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TWO THEOREMS IN THE APPROXIMATION OF FUNCTIONS OF TWO VARIABLES BY POLYNOMIALS OF THE BERNSTEIN-TYPE

by

P.V. LAMBERT
(University of Louvain)

1964

Joint Nuclear Research Centre
Ispra Establishment
Scientific Information Processing Centre (CETIS)

Work performed under Euratom contract No. 011-61-1 CETB with the University of Louvain

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This paper studies the approximation of functions of two variables in the triangle

$$S = [(x,y);\ \ o \leqslant x,\ \ o \leqslant y,\ \ x+y \leqslant 1]$$

by polynomials constructed by assigning the trinomial weights

$$p_{n,k,m}(x,y) = \frac{n!}{k!\, m!\, (n-h-m)!}\, x^k y^m (1-x-y)^{n-k-m}$$

to the functionvalues at the points

$$\left(\frac{k}{n}, \frac{m}{n}\right), \quad \begin{cases} k = o, \quad 1, \ldots, \quad n; \quad m = o, \quad 1, \ldots, \quad n; \\ k + m \leqslant n; \end{cases}$$

this being done for each fixed $n = 1, 2, \ldots\ldots$

Using the C-norm, we first establish an estimation formula of the approximation degree in function of n and afterwards prove an asymptotic relation for functions with 2nd order differential.

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36e Jaargang, Aflevering III (Februari 1963)

TWO THEOREMS IN THE APPROXIMATION OF FUNCTIONS OF TWO VARIABLES BY POLYNOMIALS OF THE BERNSTEIN-TYPE (1)

Pol V. LAMBERT

(University of Louvain, Seminarium Prof. Florin)

I. INTRODUCTION

This paper studies the approximation of functions of two variables in the triangle

$$S = \{(x, y);\ 0 \leqslant x, 0 \leqslant y, x + y \leqslant 1\}$$

by two-dimensional polynomials of the Bernstein-type.

For every positive integer n, we consider the trinomial weights defined for every (x, y) in S by

$$(1.1.1)\quad \begin{cases} p_{n,k,m}(x, y) = \dfrac{n!}{k!\,m!\,(n-k-m)!} . x^k . y^m . (1-x-y)^{n-k-m} \\ k = 0, 1, \ldots, n;\ m = 0, 1, \ldots, n;\ k+m \leqslant n. \end{cases}$$

It is known that for every point (x, y) in S

$$(1.1.2)\quad 0 \leqslant p_{n,k,m}(x, y) \leqslant 1, \quad \sum_{0 \leqslant k+m \leqslant n} p_{n,k,m}(x, y) = 1.$$

This probability distribution over the points $\left(\frac{k}{n}, \frac{m}{n}\right)$ of S has similar convergence propeities (for $n \to \infty$) as the one-dimensional Bernoulli distribution given for every x in $[0, 1]$ by

$$(1.1.3)\qquad p_{n,k}(x) = \frac{n!}{k!\,(n-k)!} \cdot x^k \cdot (1-x)^{n-k}$$

cfr [1] pp. 3-4.

(1) This research was sponsored by the *European Commission for Atomic Energy* under contract 011/61/1 DOB.

Over the set of functions f defined and finite in S we define for every positive integer n the linear operator B_n such that

$$(1.1.4) \quad B_n[f(x, y)] = \sum_{0 \leqslant k+m \leqslant n} f\left(\frac{k}{n}, \frac{m}{n}\right) . p_{n,k,m}(x, y) .$$

A convergence theorem (cfr. theorem 3.1) is given in [1] p. 51. Our aim is to prove first an estimation formula of the approximation degree and afterwards an asymptotic relation for functions with 2nd derivatives which generalizes Woronowskaja's result to the two-dimensional case.(*)

II. Lemmas

We need first some estimation formulae which are similar to the one-dimensional case because of equality-relations existing between the moments $P_{n,s}(x, y)$ and $Q_{n,s}(x, y)$ defined by

$$(2.0.1) \quad \begin{cases} P_{n,s}(x,y) = \displaystyle\sum_{0 \leqslant k+m \leqslant n} (k-nx)^s . p_{n,k,m}(x,y) \\ Q_{n,s}(x,y) = \displaystyle\sum_{0 \leqslant k+m \leqslant n} (m-ny)^s . p_{n,k,m}(x,y) \quad s=0,1,2,\ldots. \end{cases}$$

and the one-dimensional moments $T_{n,s}(x)$ defined by

$$(2.0.2) \quad T_{n,s}(x) = \sum_{k=0}^{n} (k-nx)^s . p_{n,k}(x); \qquad 0 \leqslant x \leqslant 1 .$$

We have indeed :

Lemma 2.1

For every $s=0, 1, \ldots, P_{n,s}(x, y)$ is independent of y, $Q_{n,s}(x, y)$ is independent of x and :

$$(2.1.1) \qquad P_{n,s}(x, y) = T_{n,s}(x) ; \; Q_{n,s}(x, y) = T_{n,s}(y).$$

(*) This study was part of the author's dissertation requirements to acquire his academic degree at the University of Louvain, Belgium, and he expresses his deepest feelings of gratitude to Prof. H. Florin, Dr. M. Neuts and Dr. The Tjoe Tie for valuable comments.

Proof :

We only need to examine $P_{n,s}(x, y)$. We have :

$$P_{n,s}(x, y) = \sum_{0 \leqslant k+m \leqslant n} (k-nx)^s \, p_{n,k,m}(x, y) =$$

$$= \sum_{k=0}^{n} (k-nx)^s \cdot \frac{n!}{k!\,(n-k)!} \cdot x^k \cdot$$

$$\cdot \sum_{m=0}^{n-k} \frac{(n-k)!}{m!\,(n-k-m)!} \cdot y^k . (1-x-y)^{n-k-m} .$$

But

$$\sum_{m=0}^{n-k} \binom{n-k}{m} . y^m . (1-x-y)^{n-k-m} = [y + (1-x-y)]^{n-k} =$$
$$= (1-x)^{n-k}.$$

This yields the result.

COROLLARIES :

We now recall (6) p. 15 in [1], i.e.

(2.2.1) $\quad 0 \leqslant T_{n,2s}(x) \leqslant A(s) \,.\, n^s; \; 0 \leqslant x \leqslant 1; \qquad s = 0, 1, 2, \ldots,$

A being a constant depending only on s. So by (2.2.1) and lemma 2.1 follows

(2.2.2) $\quad 0 \leqslant P_{n,2s}(x,y) \leqslant A(s) \,.\, n^s; \; (x,y) \in S; \qquad s = 0, 1, 2, \ldots,$

and similar results for $Q_{n,2s}(x, y)$.

We now have the following estimation formula for every $\delta > 0$ and (x, y) in S :

(2.2.3) $$\sum_{\substack{0 \leqslant k+m \leqslant n \\ |k/n-x| \geqslant \delta}} p_{n,k,m}(x, y) \leqslant \frac{A(s)}{n^s \,.\, \delta^{2s}}. \qquad (*)$$

In order to prove this, we first notice that $\left|\frac{k}{n} - x\right| \geqslant \delta$ implies $(k-nx) \neq 0$. Then by (1.1.2) and (2.2.2) we get :

$$\sum_{\substack{0 \leqslant k+m \leqslant n \\ |k/n-x| \geqslant \delta}} p_{n,k,m}(x,y) = \sum_{\substack{0 \leqslant k+m \leqslant n \\ |k/n-x| \geqslant \delta}} \frac{(k-nx)^{2s}}{(k-nx)^{2s}} \cdot p_{n,k,m}(x,y)$$

(*) $\quad |k/n-x|$ means $\left|\frac{k}{n} - x\right|$.

$$\leqslant \frac{1}{n^{2s} \cdot \delta^{2s}} \cdot \sum_{\substack{0 \leqslant k+m \leqslant n \\ |k/n - x| \geqslant \delta}} (k - nx)^{2s} \cdot p_{n,k,m}(x,y)$$

$$\leqslant \frac{P_{n,2s}(x,y)}{n^{2s} \cdot \delta^{2s}} \leqslant \frac{A(s) \cdot n^s}{n^{2s} \cdot \delta^{2s}} = \frac{A(s)}{n^s \cdot \delta^{2s}}, \text{ q.e.d.}$$

For $s = 1$, since $P_{n,2}(x, y) = P_{n,2}(x) = nx(1-x)$, (2.2.3) yields

(2.2.4) $$\sum_{\substack{0 \leqslant k+m \leqslant n \\ |k/n - x| \geqslant \delta}} p_{n,k,m}(x,y) \leqslant \frac{nx(1-x)}{n^2 \delta^2} \leqslant \frac{1}{4n\delta^2}.$$

Now for a fixed $k > 0$, we can find a positive integer s and a real number α in the open interval (0, 1/2) satisfying the relation $s(1-2\alpha) = k$. Then for $\delta = n^{-\alpha}$ (2.2.3) implies :

(2.2.5) $$\sum_{\substack{0 \leqslant k+m \leqslant n \\ |k/n - x| \geqslant n^{-\alpha}}} p_{n,k,m}(x,y) \leqslant \frac{A(s)}{n^{s(1-2\alpha)}} = \frac{C(\alpha,k)}{n^k}.$$

By starting from the moment $Q_{n,2s}(x, y)$ similar results can of course be obtained where m and y take the place of k and x.

III. Main theorems

We state first a known result ([1] p. 51).

Theorem 3.1

Let f and $B_n f$ be defined in S. Then

(3.1.1) $$\lim_{n \to \infty} B_n[f(x, y)] = f(x, y)$$

in every point (x, y) of S where f is continuous. The convergence is uniform if f is continuous in S.

Definitions

1) For a function f defined in S, we use the following oscillation ($\delta_1 > 0$, $\delta_2 > 0$)

(3.1.2) $$\omega(\delta_1, \delta_2) = \sup_{\substack{|x-x'| < \delta_1, |y-y'| < \delta_2 \\ (x,y) \in S, (x',y') \in S}} |f(x', y') - f(x, y)|.$$

2) We now denote as in [1] p. 20 $\lambda(x_1, x_2, \delta)$ as the greatest integer contained in $\frac{|x_1 - x_2|}{\delta}$, $\delta > 0$. Similar definition for $\lambda(y_1, y_2, \delta)$. We are now able to prove the following estimation theorem.

THEOREM 3.2

Let f and $B_n f$ be defined in S. Then for every point (x, y) of S we have :

$$(3.2.1) \qquad |B_n[f(x, y)] - f(x, y)| \leqslant \frac{3}{2} \cdot \omega(n^{-1/2}, n^{-1/2}).$$

Proof

We have by our definitions and by (1.1.2)

$$(3.2.2) \qquad |B_n[f(x, y)] - f(x, y)| \leqslant$$
$$\leqslant \sum_{0 \leqslant k+m \leqslant n} \left| f\left(\frac{k}{n}, \frac{m}{n}\right) - f(x, y) \right| \cdot p_{n,k,m}(x, y).$$

Any interval $\left[\frac{k}{n}, x\right]$ can be divided into $\left[\lambda\left(\frac{k}{n}, x, \delta_1\right) + 1\right]$ subintervals the length of each being less than δ_1. Let $\left[\lambda\left(\frac{k}{n}, x, \delta_1\right) + 1\right] = N$ and call $x_{N-1}, x_{N-2}, \ldots, x_1$ the interior subdivision points. In the same way let $\left[\lambda\left(\frac{m}{n}, y, \delta_2\right) + 1\right] = M$ and call $y_{M-1}, y_{M-2}, \ldots, y_1$, the interior subdivision points of an interval $\left[\frac{m}{n}, y\right]$. Let us now take te case $N \geqslant M$. (The case $N < M$ would be similar). We then have :

$$\left| f\left[\frac{k}{n}, \frac{m}{n}\right] - f(x, y) \right| \leqslant \left| f\left(\frac{k}{n}, \frac{m}{n}\right) - f(x_{N-1}, y_{M-1}) \right| +$$
$$+ |f(x_{N-1}, y_{M-1}) - f(x_{N-2}, y_{M-2})| + \ldots +$$
$$|f(x_{N-M+1}, y_1) - f(x_{N-M}, y)| + |f(x_{N-M}, y) - f(x_{N-M-1}, y)| +$$
$$+ \ldots + |f(x_1, y) - f(x, y)|,$$

where some terms can vanish.

This means that in every case ($N \geqslant M$ or $N < M$) we have :

$$(3.2.3) \qquad \left| f\left(\frac{k}{n}, \frac{m}{n}\right) - f(x, y) \right| \leqslant$$

$$\leqslant \left[\max \left[\lambda \left(\frac{k}{n}, x, \delta_1 \right) . \lambda \left(\frac{m}{n}, y, \delta_2 \right) \right] + 1 \right] . \omega(\delta_1, \delta_2)$$

Let $\lambda_{k,m}(x, y) = \max \left[\lambda \left(\frac{k}{n}, x, \delta_1 \right), \lambda \left(\frac{m}{n}, y, \delta_2 \right) \right]$. Then (3.2.2) and (3.2.3) imply :

$$(3.2.4) \qquad |B_n[f(x, y)] - f(x, y)| \leqslant \omega(\delta_1, \delta_2) .$$

$$. \sum_{0 \leqslant k+m \leqslant n} [1 + \lambda_{k,m}(x, y)] . p_{n,k,m}(x, y)$$

$$\leqslant \omega(\delta_1, \delta_2) . \left[1 + \sum_{\lambda_{k,m} \geqslant 1} \lambda^2_{k,m}(x, y) . p_{n,k,m}(x, y) \right] \leqslant$$

$$\leqslant \omega(\delta_1, \delta_2) . \left[1 + \sum_{\substack{0 \leqslant k+m \leqslant n \\ |k/n - x| \geqslant \delta_1}} \frac{\left(\frac{k}{n} - x \right)^2}{\delta_1^2} . p_{n,k,m}(x, y) + \right.$$

$$\left. + \sum_{\substack{0 \leqslant k+m \leqslant n \\ |m/n - y| \geqslant \delta_2}} \frac{\left(\frac{m}{n} - y \right)^2}{\delta_2^2} . p_{n,k,m}(x, y) \right]$$

$$\leqslant \omega(\delta_1, \delta_2) . \left[1 + \frac{1}{n^2 \delta_1^2} . P_{n,2}(x, y) + \frac{1}{n^2 \delta_2^2} . Q_{n,2}(x, y) \right] \leqslant$$

$$\leqslant \omega(\delta_1, \delta_2) . \left[1 + \frac{1}{4n\delta_1^2} + \frac{1}{4n\delta_2^2} \right].$$

If we now let $\delta_1 = \delta_2 = n^{-1/2}$, we get (3.2.1), q.e.d.

Remark :

This theorem implies as a particular case the 2nd part of theorem 3.1.

We now give the asymptotic formula mentioned in the introduction and its proof.

THEOREM 3.3

Let f be defined and bounded in S. In each point (x, y) of S where f has a 2^{nd}-order differential we have

$$(3.3.1) \qquad B_n[f(x, y)] = f(x, y) + \frac{x(1-x)}{2n} \cdot \frac{\partial^2 f}{\partial x^2} + \frac{y(1-y)}{2n} \cdot \frac{\partial^2 f}{\partial y^2} - \frac{xy}{n} \cdot \frac{\partial^2 f}{\partial x \partial y} + \frac{\varrho_n}{n}, \qquad \text{where } \lim_{n\to\infty} \varrho_n = 0.$$

If all the 2^{nd}-order partial derivatives of f are continuous in S, then :

$$\lim_{n\to\infty} \varrho_n = 0, \text{ uniformly in } S.$$

Proof :

By our assumptions $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ and

$$(3.3.2) \qquad f\left(\frac{k}{n}, \frac{m}{n}\right) = f(x, y) + \left(\frac{k}{n} - x\right) \cdot \frac{\partial f}{\partial x} + \left(\frac{m}{n} - y\right) \cdot \frac{\partial f}{\partial y} + \frac{1}{2}\left(\frac{k}{n} - x\right)^2 \cdot \frac{\partial^2 f}{\partial x^2} + \frac{1}{2}\left(\frac{m}{n} - y\right)^2 \cdot \frac{\partial^2 f}{\partial y^2} + \left(\frac{k}{n} - x\right) \cdot \left(\frac{m}{n} - y\right) \cdot \frac{\partial^2 f}{\partial x \partial y} + \varphi_1\left(\frac{k}{n}, x, y\right) \cdot \left(\frac{k}{n} - x\right)^2 + \varphi_2\left(\frac{m}{n}, x, y\right) \cdot \left(\frac{m}{n} - y\right)^2 + \varphi_3\left(\frac{k}{n}, \frac{m}{n}, x, y\right) \cdot \left(\frac{k}{n} - x\right) \cdot \left(\frac{m}{n} - y\right), \text{ where}$$

$$(3.3.3) \qquad \lim_{k/n\to x} \varphi_1 = 0; \ \lim_{m/n\to y} \varphi_2 = 0; \ \lim_{\substack{k/n\to x \\ m/n\to y}} \varphi_3 = 0.$$

If we recall the identities

$$(3.3.4) \qquad \begin{cases} P_{n,1}(x, y) = Q_{n,1}(x, y) = 0; \\ P_{n,2}(x, y) = nx(1-x); \ Q_{n,2}(x, y) = ny(1-y); \\ \sum\limits_{0 \leq k+m \leq n} (k - nx) \cdot (m - ny) \cdot p_{n,k,m}(x, y) = -nxy \end{cases}$$

it follows from (1.1.4) and (3.3.2) that :

$$(3.3.5) \qquad B_n[f(x, y)] = f(x, y) + \frac{x(1-x)}{2n} \cdot \frac{\partial^2 f}{\partial x^2} + \frac{y(1-y)}{2n} \cdot \frac{\partial^2 f}{\partial y^2} - \frac{xy}{n} \cdot \frac{\partial^2 f}{\partial x \partial y} + r_n, \text{ where}$$

$$(3.3.6) \qquad r_n = \sum_{0\leqslant k+m\leqslant n} p_{n,k,m}(x,y) \cdot \left[\left(\frac{k}{n}-x\right)^2 \cdot \varphi_1\left(\frac{k}{n}, x,y\right) + \right.$$

$$+ \left(\frac{m}{n}-y\right)^2 \cdot \varphi_2\left(\frac{m}{n}, x,y\right) +$$

$$\left. + \left(\frac{k}{n}-x\right) \cdot \left(\frac{m}{n}-y\right) \cdot \varphi_3\left(\frac{k}{n}, \frac{m}{n}, x, y\right)\right].$$

Since f and hence $B_n f$ are bounded in S and the partial derivatives are finite at the point (x, y), we have $|\varphi_i| \leqslant H, i = 1, 2, 3,$ uniformly when $\frac{k}{n}$ and $\frac{m}{n}$ vary in S. We also have $\left|\frac{k}{n}-x\right| \leqslant 1$ and $\left|\frac{m}{n}-y\right| \leqslant 1$ everywhere in S. On the other hand by (3.3.3) given $\varepsilon > 0$, there is a $\delta > 0$ depending only on ε and (x, y) such that $|\varphi_i| < \varepsilon, i = 1, 2, 3,$ whenever $\left|\frac{k}{n}-x\right| < \delta$ and $\left|\frac{m}{n}-y\right| < \delta$. All this and the identities (3.3.4) imply :

$$|r_n| \leqslant \frac{\varepsilon}{n^2} \cdot [nx(1-x) + ny(1-y) + nxy] +$$

$$+ 3H \cdot \left(\sum_{\substack{0\leqslant k+m\leqslant n \\ |k/n-x|\geqslant\delta}} p_{n,k,m}(x, y) + \sum_{\substack{0\leqslant k+m\leqslant n \\ |m/n-y|\geqslant\delta}} p_{n,k,m}(x, y)\right)$$

Let $\delta = n^{-\alpha}$, $0 < \alpha < \frac{1}{2}$, α fixed; then ε depends on n and $\lim_{n\to\infty} \varepsilon(n)=0$.

By now using (2.2.5) with $k > 1$ we get

$$|r_n| \leqslant \frac{3}{4} \cdot \frac{\varepsilon(n)}{n} + 6H \cdot \frac{C(\alpha, k)}{n^k} = \frac{\frac{3}{4}\varepsilon(n) + \zeta(n)}{n}$$

where $\lim_{n\to\infty} \zeta(n) = 0$. This proves the first part of the theorem. When the 2^{nd}-order partial derivatives are continuous everywhere in S, and hence bounded, the number H can be chosen the same everywhere in S. On the other hand, given $\varepsilon > 0$, we can now also choose the same δ for all points (x, y) of S. This implies that the estimation of r_n becomes uniform in S, and so proves the 2^{nd} part of the theorem.

General remark

All the results of this paper can easily be generalized to functions of n variables ($n = 1, 2, 3, \ldots$).

June 1961

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