$$

so $A^p \subset B$.

For reasons of symmetry $A = B^p$.

Thus, if the direct sum $A \oplus B = L$, A and B are bands.

<u>Definition.</u> If the band A has the property, that $A \oplus A^p = L$, then A is called a projection band.

$f = f_1 + f_2$,

f_1 is the projection of f on the band A,

f_2 " " " of f on the disjoint complement.

It results from (ii) of the next to last theorem:

In a DEDEKIND complete space, every band is a projection band.

(In C ([o,1]), $\left\{f : f(x) = o \text{ on } o \leqslant x \leqslant \frac{1}{2}\right\}$ is a band, but not a projection band :

$$A^p = \left\{f : f(x) = o \text{ on } \tfrac{1}{2} \leqslant x \leqslant 1\right\} \Longrightarrow A \oplus A^p \neq L \quad)$$

$\mathcal{A}$: collection of all bands: not a Boolean algebra in general.

$\mathcal{B}$: subcollection of all disjoint complements, is a Boolean algebra.

$\mathcal{C}$: subcollection of all projection bands, is a Boolean algebra.

The following theorem has been already partially proved.

<u>Theorem.</u>

(i) DEDEKIND complete $\Longrightarrow$ (ii) DEDEKIND σ-complete $\xrightarrow{\text{not proved}}$ (iv) Every principal band is a projection band

(i) DEDEKIND complete $\Longrightarrow$ (iii) Every band is a projection band $\Longrightarrow$ (iv)

$\Longrightarrow$ (v) ARCHIMEDEAN

No arrow can be reversed.
There are examples of DEDEKIND σ-complete spaces, that are not DEDEKIND complete.

Every band being a projection band, is not DEDEKIND complete.

Although (ii) $\not\Rightarrow$ (i) and (iii) $\not\Rightarrow$ (i), it is true that ((ii)+(iii)) $\Longrightarrow$ (ii)
(LUXEMBURG).

10^{th} Lecture

Linear functionals, linear algebra.

We recall the following definitions:

Let be a real linear vector space V. If to every $f \in V$, there is assigned a real number $\varphi(f)$, $\varphi(f)$ is called a functional.
A functional φ is called linear, if

$$\begin{aligned} \varphi(f_1+f_2) &= \varphi(f_1)+\varphi(f_2) \\ \varphi(\alpha f) &= \alpha\,\varphi(f) \end{aligned}$$

Let us take

<u>Definition</u>:
$$\begin{aligned} (\varphi_1+\varphi_2)(f) &= \varphi_1(f)+\varphi_2(f) \\ (a\varphi)(f) &= a\varphi(f) \end{aligned}$$

The linear vector space formed by the linear functionals on V is called the algebraic dual of V.

Let L be a RIESZ space, we consider the linear functionals $\varphi_1, \psi_1, \dots$

<u>Definition</u>: A linear functional φ on L is called positive, if $\varphi(f) \geq 0$ for all $f \in L^+$.

<u>Lemma</u>: (Extension lemma) Let $\tau(u)$ be a real function, defined for all $u \in L^+$, such that $\tau(u) \geq 0$ and $\tau(u+v) = \tau(u)+\tau(v)$ for all $u, v \in L^+$. Then there exists a positive linear functional φ such that $\varphi(u) = \tau(u)$ for all $u \in L^+$.

<u>Proof:</u> Let us define, for any $f \in L$, $\varphi(f) = \tau(f^+)-\tau(f^-)$.
Then $\varphi(u) = \tau(u)$ for all $u \in L^+$.

1^o We prove first that if $f = u-v$; $u, v \in L^+$, then $\varphi(f) = \varphi(u)-\varphi(v)$.
Indeed, $f = u-v \Rightarrow f^+-f^- = u-v \Rightarrow f^++v = f^-+u \Rightarrow \tau(f^+)+\tau(v) = \tau(f^-)+\tau(u)$
$\Rightarrow \tau(f^+)-\tau(f^-) = \tau(u)-\tau(v) \Rightarrow \varphi(f) = \varphi(u)-\varphi(v)$

2^{o} Now $\varphi(f+g) = \varphi(f) + \varphi(g)$ for all $f, g \in L$. Indeed

$$\varphi(f+g) = \varphi(f^{+}+g^{+}-f^{-}-g^{-}) = \varphi\{\underbrace{(f^{+}+g^{+})}_{u}-\underbrace{(f^{-}+g^{-})}_{v}\}$$

$$\underset{===}{1^{o}} \quad \varphi(f^{+}+g^{+}) - \varphi(f^{-}+g^{-}) = \tau(f^{+}+g^{+}) - \tau(f^{-}+g^{-}) =$$

$$\tau(f^{+}) + \tau(g^{+}) - \tau(f^{-}) - \tau(g^{-}) = \varphi(f) + \varphi(g)$$

3^{o} We have still to prove : $\varphi(af) = a\,\varphi(f)$ for all $f \in L$ and real a.

Applying 1^{o}, we find that we can take f in the positive cone, and for a : a non-negative number.

It is sufficient to prove that $\varphi(au) = a\,\varphi(u)$ for $u \in L^{+}$ and $a \geqslant o$.
For $a = 1, 2, 3, \dots$ it is evident by addition. Then also for

$$a = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots \quad \text{because}$$

$$\varphi(\tfrac{1}{3}u) = \tfrac{1}{3}\varphi(u)$$

$$3\varphi(\tfrac{1}{3}u) = \varphi(u) \quad , \quad \text{we set } u = 3v,$$

$$3\,\varphi(v) = \varphi(3v)$$

Then also for $a = \frac{m}{n}$ (m, n natural numbers), so for a rational.
We assume now that a is irrational. We choose

$$o \leqslant r \leqslant a \leqslant r' \quad (r\ ,\ r',\ \text{rationals}).$$

We have

$$o \leqslant \varphi(ru) \leqslant \varphi(au) \leqslant \varphi(r'u)$$

since $au-ru$ is an element in the positive cone, and hence $\varphi(au-ru) \geqslant o$.

We find

$$o \leqslant r\,\varphi(u) \leqslant \varphi(au) \leqslant r'\varphi(u)$$

Let $r \uparrow a$

$r' \downarrow a$ (keeping r, r' rational),

$r\,\varphi(u)$ and $r'\,\varphi(u)$ converge to $a\,\varphi(u)$.
Now the third term of the inequalities is constant with respect to r and r'.
So $\varphi(au) = a\,\varphi(u)$. Hence φ is a linear functional, and φ is uniquely determined.

Definition:

A linear functional φ on L is called order bounded, if for any $u \in L^+$, the number $\sup(|\varphi(f)| : |f| \leqslant u)$ is finite.

If φ is positive, then φ is order bounded. Indeed, we take $u \in L^+$, if $|f| \leqslant u$, then $\begin{matrix} f^+ \\ f^- \end{matrix} \leqslant u$, so $|\varphi(f)| = |\varphi(f^+)-\varphi(f^-)| \leqslant \varphi(f^+)+\varphi(f^-) \leqslant 2\varphi(u) \Rightarrow$ $\sup(|\varphi(f)| : |f| \leqslant u) \leqslant 2\varphi(u)$: finite.

If φ_1, φ_2 are order bounded, then $\varphi_1 + \varphi_2$ also.

$$|(\varphi_1+\varphi_2)(f)| \leqslant |\varphi_1(f)| + |\varphi_2(f)|$$

If φ is order bounded, then $a\varphi$ (a real) is it also.
Hence: the set $L^\sim$ of all order bounded linear functionals is a real linear vector space; every positive linear functional is in $L^\sim$.

Theorem: (JORDAN decomposition theorem). φ order bounded $\Longleftrightarrow$

$$\varphi = \varphi_1 - \varphi_2 \quad \text{with } \varphi_1, \varphi_2 \text{ positive.}$$

Remark. This theorem is called JORDAN theorem: not because he proved this theorem, but because he proved another analogous theorem : Every function of bounded variation is the difference of two monotone functions.

Proof. If $\varphi = \varphi_1 - \varphi_2$ with $\varphi_1, \varphi_2 \geqslant 0$, then $\varphi_1, \varphi_2 \in L^\sim$, so $\varphi_1 - \varphi_2 \in L^\sim$.
Conversely, assume that $\varphi \in L^\sim$. For any $u \in L^+$, define

$$\tau(u) = \sup(\varphi(v) : 0 \leqslant v \leqslant u)$$

Among the elements v is the null-element; evidently

$$\tau(u) \geqslant \varphi(0) = 0 ;$$

$\tau(u)$ is finite, because $\tau(u) \leqslant \sup(|\varphi(f)| : |f| \leqslant u)$ is finite.

Property that $\tau(u_1) + \tau(u_2) \leqslant \tau(u_1+u_2)$:

Let us take $\varepsilon > 0$: There exists $0 \leqslant v_1 \leqslant u_1$ such that $\varphi(v_1) > \tau(u_1) - \varepsilon$.
" " $0 \leqslant v_2 \leqslant u_2$ " " $\varphi(v_2) > \tau(v_2) - \varepsilon$.

We add now;

$$\tau(u_1+u_2) = \sup(\varphi(w) : o \leqslant w \leqslant u_1+u_2) \geqslant \varphi(v_1+v_2)$$
$$= \varphi(v_1)+\varphi(v_2) > \tau(u_1)+\tau(u_2) - 2\varepsilon$$

Property that $\tau(u_1)+\tau(u_2) \geqslant \tau(u_1+u_2)$:

We take $o \leqslant w \leqslant u_1+u_2$. Then (according to the dominated decomposition theorem) there exist w_1 and w_2 such that $o \leqslant w_1 \leqslant u_1$
$o \leqslant w_2 \leqslant u_2$

and $w_1+w_2 = w$.

$$\text{Then } \varphi(w) = \varphi(w_1)+\varphi(w_2) \leqslant \tau(u_1)+\tau(u_2)$$
$$\tau(u_1+u_2) = \sup(\varphi(w) : o \leqslant w \leqslant u_1+u_2) \leqslant \tau(u_1)+\tau(u_2)$$

We have therefore $\tau(u) \geqslant o$

$$\tau(u_1+u_2) = \tau(u_1)+\tau(u_2)$$

There exists a positive linear functional φ^+, such that $\varphi^+(u) = \tau(u)$ for all $u \in L^+$.

So $\varphi^+(u) = \tau(u) = \sup(\varphi(v) : o \leqslant v \leqslant u) \geqslant \varphi(u) \Longrightarrow (\varphi^+ - \varphi)(u) \geqslant o$ for $u \in L^+$

$\Longrightarrow \varphi^+ - \varphi$ is positive; we call it φ^-, so $\varphi^+ - \varphi = \varphi^- \Longrightarrow$

$\varphi = \varphi^+ - \varphi^-$, φ^+ and φ^- positive.

This concludes the proof.

If φ positive, then $\varphi^+(u) = \sup(\varphi(v) : o \leqslant v \leqslant u) = \varphi(u)$

$\Longrightarrow \varphi^+ = \varphi$, so $\varphi^- = o$.

Let us take two functionals φ_1 and φ_2;

If $\varphi_1 - \varphi_2$ is positive,

then $(\varphi_1 - \varphi_2)(v) \geqslant o$ for all $v \in L^+$,

so $\varphi_1(v) \geqslant \varphi_2(v)$ for $v \in L^+$.

Hence, if $u \in L^+$, then $\varphi_1{}^+(u) \geqslant \varphi_2{}^+(u) \Longrightarrow \varphi_1{}^+ - \varphi_2{}^+$ is positive.

<u>Definition</u>. If $\varphi_1, \varphi_2 \in L$, and $\varphi_1 - \varphi_2$ is positive, then we write $\varphi_1 \geqslant \varphi_2$. Then $L^\sim$ becomes an ordered vector space.

Remark. The remark above that if $\varphi_1-\varphi_2$ is positive, then $\varphi_1^+-\varphi_2^+$ is also positive, can now also be stated as follows: If $\varphi_1 \geq \varphi_2$, then $\varphi_1^+ \geq \varphi_2^+$.

Theorem. With respect to the partial order, $L^\sim$ is a RIESZ space.

Proof. Given $\varphi_1, \varphi_2 \in L^\sim$, let $\varphi_3 = (\varphi_2-\varphi_1)^+ + \varphi_1$. Then

since $\varphi^+ - \varphi \geqslant 0$ or $\varphi^+ \geqslant \varphi$ is true in any notation,

$\varphi_3 \geqslant \varphi_1$ and $\varphi_3 \geqslant (\varphi_2-\varphi_1) + \varphi_1 = \varphi_2$

$\Rightarrow \varphi_3$ is an upper bound of φ_1, φ_2. Let ψ be another upper bound of φ_1 and φ_2. Then

$$\psi-\varphi_1 \geqslant \left.\begin{array}{l} 0 \Longrightarrow \psi-\varphi_1 = (\psi-\varphi_1)^+ \\ \varphi_2-\varphi_1 \Longrightarrow (\psi-\varphi_1)^+ \geqslant (\varphi_2-\varphi_1)^+ \end{array}\right\} \Longrightarrow \psi-\varphi_1 \geqslant (\varphi_2-\varphi_1)^+ \Rightarrow \psi \geqslant \varphi_3$$

$\Longrightarrow \varphi_3$: least upper bound.

Hence every pair of elements in $L^\sim$ has a least upper bound; more precisely, we have proved that if $\varphi_1, \varphi_2 \in L^\sim$, then

$$\sup(\varphi_1, \varphi_2) = (\varphi_2-\varphi_1)^+ + \varphi_1.$$

Similarly, it can be proved that

$$\inf(\varphi_1, \varphi_2) = \varphi_2 - (\varphi_2-\varphi_1)^+.$$

Hence, $L^\sim$ is a RIESZ space.

11^{th} Lecture

Remark : In the preceeding lecture we have introduced the linear vector space $L^{\sim}$ of all order bounded linear functionals on L, and we have defined that, in $L^{\sim}$,

$$\varphi \leq \psi \quad \text{iff} \quad \psi - \varphi \quad \text{is positive.}$$

This makes $L^{\sim}$ into an ordered vector space. Given $\varphi \in L^{\sim}$, we have also defined the positive linear functional φ^+ by

$$\varphi^+(u) = \sup\ (\varphi(v)\ :\ 0 \leq v \leq u)$$

for every $u \in L^+$. Hence, the notation φ^+ was introduced before it was proved that $L^{\sim}$ is a RIESZ space. We proved that $L^{\sim}$ is indeed a RIESZ space; given $\varphi_1, \varphi_2 \in L^{\sim}$, we proved that

$$\sup\ (\varphi_1, \varphi_2) = (\varphi_2 - \varphi_1)^+ + \varphi_1$$

It must be shown now that φ^+ is indeed the positive part of φ in the sense of the RIESZ space definitions, i.e.,we must prove that $\varphi^+ = \sup\ (\varphi, 0)$ for every $\varphi \in L^{\sim}$. Taking $\varphi_1 = 0$ and $\varphi_2 = \varphi$ in the formula

$$\sup\ (\varphi_1, \varphi_2) = (\varphi_2 - \varphi_1)^+ + \varphi_1\ ,$$

we obtain

$$\sup\ (0, \varphi) = (\varphi - 0)^+ + 0 = \varphi^+\ ;$$

this shows that the notation is in agreement with the earlier notations in RIESZ spaces in the earlier lectures.

Theorem : $L^{\sim}$ is DEDEKIND complete.

Proof : First take a subset in L^+, bounded above and directed upwards; call it $\varphi_\sigma : \sigma \in \{\sigma\}$; $0 \leq \varphi_\sigma \leq \varphi_0$ for all σ, and $\varphi_\sigma \uparrow$.
To prove is that $\sup \varphi_\sigma$ exists in $L^{\sim}$. For any $u \in L^+$, define $\tau(u) = \sup_\sigma \varphi_\sigma(u)$; then $0 \leq \tau(u) \leq \varphi_0(u)$; $\tau(u)$ is finite.

If $u, v \in L^+$, then $\tau(u+v) = \sup_\sigma \varphi_\sigma(u+v) = \sup_\sigma \left\{\varphi_\sigma(u) + \varphi_\sigma(v)\right\}$

$$\leq \sup_\sigma \varphi_\sigma(u) + \sup_\sigma \varphi_\sigma(v) = \tau(u) + \tau(v)$$

Take $\varepsilon > 0$; there exists σ_1, such that $\varphi_{\sigma 1}(u) > \tau(u) - \varepsilon$
" " σ_2, " " $\varphi_{\sigma 2}(v) > \tau(v) - \varepsilon$ $\Big\} \Longrightarrow$

There exists σ_3
such that $\varphi_{\sigma 3} \geq \left.\begin{matrix}\varphi_{\sigma 1}\\ \varphi_{\sigma_2}\end{matrix}\right\}$ $\left.\begin{matrix}\varphi_{\sigma 3}(u) > \tau(u) - \varepsilon\\ \varphi_{\sigma 3}(v) > \tau(v) - \varepsilon\end{matrix}\right\} \Rightarrow \tau(u+v) \geq \varphi_{\sigma 3}(u+v) > \tau(u) + \tau(v) - 2\varepsilon$

Hence : $\left.\begin{matrix}\tau(u) \geq 0 \quad \text{for all} \quad u \in L^+\\ \tau(u+v) = \tau(u) + \tau(v) \quad \text{for } u, v \in L^+\end{matrix}\right\} \Longrightarrow$ There exists a positive

linear functional ψ, such that $\psi(u) = \tau(u)$ for all $u \in L^+$, i.e.
$\psi(u) = \sup_\sigma \varphi_\sigma(u)$ for every $u \in L^+$,
$\Longrightarrow \psi$ is an upper bound of all φ_σ.

Let ψ_1 be another upper bound, then $\psi_1 \geq \varphi_\sigma$ for all σ.

$\Longrightarrow \psi_1(u) \geq \varphi_\sigma(u)$ for every σ and every $u \in L^+$,

$\psi_1(u)$ is greater than the supremum of all $\varphi_\sigma(u) \Longrightarrow$

$\psi_1(u) \geq \psi(u)$; hence ψ is the least upper bound of all φ_σ .

Remark for the general case : Let us consider an arbitrary subset in a RIESZ space : we can make the subset larger by adding elements, such that the new subset is directed upwards and has the same upper bounds (cf. the remark in the 4th lecture). Therefore the choice of a directed set involves no loss of generality.

Definition : The element $\varphi \in L^\sim$ is called an integral, if, for every sequence $u_n \downarrow 0$ in L, $\lim_{n\to\infty} \varphi(u_n) = 0$.

The set of all integrals is called $L_c^\sim$: $L_c^\sim$ is a linear subspace of $L^\sim$.

<u>Lemma</u> : $\varphi \in L_c^\sim \Longleftrightarrow \varphi^+, \varphi^- \in L_c^\sim \Longleftrightarrow |\varphi| \in L_c^\sim$ *
(i) (ii) (iii)

<u>Proof</u> : To prove that (i)$\Longrightarrow$(ii) is the only difficult case.
To prove is that $\varphi^+ \in L_c$. Let $u_n \downarrow 0$ in L. To prove is that $\varphi^+(u_n) \downarrow 0$.
Let $0 \leq v \leq u_1$.
Then $0 \leq v-\inf(v, u_n) = \inf(v, u_1) - \inf(v, u_n)$ ** $\leq u_1 - u_n$,
by applying the BIRKHOFF's inequalities.
Now, since $\varphi^+(u) = \sup(\varphi(v) : 0 \leq v \leq u)$, it follows that

if $0 \leq v \leq u$, then $\varphi^+(u) \geq \varphi(v)$.

It results that

$$\varphi(v-\inf(v, u_n)) \leq \varphi^+(u_1 - u_n) \Longrightarrow$$

$0 \leq \varphi^+(u_n) \leq \varphi(\inf(v, u_n)) - \varphi(v) + \varphi^+(u_1)$; since $u_n \downarrow 0$, we have

$\inf(v, u_n) \downarrow \inf(v, 0) = 0$ so $\varphi(\inf(v, u_n)) \to 0$,

the quantities $\varphi(v)$ and $\varphi^+(u_1)$ are constant, and $\varphi^+(u_n)$ is a decreasing sequence of positive numbers. Thus,

$0 \leq \lim \varphi^+(u_n) \leq \varphi^+(u_1) - \varphi(v)$, this for every v, such that $0 \leq v \leq u_1$,

hence $0 \leq \lim \varphi^+(u_n) \leq \varphi^+(u_1) - \sup(\varphi(v) : 0 \leq v \leq u_1) = \varphi^+(u_1) - \varphi^+(u_1) = 0$.

This finishes the proof.
The proof of the other parts are trivial, indeed,

(ii)$\Longrightarrow$(iii), since $|\varphi| = \varphi^+ + \varphi^-$

and it is proved that $L_c^\sim$ is a linear space.

To prove that (iii)$\Longrightarrow$(i), we use the relation : $\varphi = \varphi^+ - \varphi^-$.

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* We use the RIESZ space notation.

** We can take the absolute value of this expression, but it is not necessary.

We know that $|\varphi|(u_n) \to 0$ for every sequence $u_n \downarrow 0$ in L.

$$|\varphi(u_n)| = |\varphi^+(u_n) - \varphi^-(u_n)| \leq \varphi^+(u_n) + \varphi^-(u_n) = |\varphi|(u_n) \to 0.$$

Consequence of this lemma : $L^\sim$ is not only a linear subspace, but also a band.

<u>Theorem</u> : $L_c^\sim$ is a band in $L^\sim$.

<u>Proof</u> : 1) $L_c^\sim$ is a linear subspace of $L^\sim$. Let $|\psi| \leq |\varphi|$ and $\varphi \in L_c^\sim$.
In order to prove that $L_c^\sim$ is an ideal, we have to prove that $\psi \in L_c^\sim$.

$$\varphi \in L_c^\sim \Longrightarrow |\varphi| \in L_c^\sim \Longrightarrow |\psi| \in L_c^\sim \Longrightarrow \psi \in L_c^\sim$$

by virtue of the lemma. $L_c^\sim$ is an ideal.

2) In order to prove that $L_c^\sim$ is a band, we assume that $0 \leq \varphi_\sigma \in L_c^\sim$ and $\varphi_\sigma \uparrow \varphi$.
To prove is that $\varphi \in L_c^\sim$.
For every $u \in L^+$, $\varphi(u) = \sup_\sigma \varphi_\sigma(u)$. We know that if $u_n \downarrow 0$ in L,
then $\varphi_\sigma(u_n) \downarrow 0$ as $n \to \infty$, for every σ. To prove is that $\varphi(u_n) \downarrow 0$.
Take $\varepsilon > 0$, there exists σ_1 such that $(\varphi - \varphi_{\sigma 1})(u_1) < \varepsilon \Longrightarrow$

$0 \leq (\varphi - \varphi_{\sigma 1})(u_n) < \varepsilon$ for every n; $\varphi_{\sigma 1}(u_n) < \varepsilon$ for $n \geq N(\varepsilon) \Longrightarrow$

$\varphi(u_n) < 2\varepsilon$ for $n \geq N$.

Hence $\varphi(u_n) \to 0$, so $\varphi \in L_c^\sim$.

$L^\sim$: DEDEKIND complete space.
$L_c^\sim$: band in $L^\sim$; any element in $L^\sim$ disjoint to $L_c^\sim$ is called a singular linear functional: hence the set of all singul r functionals is the disjoint complement of $L_c^\sim$, $L_s^\sim$; $L_s^\sim = (L_c^\sim)^p$.

$$L^\sim = L_c^\sim \oplus L_s^\sim \text{ (by a theorem in the } 9^{th} \text{ lecture)},$$

i.e. every $\varphi \in L^\sim$ has a unique decomposition; $\varphi = \varphi_c + \varphi_s$,

where φ_c : integral,

φ_s : singular.

The relation between φ and φ_c holds in a formula:

For every $u \in L^+$ and φ : positive linear functional, we have

$$\varphi_c(u) = \inf \left(\lim \varphi(u_n) \ : \ 0 \leq u_n \uparrow u\right)$$

(u is fixed, but we can take a different sequence u_n; then we have taken the infimum of the numbers $\lim \varphi(u_n)$).

12th Lecture

Remark

$L_c^\sim$ being the subspace of all integrals φ , we have in an analogous manner:

$L_n^\sim$ subspace of all normal integrals : φ is a normal integral, if, for any downwards directed set $u_\tau \downarrow 0$ in L, $\inf |\varphi(u_\tau)| = 0$. This definition is analogous to that of an integral.

A normal integral is always an integral, but there are many examples, where integrals are not normal integrals.

$L_n^\sim$ is a band,

$L_{sn}^\sim$: set of all φ disjoint to $L_n^\sim$,

$L^\sim = L_n^\sim \oplus L_{sn}^\sim$ (analogous to $L^\sim = L_c^\sim \oplus L_s^\sim$).

If $\varphi \geq 0$ in $L^\sim$, $u \in L^+$, then $\varphi_c(u) = \inf (\lim \varphi(u_n) : 0 \leq u_n \uparrow u)$,

but we can write also: $\qquad \inf (\sup \varphi(u_n) : 0 \leq u_n \uparrow u)$,

$$\varphi = \varphi_n + \varphi_{sn}$$

If $\varphi \geq 0$ in $L^\sim$, $u \in L^+$, then $\varphi_n(u) = \inf (\sup \varphi(u_\tau) : 0 \leq u_\tau \uparrow u)$.

This proposition is true, but very difficult to prove.

There exist examples where $L^\sim = \{0\}$, although L is infinite dimensional.

Example : X : non empty point set ,

μ : countably additive measure in X such that $0 < \mu(X) < \infty$,

L : set of all real (finite valued) μ-measurable functions on X, with identification of functions, which are μ-almost equal.

Then L is a RIESZ space, $L^\sim = \{0\}$.

We take the LEBESGUE measure in $(-\infty, +\infty)$, we take $0 < p < 1$.

L : all LEBESGUE measurable real functions $f(x)$ on $(-\infty, +\infty)$

such that $\int_{-\infty}^{+\infty} |f(x)|^p dx$ is finite $\Longrightarrow$ L : RIESZ space $\Longrightarrow L^{\sim} = \{0\}$.

<u>Definition</u> : L is a RIESZ space. If to every $f \in L$ is assigned a real number $\rho(f)$ such that

(i) $0 \leqslant \rho(f) < \infty$ and $\rho(f) = 0$ iff $f = 0$,

(ii) $\rho(f_1 + f_2) \leqslant \rho(f_1) + \rho(f_2)$,

(iii) $\rho(af) = |a| \, \rho(f)$ for every real a,

(iv) (compatibility of order and norm) :
if $|f| \leqslant |g|$, then $\rho(f) \leqslant \rho(g) \Longrightarrow$ (iv) equiv. to $\begin{cases} \rho(f) = \rho(|f|) \\ \text{if } 0 \leqslant u \leqslant v, \\ \text{then } \rho(u) \leqslant \rho(v), \end{cases}$

then ρ is called a RIESZ norm, and L is now called a normed RIESZ space.

If $g = |f|$, (iv) is satisfied, hence $\begin{cases} & \rho(|f|) \leqslant \rho(f) \\ \text{and} & \\ & \rho(f) \leqslant \rho(|f|) \end{cases}$

Many examples of normed spaces taken in functional analysis are RIESZ spaces.

$C([0,1])$: $\rho(f) = \max\,(|f(x)| : 0 \leqslant x \leqslant 1)$ is a RIESZ norm.

L^p : $1 \leqslant p < \infty$: $\rho(f) = \left(\int_X |f(x)|^p \, d\mu \right)^{1/p}$ is a RIESZ norm.

L^∞ : $\rho(f) = \text{ess. sup}\,(|f(x)| : x \in X)$: RIESZ norm.

(These spaces cover a great part of functional analysis).

<u>Definition</u> : L is a <u>normed</u> RIESZ space, L^* is the space of all norm bounded linear functionals φ with:

$$||\varphi|| = \sup\,(|\varphi(f)| : \rho(f) \leqslant 1)$$

Here is no order, this definition is valid for a normed linear space. As examples of normed spaces we have the BANACH spaces.

L does not only consist of the null-element $\Longrightarrow$

L^* also does not only consist of the null-element.

The dimension of L^* is never less than the dimension of L.

<u>Theorem</u> : L : normed RIESZ space.

(i) L^* is an ideal in $L^\sim$, not necessarily a band.

(ii) If L is norm complete (i.e. L is a BANACH space), then $L^* = L^\sim$.
(Every CAUCHY sequence has a limit in L).
It is true that any element of L^* is an element of $L^\sim$, or $L^* \subset L^\sim$.

<u>Part of the proof</u> : We have to prove that if $\varphi \in L^*$, then $\varphi \in L^\sim$. To prove is that, for any $u \in L^+$, $\sup (|\varphi(f)| : |f| \leq u)$ is finite.

Since, $|f| \leq u \Longrightarrow \rho(f) \leq \rho(u)$, we have that

if $|f| \leq u$, then $|\varphi(f)| \leq ||\varphi|| \, \rho(f) \leq ||\varphi|| \, \rho(u)$

$$\Longrightarrow \sup (|\varphi(f)| : |f| \leq u) \leq ||\varphi|| \, \rho(u), \text{ finite.}$$

Is L^* an ideal?

We have to prove that if $\varphi \in L^*$, and $|\psi| \leq |\varphi|$, $\psi \in L^*$.

For φ, $|\varphi|$, we have

$$||\varphi|| = || \, |\varphi| \, ||$$

The continuation of the proof is easy.

We consider now the theory of bounded linear functionals.

It is not difficult to prove that any ideal in a DEDEKIND complete RIESZ space, is, by itself, a DEDEKIND complete RIESZ space $:\Longrightarrow L^*$, by itself : DEDEKIND complete RIESZ space.

$$\left.\begin{array}{ll} & L^{\sim} = L^{\sim}_{c} \oplus L^{\sim}_{s} \\ \text{Define :} & L^{*}_{c} = L^{\sim}_{c} \cap L^{*} \\ \text{and} & L^{*}_{s} = L^{\sim}_{s} \cap L^{*} \end{array}\right\} \Longrightarrow L^{*} = L^{*}_{c} \oplus L^{*}_{s}$$

Similarly, $L^{\sim} = L^{\sim}_{n} \oplus L^{\sim}_{sn}$ implies that

$$L^{*} = L^{*}_{n} \oplus L^{*}_{sn}$$

Examples

1) L^{p} spaces : e.g. with respect to the LEBESGUE measure; in the real line.

Hence , for $1 \leq p < \infty$, then

$$\rho(f) = \left(\int_{-\infty}^{+\infty} |f(x)|^{p} \, dx \right)^{1/p} : L^{\sim} = L^{*} = L^{*}_{c} = L^{*}_{n}$$

Every element is an integral, the only singular functional is the null functional.

2) L^{∞} , for $p = \infty$:

$$\rho(f) = \text{ess. sup } (|f(x)| : -\infty < x < +\infty) : L^{\sim} = L^{*},$$

L^{*}_{c} and L^{*}_{s} are both proper subspaces of L^{*}, $L^{*}_{s} \neq \{0\}$; $L^{*}_{sn} \neq \{0\}$.

L^{*} : isometric with L^{q} $(p^{-1} + q^{-1} = 1)$ for $1 \leq p < \infty$;

L^{*}_{c} : isometric with L^{1} in the case that $L = L^{\infty}$.

3) $C([0,1])$: $L^{\sim} = L^{*} = L^{*}_{s}$

Every linear functional is singular; there are no integrals except the null functional.

Last definition : The norm ρ is called normal norm, if for any downwards directed set, $u_{\tau} \downarrow 0$, we have $\inf \rho(u_{\tau}) = 0$.

<u>Theorem</u> : ρ normal norm $\Longleftrightarrow L^* = L^*_n$

Proof of $\Longrightarrow$ Assume ρ normal norm, and let $\varphi \in L^*$. To prove is that, for any $u_\tau \downarrow 0$, $\inf |\varphi(u_\tau)| = 0$.

$$|\varphi(u_\tau)| \leq ||\varphi||\ \rho(u_\tau) \Longrightarrow \inf |\varphi(u_\tau)| \leq ||\varphi||\ \inf \rho(u_\tau) = 0 .$$

Proof of $\Longleftarrow$ founded upon HAHN - BANACH extension theorem.

This part is not elementary.

Remark : In $L^p (1 \leq p < \infty)$ ρ is a normal norm, but not in L^∞, also not in $C([0,1])$.

<u>Theorem</u> : We consider the following properties in a normed RIESZ space L, ρ : RIESZ norm.

(i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii)

- (i) $L^* = L^*_n$ (i.e., ρ normal)
- (ii) Every norm closed ideal is a band.
  (It is easy to prove that in every normed RIESZ space, every band is always norm closed)
- (iii) In L^* every band is weak* closed.
  (It is easy to prove that in every normed RIESZ space, every weak* closed linear subspace is always a band)

Some RIESZ spaces have the property (i), but not all RIESZ spaces have this property.

This theorem was proved by (T. ANDO , W.A.J. LUXEMBURG, A.C. ZAANEN).

It can be proved that this theorem is a particular case of a more general theorem, and that theorem is true in any RIESZ space.

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To disseminate knowledge is to disseminate prosperity — I mean general prosperity and not individual riches — and with prosperity disappears the greater part of the evil which is our heritage from darker times.

Alfred Nobel

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