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STATISTICAL FLUCTUATIONS AND THEIR CORRELATION IN REACTOR NEUTRON DISTRIBUTIONS

by W. MATTHES

AUGUST 1962

JOINT NUCLEAR RESEARCH CENTER
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Reactor Physics Department

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The following work is divided in three main chapters. In parts A and B we give for the convenience of the reader, a short summary of definitions and relations concerning correlation functions. Part C is devoted to a generalization of a method elaborated by F. DE HOFFMANN, for determining the physical explanation, as to how correlation between two count-impulses can arise.

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CONTENTS

STATISTICAL FLUCTUATIONS AND THEIR CORRELATION IN REACTOR NEUTRON DISTRIBUTIONS

SUMMARY

The purpose of this paper is to establish a method for the determination of the autocorrelation function of the output of a detector placed in a reactor.

The analytical formulas derived for the autocorrelation function contain the physical parameters characterizing the kinetic behaviour of the neutrons in the reactor system. As the autocorrelation function can easily be measured we have a useful tool for the experimental determination of these parameters.

The following work is divided into three main chapters. In parts A and B we give, for the convenience of the reader, a short summary of definitions and relations concerning correlation functions. Part C is devoted to a generalization of a method elaborated by F. De Hoffmann, for determining the physical explanation, as to how correlation between two count-impulses can arise.

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1 — DEFINITION OF THE CORRELATION FUNCTIONS

We consider a reactor as being in a steady state with a small neutron detector D_a placed at some point in it.

We assume that each time a neutron gives rise to a count, a very sharp (voltage or current) impulse occurs at the detector output. The height h of the impulse is taken as constant during the infinitesimal duration dt of the impulse. h goes to infinity if dt goes to zero in such a way that the area $q = h.\ dt$ under the impulse (which can be, for instance, the total charge delivered from the detector through this pulse) remains finite.

If we consider the electrical quantity (voltage or current) at the detector output as a function of time, this function $x(t)$ will consist of a sequence of pulses. It is assumed that the distribution of the heights of the impulses is independent of the distribution of the pulse arrival time-points. This is due to the fact that a neutron, when entering the detector, can release different amounts of charges according to a certain probability distribution.

If the counter is opened for a very long time, a large piece of the function $x(t)$ is obtained. We call this piece a record and denote it by $x^k(t)$. If we repeat this procedure very often, we get an ensemble of such records $\{x^k(t)\}$, $k = 1, 2 \ldots$ where we can approximate the range of t by: $-\infty < t < +\infty$.

Such an ensemble of time functions $\{x^k(t)\}; -\infty < t < +\infty : k = 1, 2, 3 \ldots$ (perhaps even uncountable) is called a stochastic, or a random process.

The form of the function $x^k(t)$ varies at random from experiment to experiment. For any particular value of k the function $x^k(t)$ is a determinate function. If the time t is fixed, the function $x^k(t)$ becomes a random variable which we will call $x(t)$.

The expectation value of the random variable $x(t)$ is given by averaging $x^k(t)$ over the ensemble of the records:

$$E\{x(t)\} = \langle x^k(t) \rangle_k \tag{1}$$

and gives us the mean value of the function $x(t)$ at the fixed time-point t.

Instead of considering the function $x(t)$ only at one time-point t, we consider the values of $x(t)$ simultaneously at two time-points t_1 and t_2 and compute the second moment given as the expectation value of $x(t_1)\,x(t_2)$:

$$E\{x(t_1)x(t_2)\} = <x^k(t_1)x^k(t_2)>_{Av.k} \tag{2}$$

If this second moment is invariant against a displacement of the time-points t_1, t_2 over the time axis, keeping their relative position fixed, then the random process is called *stationary.*

For a stationary random process the second moment (2) is therefore only a function of the length of the interval between the two time-points $\tau = |t_2 - t_1|$ and we write:

$$\Phi(\tau) = <x^k(t_1)x^k(t_2)>_k \tag{3}$$

The second *central* moment, defined by (we write x_i for $x(t_i)$)

$$E\{(x_1 - E\{x_1\})(x_2 - E\{x_2\})\} = <(\overset{k}{x_1} - <\overset{k}{x_1}>_k)(\overset{k}{x_2} - <\overset{k}{x_2}>_k)>_k \tag{4}$$

is called the autocorrelation function for the random process $\{x^k(t)\}$ and for a stationary random process this takes the form:

$$\Gamma(\tau) = \Phi(\tau) - E\{x_1\}E\{x_2\} \tag{5}$$

or

$$\Gamma(\tau) = \Phi(\tau) - E\{x\}^2 \tag{6}$$

as for a stationary random process the first moment (mean value of the random variable) $<x^k(t)>_k$ is also independent of t.

For our special case of a random process (series of δ pulses) it is very easy to calculate the autocorrelation function.

We denote by

$$P'(\tau)\,d\tau\,dt$$

the probability of having a pair of pulses separated by the time distance τ where the one impulse appears in dt at time t and the other pulse appears in the time element $d\tau$ at the later time $t + \tau$.

According to (3) we have for the second moment

$$\Phi(\tau) = <q>^2 P'(\tau) \tag{7}$$

This is only valid for $\tau > 0$; for $\tau = 0$ we have

$$\Phi(0) = <x^k(t)^2> = <q^2> P\delta(\tau) \tag{8}$$

where P is the mean counting rate of the detector.

The second moment for the whole range of τ is therefore:

$$\Phi(\tau) = <q^2> P\delta(\tau) + <q>^2 P'(\tau) \tag{9}$$

As the pulse pairs separated by τ may be correlated or uncorrelated, we can split $P'(\tau)$ into two parts:

$$P'(\tau) = P_n(\tau) + P(\tau) \tag{10}$$

where

$P_n(\tau)$ is the probability that the two pulses of the pair are uncorrelated, that means

$$P_n(\tau) = P\,P \quad \text{and}$$

$P(\tau)$ is the probability that the two pulses of the pair are correlated.

The uncorrelated pulses give us in:

$$<q>^2 P P = <x^k(t)>^2_k \tag{11}$$

the square of the mean value of the detector output and formula (6) gives us for the autocorrelation function:

$$\Gamma(\tau) = <q^2> P\delta(\tau) + <q>^2 P(\tau) \tag{12}$$

In the case of two detectors placed at different points in the reactor, we can proceed in a similar way. The cross correlation between the detector outputs of the detectors D_a and D_b is now defined by the ensemble average:

$$\Phi_{ab}(\tau) = <\overset{k}{x}_a(t)\, \overset{k}{x}_b(t+\tau)>_k \tag{13}$$

corresponding to the two random processes $\{\overset{k}{x}_a(t)\}$ and $\{\overset{k}{x}_b(t)\}$.

Analog to $P' = P'_{aa}$ we define by:

$$P'_{ab}(\tau)\, d\tau\, dt$$

the probability of having a pair of pulses separated by the time distance τ where the one impulse appears at the output of D_a in the time element dt at t and the other impulse appears at the output of D_b in the time element $d\tau$ at the later time $t+\tau$, and we find:

$$\Phi_{ab}(\tau) = <q_a>\ <q_b>\ P'_{ab}(\tau)$$

The separation of P'_{ab} *in* $P'_{ab} = P^u_{ab} + P_{ab}$, where:

P^u_{ab} is the probability that the two impulses of the pair are uncorrelated; that means $P^u_{ab} = P_a . P_b$ and

P_{ab} is the probability that the two impulses of the pair are correlated gives us in $<q_a>\ <q_b> P_a P_b = <\overset{k}{x}_a>_k <\overset{k}{x}_b>_k$ the product of the mean values of the counter outputs. Subtracting this product form Φ_{ab} gives us another form for the cross-correlation function:

$$\Gamma_{ab}(\tau) = <q_a>\ <q_b>\ P_{ab}(\tau) \tag{14}$$

The Fourier transform of the autocorrelation function (second moment) has a simple physical meaning. For this we take the Fourier transform to be given by

$$\chi(\omega) = \int_{-\infty}^{+\infty} \Phi(\tau) e^{i\omega\tau}\, d\tau$$

$$\Phi(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \chi(\omega)\, e^{-i\omega\tau}\, d\omega \tag{15}$$

where we take $\Phi(\tau)$ to be an even function of τ, as according to (3) $\Phi(\tau)$ depends only on the absolute value of the difference of the two time-points $|t_2 - t_1|$.

We see that:

$$\Phi(0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \chi(\omega)\, d\omega = <x^k(t)^2>_k \tag{16}$$

If $x(t)$ is a voltage or a current for a resistor of one ohm, the mean square of $x(t)$ is the *mean power* given to the resistor. Consequently $\chi(\omega)$ can be interpreted as the power density spectrum of $x(t)$. The total power given to the resistor from the spectral components of $x(t)$ with frequencies between, for instance, ω_a and ω_b would be given by:

$$L_{ab} = \frac{1}{\pi}\int_a^b \chi(\omega)\, d\omega \tag{17}$$

For the special form (9) using (11) for the second moment, equation (13) would lead to:

$$\chi(\omega) = <q^2>P + 2\pi <x^k(t)>^2_k \delta(\omega) + <q>^2 P(\omega) \tag{18}$$

where we have put:

$$2\pi\, \delta(\omega) = \int_{-\infty}^{+\infty} e^{i\omega\tau}\, d\tau$$

$$P(\omega) = \int_{-\infty}^{+\infty} e^{i\omega\tau} P(\tau)\, d\tau \tag{19}$$

We see that the first term

$<q^2>P$ is constant over the whole frequency range and corresponds to the white noise present in the pulse series,

that the second term

$2\pi <x^k(t)>^2_k \delta(\omega)$ gives only a contribution for $\omega = 0$ in the form:

$\chi(0)\, d\omega = 2\pi <x^k(t)>^2_k$ and corresponds to the power given to the resistor by the *d-c*-component of $x(t)$ and

that the third term $<q>^2 P(\omega)$ is a function of ω and is due to the correlation between pulse-pairs.

As the correlation function is given by

$$\Gamma(\tau) = \Phi(\tau) - E\{x(t)\}^2 = <[x^k(t) - \bar{x}][x^k(t+\tau) - \bar{x}]>_k$$

the Fourier transform of $\Gamma(\tau)$ gives us the continuous power density spectrum (mean value of $x(t)$ subtracted !) and we obtain:

$$\Phi(\omega) = <q^2>P + <q>^2 P(\omega)$$

where (20)

$$\Phi(\omega) = \int_{-\infty}^{+\infty} \Gamma(\tau)\, e^{i\omega\tau} d\tau$$

We therefore arrive at the important result: The power density spectrum and the autocorrelation function are coupled by the Fourier transformation. An analogous formulation defines the "power density" for the cross-correlation, which has no immediate physical meaning.

1.1 — *Effect of linear systems on random inputs*

We know, that the output of a linear system is given by:

$$y(t)=\int_o^t x(\xi)\,R(t-\xi)\,d\xi \tag{21}$$

where $x(\xi)$ is the input at time ξ and $R(t)$ is the output at time t if at time 0 a δ-impulse with area 1 appeared at the input.

We shall first explain relation (21) if a random input $x(t)$ is in action.

We shall assume that a determinate function $R(t)$ is given and the integration interval (o,t) fixed. In that case operation (21) gives a certain numerical value of $y(t)$ for every ensemble $x^k(t)$ of the random variable $x(t)$:

$$y^k(t)=\int_o^t x^k(\xi)\,R(t-\xi)\,d\xi \tag{22}$$

This value varies in a random manner from one realization of $x^k(t)$ to another. This means that for given $R(t)$ and integration limits the integral transformation (21) is a random function of the realization number k of the random function $x^k(t)$.

As we can interpret an integral as a limit of a sum, we can interchange averaging over k and integration and find in this way, for instance, for the first moment of $y(t)$ (the mean value of the output):

$$<y^k(t)>_k=\int_o^t <x^k(\xi)>_k R(t-\xi)\,d\xi \tag{23}$$

For a stationary random process we obtain (as the process is stationary, the input is present since $t=-\infty$)

$$<y> = <x>\int_{-\infty}^t R(t-\xi)\,d\xi = <x>\int_o^\infty R(\xi)\,d\xi \tag{24}$$

For our case (series of impulses) we found (11)

$$<x> = <q>\,P$$

and therefore have for the mean value of the output of the linear system

$$<y> = <q>\,P\int_o^\infty R(t)\,dt \tag{25}$$

In the same way, we can compute the second moment (autocorrelation) of the output:

$$<y^k(t_1)y^k(t_2)>_k=\int_o^{t_1} d\xi_1\int_o^{t_2} d\xi_2\,<x^k(\xi_1)x^k(\xi_2)>R(t_1-\xi_1)R(t_2-\xi_2) \tag{26}$$

For a stationary process the second moments for output and input depend only on the absolute values of the time differences $|t_1-t_2|$ resp $|\xi_1-\xi_2|$ and after some transformation equation (24) gives

$$\Phi_o(\tau)=\int_{-\infty}^{+\infty}\Phi_{kk}(t)\ \Phi_i(\tau-t)\ dt \tag{27}$$

where

$$\Phi_{kk}(t)=\int_{-\infty}^{+\infty}R(\xi)\ R(\xi+t)\ d\xi$$

and $\Phi_o(\tau)$ is the second moment for the output and

$\Phi_i(\tau)$ is the second moment for the input

For the autocorrelation functions we find the relation:

$$\Gamma_o(\tau)=\int_{-\infty}^{+\infty}\Phi_{kk}(t)\Gamma_i(\tau-t)dt \tag{28}$$

The power density spectrum of the output can be found by taking the Fourier transform of the autocorrelation function of the output and with the aid of (26) we can express this in the form:

$$\Gamma_o(\omega)=|R(\omega)|^2\Gamma_i(\omega) \tag{29}$$

where:

$\Gamma_i(\omega)$ power density spectrum of the input (Fourier transform of the input autocorrelation function according to (13))

$\Gamma_o(\omega)$ power density spectrum of the output

$$|R(\omega)|^2=|R(\omega)R^*(\omega)| \quad \text{with} \tag{30}$$

$$R(\omega)=\int_{-\infty}^{+\infty}R(\xi)e^{i\omega\xi}\,d\xi=\int_{0}^{\infty}R(\xi)\ e^{i\omega\xi}d\xi \quad \text{as}$$

$R(t)$ is equal to zero for $t<0$.

1.2 — *Random transfer systems*

We can generalize somewhat from the results of the foregoing chapter if we allow for the possibility that the response function $R(t)$ used in the integral transformation (21) is itself a random variable. By this we mean that the response function $R(t)$ is also a function of a set of parameters $(a_1, a_2, \dots a_n)$ which are random variables with a joint probability distribution function: $R(t, a_1, a_2, \dots a_n)$. For a given set of a's, this function is a determined function over the whole range of t. To find the response of a δ-unit impulse, we have therefore to draw a batch of a's out of the corresponding distribution for the a's. We take the integral transformation (22) now in the form

$$y^k(t)=\int_{-\infty}^{+\infty}x^k(t-\xi)R(\xi,a_\xi)d\xi \tag{22'}$$

where a_ξ stands for the *set* $(a_1, a_2, \dots a_n)$ and the index ξ indicates that we have chosen the values a_ξ for the δ-impulse arriving at time $(t-\xi)$.

The mean value of the output becomes:

$$<y^h(t)>_h = \int_{-\infty}^{+\infty} <x^h(t-\xi)>_h <R(\xi, a_\xi)> d\xi$$

$$<y> = <x> \int_0^\infty R_o(\xi)\, d\xi$$

where $<R(\xi, a_\xi)> = R_o(\xi)$ denotes the mean value of $R(\xi, a_\xi)$, averaged over the distribution of the random variables a_ν. For the second moment we find the relation:

$$<y^h(t+\tau)y^h(t)> = \int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} d\eta <x^h(t-\xi)x^h(t+\tau-\xi> <R(\xi, a_\xi)R(\eta, a_{\eta-\tau})>$$

and for stationary processes:

$$\Phi_o(\tau) = \int_{-\infty}^{+\infty} \Phi_i(\tau - t)\,\Phi_{hh}(t)\, dt \tag{27'}$$

$$\Phi_{hh}(t) = \int_{-\infty}^{+\infty} <R(\xi, a_\xi)\, R(\xi + t, a_{\xi+t-\tau})> d\xi$$

As the choice for the values of the a's at the time-point ξ is independent of the choice for the values of the a's at the time-point $\xi + t - \tau$ if $t \neq \tau$, we have:

$$\Phi_{hh}(t) = \int_{-\infty}^{+\infty} R_o(\xi) R_o(\xi + t)\, d\xi \qquad t \neq \tau$$

$$\Phi_{hh}(\tau) = \int_{-\infty}^{+\infty} <R(\xi) R(\xi + \tau)> d\xi$$

An analogous formulation holds good for $\Gamma_o(\tau)$. We have only to replace $\Phi_i(\tau)$ by $\Gamma_i(\tau)$.

In an example we consider a transfer system whose statistical properties are given by the following probability distribution for the output:

$p(\xi, h)dh\, d\xi$ is the probability that the system emits an impulse in the time element $d\xi$ at time ξ with an amplitude in the range $h \,..\, h + dh$ due to an initiating δ-impulse entering the system at time $\xi = 0$.

We obviously have:

$$<R(\xi)> = R_o(\xi) = \int_{-\infty}^{+\infty} p(\xi, h) h\, d\, h\, d\xi = <q>\, p(\xi)$$

with $h\,d\xi = <q> =$ mean area under the (δ) impulse, and

$$<R(\xi)R(\xi+\tau)> = <p(\xi,h_1)\,dh_1\,d\xi\;p(\xi+\tau,h_2)\,d\,h_2\,d\tau>$$
$$= <q^2>\delta(\tau)\,p(\xi) + <q>^2 p(\xi)\,p(\xi+\tau)$$

Here we make the assumption that the distribution for the height h of the impulse is independent of the distribution for the time-points of impulse-emission.

For an uncorrelated input of δ impulses (all of the same height) we obtain for the correlation function of the output:

$$\Gamma_o(\tau) = \int_{-\infty}^{+\infty} \Gamma_i(\tau - t)\,\Phi_{hh}(t)\,dt;$$
$$= S\,\Phi_{hh}(\tau)$$

because:

$\Gamma_i(\tau) = S\,\delta(\tau)$ with S as the mean rate of the arriving δ-pulses.

With the formulas for $\Phi_{hh}(\tau)$ for our special system we obtain:

$$\Gamma_o(\tau) = S <q^2>\;\delta(\tau)\int_o^\infty p(\xi)d\xi + S<q>^2\int_o^\infty p(\xi)p(\xi+\tau)\;d\xi$$

$\int_o^\infty p(\xi)d\xi$ is the mean number of pulses emitted by the system due to an initiating δ-pulse.

Then $S\int_o^\infty p(\xi)d\xi = C =$ mean rate of pulses at the output of the system.

$S\int_o^\infty p(\xi)p(\xi+\tau)\,d\xi =$ probability to have a pulse pair separated by the time distance τ at the output $[P(\tau)]$

Finally we can write:

$$\Gamma_o(\tau) = C <q^2>\;\delta(\tau) + P(\tau)<q>^2$$

We see now that we have at the output of our system a random series of δ-impulses with a correlation term. The important conclusion is that two pulses at the output are correlated only if they belong to one and the same response of the system initiatied by one of the incoming δ-impulses. Pulses at the output generated by two different incoming pulses are uncorrelated.

We now consider our system to consist of a subcritical reactor with a neutron detector. An external source injects neutrons and each source neutron starts the evolution of a neutron cloud which gives rise to a series of impulses at the detector output. In this case our result reads: two pulses at the output of our detector are correlated only if the two neutrons which gave rise to the counts belong to one and the same neutron chain initiated by one source neutron. Two count impulses arising from two neutrons belonging to two neutron chains initiated by two different source neutrons are uncorrelated.

Up to now, we have considered only small neutron detectors, which simultaneously represented the counters. In the general case, we can have many small detectors, all feeding in one counter. We allow for the possibility that the different detectors have varying degrees of efficiency for different energy regions. To account for this we divide the reactor volume into small space cells S_ν and the lethargy axis into small lethargy-intervals U_μ and imagine the detectors to be distributed over these space-lethargy intervals (ν, μ).

The correlation function for the counter output now takes the form:

$$\Gamma(\tau) = \sum_{\substack{\nu\mu \\ \nu'\mu'}} \Gamma_{\nu\mu\nu'\mu'}(\tau) \tag{31}$$

where $\Gamma_{\nu\mu\nu'\mu'}(\tau) = <q_{\nu\mu}\, q_{\nu'\mu'}> P_{\nu\mu}\, \delta(\tau)\, \delta_{\nu\mu\nu'\mu'} + <q_{\nu\mu}> \; <q_{\nu'\mu'}> P_{\nu\mu\,\nu'\mu'}(\tau)$

and the summation goes in the case of

autocorrelation:

in both index pairs over the intervals of one counter, and in the case of

cross-correlation:

in the one index pair over the intervals of the one counter and in the other index pair over the intervals of the other counter.

The corresponding power density spectrum becomes:

$$\Gamma(\omega) = \sum_{\nu\mu} <q^2{}_{\nu\mu}> P_{\nu\mu} + \sum_{\substack{\nu\mu \\ \nu'\mu'}} <q_{\nu\mu}> \; <q_{\nu'\mu'}> P_{\nu\mu\,\nu'\mu'}(\omega) \tag{32}$$

and for later use we note the Laplace transform of the correlation function:

$$\Gamma(s) = \frac{1}{2} \sum_{\nu\mu} <q^2_{\nu\mu}> P_{\nu\mu} + \sum_{\nu\mu\,\nu'\mu'} <q_{\nu\mu}> \; <q_{\nu'\mu'}> P_{\nu\mu\,\nu'\mu'}(s)$$

$$= \frac{1}{2} P\, q^2 + \bar{q}^2 P(s)$$

where:

$$P = \sum_{\nu\mu} P_{\nu\mu};\; P\,\overline{q^2} = \sum_{\nu\mu} <q^2_{\nu\mu}> P_{\nu\mu};\; P(s) = \sum_{\nu\mu\,\nu'\mu'} P_{\nu\mu\,\nu'\mu'}(s)$$

We repeat the meaning of the symbols:

$q_{\nu\mu}$ height of an impulse initiated in the detector $D_{\nu\mu}$ placed in the space cell S_ν by a neutron with a lethargy in the interval U_μ.

$P_{\nu\mu\nu'\mu'}(\tau)$ probability density for pairs of *correlated* impulses separated by the time distance τ where the first count is recorded in $D_{\nu\mu}$ produced by a neutron with a lethargy in the interval U_μ and the second count is recorded in $D_{\nu'\mu'}$ produced by a neutron with a lethargy in the interval $U_{\mu'}$.

The fact that two impulses can be correlated is due to the mechanism of the neutron chains.

A source neutron injected in the reactor generates a neutron chain. This neutron chain spreads out in space and time and two neutrons of this chain can give rise to two counts: one count in $D_{\nu\mu}$ in the time interval dt at time t and the other count in $D_{\nu'\mu'}$ (or in $D_{\nu\mu}$ again) in the time element $d\tau$ at a later time $T = t + \tau$. These two counts are correlated in that the recorded neutrons belong to the same chain. Two counts arising from two neutrons belonging to two neutron chains initiated by two different source neutrons are uncorrelated.

The measurement of the correlation functions therefore gives us, apart from some constant factors, the probability density $P_{\nu\mu\nu'\mu'}(\tau)$. If we succeed in deriving an analytical formula for $P_{\nu\mu\,\nu'\mu'}(\tau)$ in terms of the physical parameters characterizing the kinetic behaviour of the neutrons which produce the counts in $D_{\nu\mu}$ and $D_{\nu'\mu'}$, we obtain some information about these parameters by measuring the correlation function.

2 — DETERMINATION OF THE PROBABILITY DENSITY FOR CORRELATED PAIRS BY A COLLECTING DEVICE

Instead of measuring the correlation function to obtain some information about the probability density $P_{\nu\mu\,\nu'\mu'}(\tau)$, we can proceed in another way.

We lead all the charges carried with the pulses emitted at the counter output into a collecting device and denote by $Q(T)$ the total charge assembled in this device in the time interval T.

$Q(T)$ as a function of time can be represented in the following way:

$$Q(T)=\sum_{\alpha} q_{\alpha}\, Y(T-t_{\alpha}) \tag{1}$$

where $Y(T)$ is the unit step function.

We then have:

$$\overline{Q^2(T)}=\overline{q^2} < \sum_{\alpha} Y(T-t_{\alpha}) > + \bar{q}^{2} < \sum_{\alpha\,\beta} Y(t-t_{\alpha})\, Y(T-t_{\beta}) >$$

where in the first sum each term gives a one for each impulse and in the second sum we have a one for each different pair of pulses.

Therefore:

$$< \sum_{\alpha} Y(T-t_{\alpha}) > = \overline{M} = \text{mean total number of counts in the time interval } T$$

$$< \sum_{\alpha\,\beta} Y(T-t_{\alpha})\, Y(T-t_{\beta}) > = \overline{M(M-1)} = \overline{M^2}-\overline{M}$$

The mean value of the total number of ordered pairs is given by:

$$\binom{\overline{M}}{2}=\frac{\overline{M^2}-\overline{M}}{2}$$

and is made up of the mean number of uncorrelated pairs $\frac{\overline{M}^{2}}{2}$ and the mean number of correlated pairs M_c.

This gives us:

$$\overline{M^2}-\overline{M}=\overline{M}^{2}+2M_c \tag{2}$$

For the mean number of correlated pairs we have the expression:

$$M_c(\tau)=\int_0^T \int_0^{T-\tau} P(\tau)\, dt\, d\tau \tag{3}$$

$P(\tau)$ is here the probability density for a pair of pulses separated by the time distance τ.

We insert this in the expression for $\overline{Q^2}$ and use the fact that the mean value of the total charge assembled up to time T is given by:

$$\overline{Q}=\overline{M}.\bar{q}=P.T.\bar{q}.(P=\text{counting rate})$$

This leads to:

$$\overline{Q^2}-\overline{Q}^2 = T\,P\,\overline{q^2}+2\,\bar{q}^2\int_0^T (T-\tau)P(\tau)\,d\tau \tag{4}$$

We take the Laplace transform of this equation and obtain:

$$L\{\overline{Q^2}-\overline{Q}^2\} = \frac{1}{s^2}\{P\,\overline{q^2}+2\,\bar{q}^2P(s)\} = \frac{2\Gamma(s)}{s^2} \tag{5}$$

As usual, we put:

$$\overline{Q^2} = \overline{Q}^2 + G(T) + \overline{Q}$$

and have

$$\overline{Q^2}-\overline{Q}^2 = G(T)+\overline{Q}$$

$\bar{Q}$ is given by $T.P\,\bar{q} = T.I_e$ (I_e is the mean current into the collecting device). We have:

$$L\{\overline{Q^2}-\overline{Q}^2\} = G(s)+\frac{I_e}{s^2}$$

and therefore:

$$2\,\Gamma(s) = s^2G(s)+I_e \tag{5a}$$

All we need therefore, is a method to compute the function $G(T)$.

The power density spectrum $\Gamma(\omega)$ can be found from the Laplace transform $\Gamma(s)$ by:

$\Gamma(\omega) = 2$ Real $\{\Gamma(s)\}$ where s has to be replaced by $(-i\omega)$.

The equation $\frac{1}{2}s^2L\,[\overline{Q^2}-\overline{Q}^2] = \Gamma(s)$ is naturally equivalent to (by taking the inverse transformation)

$$\frac{1}{2}\frac{\partial^2}{\partial\tau^2}\,[\overline{Q^2}(\tau)-\overline{Q}^2(\tau)] = \Gamma(\tau)$$

which can immediately be seen by differentiating twice the equation (4).

This is a general result. We consider the stochastic process $\{x^k(t)\}$ and introduce the following integral transform of $x^k(t)$ as a new stochastic process:

$$Q^k(\tau) = \int_0^\tau x^k(\xi)\,d\,\xi$$

We then have:

$$\overline{Q^2}(\tau) = <Q^k(\tau)^2>_k = <\left[\int_0^\tau x^{(k)}(\xi)d\xi\right]^2>_k$$

$$= <\int_0^\tau x^k(\xi)\,d\xi\int_0^\tau x^k(\eta)\,d\eta>_k$$

$$= \int_0^\tau d\xi\int_0^\tau d\eta<x^k(\xi)x^k(\eta)>_k$$

With the definition of the correlation function this can be written:

$$<Q^k(\tau)^2>_k = \int_0^\tau d\xi \int_0^\tau d\eta \, \Phi(\tau-\xi)$$

and after a change of the integration variable $\eta - \xi = \theta$

$$<Q^k(\tau)^2>_k = 2 \int_0^\tau (\tau-\theta)\, \Phi(\theta)\, d\theta$$

This formula gives us:

$$\frac{1}{2} \frac{\partial^2}{\partial \tau^2} \overline{Q^2}(\tau) = \Phi(\tau) \qquad \text{and}$$

$$\frac{1}{2} \frac{\partial^2}{\partial \tau^2} [\overline{Q^2}(\tau) - \overline{Q}^2(\tau)] = \Gamma(\tau)$$

If we apply our consideration to the specific case $q = 1$, which means that the collecting device simply stores the number of pulses up to time T, we obtain:

$$\frac{\overline{M^2} - \overline{M}^2}{\overline{M}} - 1 = \frac{2\, M_c}{\overline{M}} = \Psi(T) \tag{6}$$

The principle of an experimental method for the measurement of $\Psi(T)$ is described by A.I. Mogilner (5).

We notice that the quantity Ψ would be equal to zero, if the pulses arrived statistically independent of each other, i.e., if the time-points of the arrival of the impulses are distributed according to Poisson's distribution. The deviation of the real arrival distribution from a Poisson distribution gives an indication for a correlation between the individual events. Ψ therefore gives us a measure for the correlation between the pulses.

We see that each of the quantities

(a) the correlation function $\Gamma(\tau)$

(b) $\overline{Q^2} - \overline{Q}^2$

(c) $\Psi(T) = \dfrac{\overline{M^2} - \overline{M}^2}{\overline{M}} - 1$

stands in relation to the probability density P. The measurement of each of these quantities therefore gives us some information about this probability density. To compare the experiment with theory, we must have some analytical formula for some of these quantities. In the following chapters we give some methods of deriving the necessary analytical formulas.

3 — SIMPLIFIED COMPUTATION OF THE PROBABILITY DENSITY P

The two neutrons which produce the pair of correlated counts derive from the same chain. These two neutrons can therefore be traced back to one common fission process. We take this common fission process to be the most recent common fission process. Let ν fission neutrons be released at this fission process with probability p_ν. Some of these fission neutrons appear as prompt neutrons, the other fission neutrons appear as delayed neutrons. We consider the place of the common fission process as a neutron source which at the moment of the fission process emits $(1-\beta)\nu$ neutrons as prompt neutrons and subsequently $\beta\nu$ neutrons as delayed neutrons according to a certain emission distribution.

We introduce the quantities:

x the place of the common fission process

t' the time-point at which the common fission process occurred

s the time-point of the emission of the neutron which generates the chain out of which one neutron gives rise to a count in $D_{\nu\mu}$

r the time-point of the emission of the neutron which generates the chain out of which one neutron gives rise to a count in $D_{\nu'\mu'}$ (or in $D_{\nu\mu}$) at the time τ later.

According to the properties of the delayed neutron emitters we have for the emission probability of neutrons from the place where a fission process occurred:

$$e(t) = (1-\beta)\,\delta(t) + \sum_l \beta_l\,\lambda_l\,e^{-\lambda_l t}; \quad \sum \beta_l = \beta \tag{1}$$

We account for the fission spectrum by the following definition: $R(\nu, \mu; x; t)$ is the number of neutrons in the interval $S_\nu U_\mu$ at time t due to one fission neutron starting at time zero in the cell S_x.

We now calculate the probability of having one count in $D_{\nu\mu}$ in the time element dt at t and another count (from the same chain) in $D_{\nu'\mu'}$ in the time element $d\tau$ at the time $t+\tau$.

The fission neutron which leads to the count in the interval $D_{\nu\mu}$ in dt at t has produced $R(\nu,\mu; x; t-s)$ neutrons in $D_{\nu\mu}$ at time t.

The probability that we shall have a count in dt due to one of these neutrons is given by:

$$R(\nu,\mu; x; t-s)\,v(\mu)\sum_{\text{detector}}(\nu,\mu)\,dt = R(\nu,\mu; x; t-s)\frac{dt}{l_c(\nu,\mu)}$$

The fission neutron which leads to the count in $D_{\nu'\mu'}$ at time $t+\tau$ has produced $R(\nu'\mu',x; t+\tau-r)$ neutrons in the interval $D_{\nu'\mu'}$ at time $t+\tau$.

The probability of having a count of one of these neutrons in $d\tau$ in $D_{\nu'\mu'}$ is given by:

$$R(\nu',\mu'; x,t+\tau-r)\,\frac{dt}{l_c(\nu'\mu')}$$

The probability for both of these events is then:

$$R(\nu,\mu; x; t-s)\,R(\nu'\mu', x; t+\tau-r)\frac{dt\,d\tau}{l_c(\nu,\mu)\,l_c(\nu'\mu')} \tag{2}$$

We now have to sum this expression over all conditions which lead to this pair of impulses:

(1) The probability that ν neutrons are released is p_ν

(2) A pair of two of these neutrons generate the chains out of which the counts are produced. There are $2\binom{\nu}{2} = \nu(\nu-1)$ such pairs of fission neutrons which can assume the role of the initiating neutrons leading to the counts.

(3) The two neutrons leading to the counts can be emitted between t' and t (resp. t' and $t+\tau$) according to the emission probability $e(z)$:

$$\int_{t'}^{t} ds \int_{t'}^{t+\tau} dr\, e(s-t')\, e(r-t') \ldots$$

(4) The time point of the fission process can lie somewhere before t:

$$\int_{-\infty}^{t} dt' \ldots$$

(5) The position x of the common fission process is distributed according to the distribution $\Phi(x)\Sigma_F(x)$, where Φ is the neutron flux in the reactor for the steady state. The steady state can, for instance, be maintained by a source of neutrons in the subcritical reactor or it can be represented by the critical state.

We first collect the terms due to time integration:

$$\int_{-\infty}^{t} dt' \int_{t'}^{t} ds \int_{t'}^{t+\tau} dr\, e(s-t')e(r-t')R(\nu,\mu;x,t-s)R(\nu',\mu',x,t+\tau-r)$$

The substitutions $s-t'=\xi, r-t'=\eta$ change this expression into:

$$\int_{-\infty}^{t} dt' \int_{0}^{t-t'} d\xi\, e(\xi)\, R(\nu,\mu,x,t-t'-\xi) \int_{0}^{t+\tau-t'} d\eta\, e(\eta) R(\nu'\mu',x,t+\tau-t'-\eta)$$

As $R(\nu,\mu;x,t)$ is zero for $t<0$ and $e(z)$ is also zero for $z<0$ we can transform this expression into:

$$\int_{-\infty}^{t} dt' \int_{-\infty}^{+\infty} d\xi\, e(\xi) R(\nu,\mu,x,t-t'-\xi) \int_{-\infty}^{+\infty} d\eta\, e(\eta)\, R(\nu',\mu',x,t+\tau-t'-\eta)$$

and with $t-t'=\alpha$ we come to:

$$\int_{0}^{\infty} d\alpha \int_{-\infty}^{+\infty} d\xi e(\xi) R(\nu,\mu,x;\alpha-\xi) \int_{-\infty}^{+\infty} d\eta\, e(\eta) R(\nu'\mu'x,\alpha+\tau-\eta)$$

The lower boundary (0) in the integral over α can now again be replaced by $(-\infty)$, as, if α becomes negative, in the second integral over ξ, ξ must become negative to have $R(\alpha-\xi)$ different from zero. But for ξ negative $e(\xi)$ is zero, so that we get no contribution at all for $\alpha<0$. Finally we arrive at:

$$\int_{-\infty}^{+\infty} \Psi(\nu,\mu,x;\alpha)\, \Psi(\nu'\mu',x,\alpha+\tau)$$

where

$$\Psi(\nu,\mu,x,\alpha)=\int_{-\infty}^{+\infty} R(\nu,\mu,x,\alpha-\xi)\,e(\xi)\,\mathrm{d}\xi$$

$$\Psi(\nu,\mu,x,\alpha)=0 \text{ for } \alpha<0. \tag{3}$$

For the probability density we obtain:

$$P_{\nu\mu,\mu'\nu'}(\tau)=\sum_{\nu} p_{\nu}\nu(\nu-1)\frac{1}{l_c(\nu\mu)\,l_c(\nu'\mu')}\int_{\text{space}}\Phi(x)\Sigma_F(x)\,dx\int_{-\infty}^{+\infty}\Psi(\nu,\mu,x,\alpha).\Psi(\nu'\mu',x,\alpha+\tau)\,d\alpha$$

and we have the property that

$$P_{\nu\mu,\nu'\mu'}(-\tau)=P_{\nu'\mu',\nu\mu}(+\tau)$$

The sum $\Sigma p_{\nu}\nu(\nu-1)$ gives us the mean value $\overline{\nu(\nu-1)}$ if there is only one fissionable isotope (for instance U235) present. If we have more than one isotope and if we assume that all parameters determining the fission neutrons (fission spectrum, decay-time constants) are the same for all isotopes, we have to replace $\Sigma p_{\nu}\nu(\nu-1)$ by $\Sigma_i\,\gamma_i\sum_{\nu} p_i(\nu)\nu\,(\nu-1)$, with γ_i as the fraction of the fissionable material due to the isotope i and $p_i(\nu)$ as the probability for releasing ν fission neutrons by a nucleus of the isotope i in a fission act. We then have:

$$\sum_i\gamma_i\sum_{\nu}p_i(\nu)\nu(\nu-1)=\sum_i\gamma_i\ \overline{\nu(\nu-1)}^{(i)}$$

To calculate the power density spectrum we need the Fourier transformation of P. For this we consider first the following transformation:

$$\int_{-\infty}^{+\infty} d\tau\ e^{i\omega\tau}\left[\int_{-\infty}^{+\infty} f_1(\alpha)f_2(\alpha+\tau)\,d\alpha\right]$$

which gives us: $f_1^*(\omega)\,f_2(\omega)$ where a^* is the conj. complex of a. If $f(y)$ is of the form:

$$f(y)=\int_{-\infty}^{+\infty} h(y-\xi)g(\xi)\,d\xi$$

then the convolution theorem leads to:

$$f(\omega)=h(\omega)\,g(\omega)$$

Using these results for the transformation of $P(\nu,\mu,\nu',\mu'\tau)$, we obtain:

$$P(\nu,\mu,\nu',\mu',\omega)=\frac{\overline{\nu(\nu-1)}\,|e(\omega)|^2}{l_c(\nu,\mu)\,l_c(\nu'\mu')}\int_{\text{space } x}\Phi(x)\sum_F(x)R^*(\nu,\mu,x,\omega)R(\nu',\mu',x,\omega)\,dx \tag{3a}$$

The integration over the space has to be extended only over those regions of the space where the macroscopic fission cross-section is different from zero. This is obvious, because the source of the correlation is concentrated in the fission processes. Our derivation of formula (3a) can therefore be applied to a situation where we have, for instance, a small space region in which fissions can occur (converter plate) and the space between the fission source and the detectors has only moderating properties for the neutrons. In this case the function R contains those parameters of the medium which are important for the diffusion and moderation of neutrons.

The same method could be used for an experimental arrangement with a pulsed neutron source. A neutron pulse coming in this case from the source is in analogy to the fission burst in our derivation. Apart from this point, which gives us that part of the correlation which is due to the spatial distribution of the neutron source (at the fixed time-point of a pulse) we have to take account of the fact that the individual pulses are now no longer independent of each other; instead the time-points of the bursts are strongly correlated because these time points are controlled by electronic devices.

This gives us another source for correlation and we must consider not only neutrons from one burst but also neutron pairs from bursts at different times.

For the application of the formulas (3) and (3a) for P, we consider only the simplest space- and energy-independent case. The formulas take the form:

$$P(\tau) = \frac{\overline{\nu(\nu-1)}}{l_c^2} F \int_{-\infty}^{+\infty} d\alpha \, \Psi(\alpha) \Psi(\alpha+\tau) \tag{4}$$

with

$$\Psi(\alpha) = \int_{-\infty}^{+\infty} R(\alpha-\xi) e(\xi) d\xi$$

and

$$P(\omega) = \frac{\overline{\nu(\nu-1)}}{l_c^2} F |R(\omega) e(\omega)|^2 \tag{5}$$

here we have put $\Phi \Sigma_F = F =$ total fission rate.

If we neglect the delayed neutrons ($\beta = 0$) we have for $R(t)$ the number of neutrons present in the reactor at time t generated by one fission neutron starting at time $t = 0$.

$$R(t) = e^{-\alpha t} \tag{6}$$

with

$$\alpha = \frac{1-k}{l}$$

$k = \bar{\nu} \dfrac{\Sigma_F}{\Sigma_A} \dfrac{1}{1+L^2 B^2}$ (multiplication factor) $\dfrac{k}{l} = \bar{\nu}\, v\, \Sigma_F$

$l = l_o(1 + L^2 B^2)^{-1}$ (total neutron lifetime) $l_o = \dfrac{1}{v \Sigma_A}$

We define the detector efficiency as the probability that a fission process will be recorded ($\varepsilon = \Sigma_c / \Sigma_F$).

The counting rate P of the detector is then given by $P = \varepsilon F$.

Now we find from equation (4) with

$$\Psi(\alpha) = R(\alpha)$$

$$P(\tau) = \frac{\overline{\nu(\nu-1)}}{l_c^2} F\ e^{-\alpha\tau} \int_0^\infty e^{-2\alpha\xi}\, d\xi$$

or:

$$P(\tau) = P \frac{\overline{\nu(\nu-1)}}{\bar{\nu}^2} \frac{k^2}{2\alpha\, l^2}\, e^{-\alpha\tau} \tag{7}$$

This is the result given by F. De Hoffman (1) and J.D. Orndorff (2).

For the quantity $\Psi(T)$ we find with the aid of (7)

$$\Psi(T) = \varepsilon \frac{\overline{\nu(\nu-1)}}{\bar{\nu}^2} \frac{k^2}{(1-k)^2} \left(1 - \frac{1-e^{-\alpha T}}{\alpha T}\right) \tag{8}$$

If we take into account the delayed neutrons, we have to find $R(t)$ as the solution of the kinetic equations

$$\begin{aligned} \frac{dn}{dt} &= \frac{k-1}{l} n - \frac{k\beta}{l} n + \sum_p \lambda_p C_p \\ \frac{dC_p}{dt} &= -\lambda_p C_p + \frac{k\beta_p}{l} n \end{aligned} \tag{9}$$

under the initial conditions

$$\begin{aligned} n(0) &= 1 \\ C_p(0) &= 0 \end{aligned} \tag{10}$$

The Fourier transformation of the equations (9) under the conditions (10) gives us:

$$-i\omega R(\omega) - 1 = \frac{k-1}{l} R(\omega) = \frac{k\beta}{l} R(\omega) + \sum_p \lambda_p C_p(\omega)$$

$$-i\omega C_p(\omega) = -\lambda_p C_p(\omega) + \frac{k\beta_p}{l} R(\omega)$$

Upon solving for $R(\omega)$, we have

$$R(\omega) = \frac{l}{k} \frac{1}{\dfrac{k-1}{k} - i\omega \left(\dfrac{l}{k} + \sum_p \dfrac{\beta_p}{\lambda_p - i\omega}\right)} \tag{11}$$

The Fourier transformation of the emission probability $e(t)$ is found to be

$$\begin{aligned} e(\omega) &= (1-\beta) + \sum_p \frac{\lambda_p \beta_p}{\lambda_p - i\omega} \\ &= 1 + i\omega \sum_p \frac{\beta_p}{\lambda_p - i\omega} \end{aligned} \tag{12}$$

Finally we arrive at:

$$R(\omega) e(\omega) = \frac{l}{k} T(-i\omega)$$

where

$$T(S)=\frac{1-S\sum_{p}\frac{\beta_p}{\lambda_p+S}}{\frac{k-1}{k}+S\left(\frac{l}{k}+\sum_{p}\frac{\beta_p}{\lambda_p+S}\right)} \tag{13}$$

We again have $\varepsilon F=P$ and equation (5) gives us for the power density (E.F. Bennet (3))

$$\Phi(\omega)=<q^2>P+<q>^2P\varepsilon\frac{\overline{\nu(\nu-1)}}{\bar{\nu}^2}|T(i\omega)|^2 \tag{14}$$

The formulas for the probability density $P(\nu,\mu;\nu'\mu';\tau)$ can be generalized if there are more possibilities which lead to a correlation between two pulses.

We shall mention one generalization which arises if fission chambers are used as neutron detectors.

In this case we have the fact that a fission process can be recorded immediately it occurs and the released fission neutrons travel outwards starting neutron chains, and neutrons out of these chains may possibly lead to counts at a later time. These two counts, the first at the moment of the fission process, and the second through a neutron out of the chains generated by the fission neutrons, are therefore again correlated, in that their cause lies in the same fission process.

The derivation of the probability density for this correlation goes as follows:

The probability of having a count in dt at time t (a fission process) is given by:

$$\Phi(\nu,\mu)\sum_F(\nu,\mu)\frac{\Sigma_c(\nu,\mu)}{\Sigma_A(\nu,\mu)}dt;\ \Phi(\nu,\mu)=v.N(\nu,\mu)$$

The probability that one of the released ν fission neutrons will lead to a count in the detector $D_{\nu'\mu'}$ in the time element $d\tau$ at the later time $t+\tau$ is:

$$\int_0^\tau e(\xi)R(\nu'\mu',\nu,\tau,-\xi)d\xi\, v(\mu')\Sigma_c(\nu'\mu')\,d\tau$$

For the probability density $P_{\nu\mu\,\nu'\mu'}(\tau)$ we find:

$$P_{\nu\mu\nu'\mu'}(\tau)=\bar{\nu}\,\Phi\,(\nu,\mu)\Sigma_F\,(\nu,\mu)\frac{\Sigma_c(\nu,\mu)}{\Sigma_A(\nu,\mu)}\int_0^\tau R(\nu',\mu',\nu,\tau-\xi)\ \ e(\xi)d\xi v(\mu')\Sigma_c(\nu'\mu') \tag{15}$$

In the space and energy-independent case, formula (15) gives us the term (first without delayed neutrons, $\Phi=vN$):

$$P(\tau)=\bar{\nu}\frac{\Sigma_F}{\Sigma_A}v^2\Sigma_c^2N\,e^{-\alpha\tau}$$

$$P(\tau)=\mathrm{P}\frac{k\varepsilon}{\tau_f}e^{-\alpha\tau}\left(\tau_f=\frac{1}{v\Sigma_F}\right) \tag{16}$$

The presence of delayed neutrons changes equation (15) in this case into the form:

$$P(\tau) = v^2 \Sigma_c^2 N k \int_0^\tau R(\tau - \xi)\ e(\xi)\, d\xi$$

and for the power spectrum we obtain:

$$P(\omega) = P \frac{\varepsilon k}{\bar{\nu}} 2 \,\text{Real} \left(T(-i\omega)\right) \tag{17}$$

where we have made use of the fact that we must have

$$P(\tau) = P(-\tau)$$

We must point out that the method of calculating the probability density described in this chapter is only a rough approximation and is only for the purpose of giving some insight into the cause of the correlation.

We have made some approximations, for instance:

(a) that it is always the *same* portion of fission neutrons that is released as prompt neutrons and the *same* portion (β) of fission neutrons that is released as delayed neutrons, and

(b) we have replaced the number of neutrons in the interval (ν,μ) at time t due to one fission neutron starting at time $t = 0$ at the place x by their *mean number*, which could be calculated from the Boltzmann equation.

To remove, for instance, the approximation mentioned under (b), we should replace the product:

$$R(\nu,\mu,x,\alpha-\xi)\ R(\nu',\mu',x,\alpha+\tau-\xi)$$

in formula (3) for $P(\tau)$ by the mean value:

$$< R^k(\nu,\mu,x,\alpha-\xi)\ R^k(\nu',\mu',x,\alpha+\tau-\xi) >_k$$

the computation of which is a more involved problem. We shall see that the study of this problem is connected with the problem of calculating the quantity $(\overline{Q^2} - \overline{Q}^2)$ which we considered in chapter B.

But as the exact computation of $(\overline{Q^2} - \overline{Q}^2)$ already gives us the solution to the task of finding the correlation function, we do not need to revert to the formulation showing this chapter. In the next chapter, we start to deal more precisely with the treatment of the statistical fluctuations in a reactor.

4 — STATISTICAL FLUCTUATIONS OF THE NUMBER OF NEUTRONS IN A REACTOR

We mentioned in chapter 3 that we want to find an exact expression for the quantity $\overline{Q^2}-\overline{Q}^2$, where $Q(T)$ is the charge in a collecting device and is the sum of all charges given to this device in the time interval T by the small detectors distributed over the reactor. We have therefore to consider the whole system consisting of the reactor and the collecting device, which we can characterize at any time-point t by the following quantities:

(a) $N_{\nu\mu}$: number of neutrons in the interval $S_\nu U_\mu$

(b) $\Gamma_{\nu\mu l}$: number of delayed neutron emitters of type l in the interval S_ν which on decaying give neutrons in the interval U_μ

(c) $Q_{\nu\mu}$: total charge delivered from the detector $D_{\nu\mu}$ up to time t.

What we have to do now is to write down the probability for the change of the state of the system in a time element dt and then to construct the equation for the probability generating function (p.g.f.) for the probability of finding the system in a given state at a given time.

We know that the p.g.f. gives us the equation for the mean values for the quantities which characterize the state of the system. This equation must coincide with the Boltzmann equation for the neutron population of the reactor. We therefore choose first the form of the Boltzmann equation which we want to derive from the general equation for the p.g.f. We take the Boltzmann equation to be of the form:

$$\frac{\partial n(x,u,t)}{\partial t} = -v(u)\Sigma_T(x,u)n(x,u,t) + \int_{-\infty}^{+\infty} n(x,u',t)v(u')\Sigma_s(x,u'\rightarrow u)\,du'$$

$$+ \Sigma_l \lambda_l \Gamma_l(x,u,t) + D(u)\Delta n(x,u,t) + S(x,u,t) + F n \tag{1}$$

$$\frac{\partial \Gamma_l(x,u,t)}{\partial t} = -\lambda_l \Gamma_l(x,u,t) + F_l n$$

The symbols employed have their usual meaning.

$$F n = (1-\beta) f_0(u) \int_{-\infty}^{+\infty} n(x,u',t)v(u')\Sigma_F(x,u')\nu(u')\,du'$$

$$F_l n = \beta_l f_1(u) \int_{-\infty}^{+\infty} n(x,u',t)v(u')\Sigma_F(x,u')\nu(u')\,du'$$

$n(x,u,t)\,dx\,du$: number of neutrons in the volume element dx and in the lethargy interval du at time t.

$v(u)$: velocity of the neutron with lethargy u.

$\Sigma_t(x,u)$: macroscopic total cross-section at the point x for neutrons with lethargy u.

$\Gamma_l(x,u,t)dxdu$: number of delayed neutron emitters of type l in the volume element dx which give neutrons with lethargy in the interval du.

$S(x,u,t)\,dx\,du\,dt$: number of neutrons in the element $dx\,du$ emitted by an external source in the time element dt.

$\int_{-\infty}^{+\infty} n(x,u',t)\,v(u')\Sigma_s(x,u'\rightarrow u)\,du'\,du\,dt\,dx$: number of neutrons scattered into the element $dx\,du$ in the time element dt from all the other lethargy elements du'.

$D(u)\Delta n(x,u,t)dx\,du$: number of neutrons leaving the element $dx\,du$ per sec by leakage.

$Fndxdu = (1-\beta)f_0(u)\int_{-\infty}^{+\infty} n(x,u',t)v(u')\Sigma_F(x,u')\nu(u')\;\;du'\,dx\,du$: number of prompt neutrons generated in $dx\,du$ per sec. by fissions in all the other lethargy elements du ($f_0(u)$ fission spectrum for prompt neutrons.

We now use the following approximation:

$$\int_{\substack{\text{all}\\ \text{space}}} f(x')\,\frac{e^{-a|x'-x|}}{4\pi|x'-x|^2}\,d\tau - f(x)\int_{\substack{\text{all}\\ \text{space}}} \frac{e^{-a|x'-x|}}{4\pi|x-x'|^2}\,d\tau = \frac{1}{3\alpha^2}\,\Delta f(x)$$

If we take the arbitrary function $f(x)$ as $n(x)v\,\Sigma_s = \Phi(x)\,\Sigma_s$ (Σ_s independent of x) we obtain:

$$\int_{\substack{\text{all}\\ \text{space}}} \Phi(x')\Sigma_s\frac{e^{-a|x'-x|}}{4\pi|x'-x|^2}\,d\tau - \Phi(x)\Sigma_s\int_{\substack{\text{all}\\ \text{space}}} \frac{e^{-a|x'-x|}}{4\pi|x-x'|^2}\,d\tau = \frac{\Sigma_s}{3\alpha^2}\,\Delta\Phi(x) \tag{2}$$

where we can choose α in such a way as to obtain the used D through

$$D(u) = \frac{v(u)\Sigma_s(u)}{3\alpha^2(u)}$$

These two integrals give us a simple formulation for the leakage term:

A neutron can become scattered *without energy change* from one point x in space to another point x' in space.

The first term in (2) gives us all the neutrons which are scattered into the unit volume around x per sec from scattering collisions in all the other volume elements of the reactor, and the second term in (2) gives us all the neutrons which are scattered out of the unit volume around x per sec to all the other space cells of the reactor.

This is in symmetry with the scattering collisions in a volume element which lead to a change of energy where the neutrons remain in the same volume element and only change their lethargy interval.

Now we are ready to establish the equation for the neutron balance. We look for the probability of finding the system at time $(t+dt)$ in the state given by the numbers $(N_{\nu\mu}, \Gamma_{\nu\mu c}, Q_{\nu\mu})$:

$$P(N_{\nu\mu}, \Gamma_{\nu\mu c}, Q_{\nu\mu}, t)$$

We have in general:

$$P(N_{\nu\mu}; \Gamma_{\nu\mu c}; Q_{\nu\mu}; t+dt) = P(N_{\nu\mu}; \Gamma_{\nu\mu c}; Q_{\nu\mu}; t) \times$$

(Probability that nothing happens in dt) $+ \sum\limits_{\substack{\text{changed}\\ \text{states}}}$ (Probability for a changed state) $\times$

Probability for the change in dt which after dt gives the right state.

Now:

Probability that nothing happens in $dt =$

$1 - \sum\limits_{\substack{\text{overall}\\ \text{possible}\\ \text{changes in } dt.}}$ Probability for a change out of the state $(N_{\nu\mu}; \Gamma_{\nu\mu c}; Q_{\nu\mu})$ in dt

We collect now all the possible state-changes in dt out of the state $(N_{\nu\mu}; \Gamma_{\nu\mu c}; Q_{\nu\mu})$ (with a minus sign) and *into* this state:

(a) We have at time t the state $(N_{\nu\mu}; \Gamma_{\nu\mu c}; Q_{\nu\mu})$ and the source emits in dt a neutron, or we have at time t the state $(N_{\nu\mu}-1; \Gamma_{\nu\mu c}, Q_{\nu\mu})$ and the source $S_{\nu\mu}$ emits one neutron in dt:

$$P(N_{\nu\mu}; \Gamma_{\nu\mu c}; Q_{\nu\mu}) \left(-\sum_{\nu\mu} S_{\nu\mu}\, dt\right) + \sum_{\nu\mu} P(N_{\nu\mu}-1; \Gamma_{\nu\mu c}, Q_{\nu\mu}; t) S_{\nu\mu}\, dt$$

In the same way we can immediately write down the other possibilities (we mention only those quantities which change):

(b) $P\left(-\sum\limits_{\nu\mu c} \lambda_c \Gamma_{\nu\mu c}\, dt\right) + \sum\limits_{\nu\mu c} P(N_{\nu\mu}-1; \Gamma_{\nu\mu c}+1)(\Gamma_{\nu\mu c}+1)\lambda_c\, dt$

(c) $P\left(\sum\limits_{\substack{\nu\\ \mu \neq \mu'}} N_{\nu\mu}\, v(\mu) \Sigma_s(\nu\mu \to \mu')\right) + \sum\limits_{\substack{\nu\\ \mu \neq \mu'}} P(N_{\nu\mu}+1; N_{\mu\nu'}-1)(N_{\nu\mu}+1)\; v(\mu)\Sigma_s(\nu, \mu \to \mu') dt$

(d) $P\left(-\sum_{\mu,\nu\neq\nu'} N_{\nu\mu}\, v_\mu\, \Sigma_T(\mu,\nu\to\nu')\,dt\right) + \sum_{\mu,\nu\neq\nu'} P(N_{\nu\mu}+1, N_{\nu'\mu}-1)(N_{\nu\mu}+1)\;\; v_\mu\, \Sigma_T(\mu\,\nu\to\nu')\,dt$

(e) $P\left(-\sum_{\nu\mu} N_{\nu\mu}\, v_\mu\, \Sigma_c(\nu\,\mu)\,dt\right) + \sum_{\nu\mu q_{\nu\mu}} P(\mathrm{N}_{\nu\mu}+1, C_{\nu\mu}-q_{\nu\mu})\,(N_{\nu\mu}+1)\, v_\mu\, \Sigma_c(\nu,\mu)\,\pi_{\nu\mu}(q)\,dt$

(f) $P\left(-\sum_{\nu\mu} N_{\nu\mu}\, v_\mu \sum_a (\nu\,\mu)\,dt\right) + \sum_{\substack{\nu\mu\\ \eta\gamma}} P(N_{\nu\mu}+1-\eta_\mu;\, N_{\nu\mu'}-\eta_{\mu'};\, \Gamma_{\nu\rho e}-\gamma_{\rho e})$

$(N_{\nu\mu}+1-\eta_\mu)v_\mu\, \Sigma_a(\nu\,\mu)\,dt\; P_{\nu\mu}(\eta_{\mu'}, \gamma_{\rho e})$

Here we have introduced:

$\Sigma_c(\nu\,\mu)$: macroscopic cross-section of the detector material in the space cell ν for neutrons with lethargy in the interval μ

$\pi_{\nu\mu}(q)$: probability for the generation of the charge q through a neutron capture in the detector material

$\Sigma_a(\nu\,\mu)$: macroscopic absorption cross-section for fission and γ absorption

$$\Sigma_a = \Sigma_F + \Sigma_\gamma$$

$\Sigma_T(\mu,\nu\to\nu')$: total macroscopic scattering cross-section for a scattering out of space cell ν into space cell ν'

$\Sigma_S(\nu,\mu\to\mu')$: total scattering cross-section for a scattering out of lethargy interval μ into lethargy interval μ'

$P_{\nu\mu}(\eta_{\mu'}, \gamma_{\rho e})$: probability that a neutron capture in reactor material leads to the generation of $\eta_{\mu'}$ prompt neutrons with lethargy in the interval μ' and to the generation of $\gamma_{\rho e}$ delayed neutron emitters of type e which leads to neutrons in the lethargy interval ρ.

The neutron balance equation now reads:

$$P(t+dt) - P(t) = \text{expressions (a)} + \text{(b)} + \ldots\ldots\ldots\ldots\text{(f)}. \qquad (3)$$

The p.g.f. for the distribution P is defined by:

$$F(X_{\nu\mu}, Y_{\nu\mu}, Z_{\nu\mu e}) = \sum_{N_{\nu\mu}\, Q_{\nu\mu}\, \Gamma_{\nu\mu e}} P(N_{\nu\mu}, \Gamma_{\nu\mu e}, Q_{\nu\mu}) X_{\nu\mu}^{N_{\nu\mu}}\; Y_{\nu\mu}^{Q_{\nu\mu}} \prod_e Z_{\nu\mu e}^{\Gamma_{\nu\mu e}}$$

We therefore multiply equation (3) by the factor of P in equation (4) and sum over all values of $N_{\nu\mu}$, $\Gamma_{\nu\mu e}$, $Q_{\nu\mu}$.

In this way we obtain the Fokker-Planck equation as an analogous stochastic formulation for the Boltzmann-equation (1) (5)

$$\frac{\partial F}{\partial t} = \sum_{\nu\mu} S_{\nu\mu}(X_{\nu\mu}-1)F + \sum_{\nu\mu e} \lambda_e (X_{\nu\mu} - Z_{\nu\mu e}) \frac{\partial F}{\partial Z_{\nu\mu e}} +$$

$$+ \sum_{\nu\mu\mu'} v_\mu (X_{\nu\mu'} - X_{\nu\mu}) \Sigma_s(\nu, \mu \to \mu') \frac{\partial F}{\partial X_{\nu\mu}} + \sum_{\nu\mu\nu'} v_\mu \Sigma_T(\mu, \nu \to \nu') (X_{\nu'\mu} - X_{\nu\mu}) \frac{\partial F}{\partial X_{\nu\mu}}$$

$$+ \sum_{\nu\mu} v_\mu \Sigma_c(\nu,\mu) (\chi_{\nu\mu} - X_{\nu\mu}) \frac{\partial F}{\partial X_{\nu\mu}} + \sum_{\nu\mu} v_\mu \Sigma_a(\nu,\mu) (\Phi_{\nu\mu} - X_{\nu\mu}) \frac{\partial F}{\partial X_{\nu\mu}}$$

where $\Phi_{\nu\mu}(\chi_{\nu\mu})$ is the p.g.f. for the distribution $P_{\nu\mu}(\pi_{\nu\mu})$:

$$\Phi_{\nu\mu} = \sum_{\eta\gamma} P_{\nu\mu}(\eta_{\mu'}, \gamma_{\rho e}) \prod_{\mu'} X_{\nu\mu'}^{\eta_{\mu'}} \prod_{\rho e} Z_{\nu\rho e}^{\gamma_{\rho e}}$$

$$\chi_{\nu\mu} = \sum_q \prod_{\nu\mu}(q) Y_{\nu\mu}^q$$

We have here:

$$P_{\nu\mu} = \left(1 - \frac{\Sigma_F}{\Sigma_a}\right) \delta_{oo.\eta\gamma} + \frac{\Sigma_F}{\Sigma_a} p_{\nu\mu}(\eta,\gamma)$$

with $p_{\nu\mu}(\eta_{\mu'}, \gamma_{\rho e})$ as the probability for the generation of $\eta_{\mu'}$ prompt neutrons in the interval μ' and $\gamma_{\rho e}$ delayed neutron emitter of type e which give neutrons in the lethargy interval ρ by a fission act in the interval (ν, μ).

As:

$$\sum_q \prod_{\nu\mu}(q) = \sum_{\eta\gamma} p_{\nu\mu}(\eta,\gamma) = 1$$

we have, if we give the value 1 to all variables X, Y, Z:

$$\chi_{\nu\mu} = 1; \qquad \Phi_{\nu\mu} = 1.$$

We notice that we have in (5) one term for the detectors and one term for the fissions (absorptions). This is due to the explicitly stated assumption that we can have an absorption in the detector material (Σ_c) which can lead to a count (according to $\pi_{\nu\mu}$: we can have $\pi_{\nu\mu}(o) \neq 0$) and an absorption which can lead to fission (or parasitic capture). Here we consider every fission process which leads to a fission, whether we have or we have not fission neutrons, as a fission process.

It can happen, as we mentioned on page 23, that an absorption process leads simultaneously to the generation of fission neutrons and charges (counts). In this case we have in equation (5) only one term for the absorption process instead of the last two terms.

This term is then:

$$\sum_{\nu\mu} v_{\mu} \Sigma_A(\nu,\mu) \left(\Phi_{\nu\mu}\, x_{\nu\mu} - X_{\nu\mu} \right) \frac{\partial F}{\partial X_{\nu\mu}} \tag{6}$$

with $\Phi_{\nu\mu}$ as in (5) with $\Sigma_A = \Sigma_C + \Sigma_a$ and $x_{\nu\mu}$ is changed into:

$$x_{\nu\mu} = \sum_q \overset{*}{\underset{\nu\mu}{\Pi}} (q) Y^q_{\nu\mu}$$

where now:

$$\overset{*}{\underset{\nu\mu}{\Pi}} = \left(l - \frac{\Sigma_c}{\Sigma_A} \right) \delta_{oq} + \frac{\Sigma_c}{\Sigma_A} \underset{\nu\mu}{\Pi} (q)$$

Before we proceed to draw some conclusions from the general equation (5) we make some assumptions about the p.g.f. $\Phi_{\nu\mu}$ for the distribution $p_{\nu\mu}$. This is now the place to remove the approximation (a) shown on page 24, where we assumed that the same portion of prompt neutrons $((1-\beta)\bar{\nu})$ is always released in any fission process. But this statement is only true in the mean and we must take account of this.

A plausible physical hypothesis (Raievski, (4)) enables us to calculate in a simple way the p.g.f. $\Phi_{\nu\mu}$. As in a fission process we almost always have two fragments, it is improbable that both fragments are excited to such an extent that they could emit delayed neutrons. We assume, therefore, that in a fission process there is at most one fission fragment which emits one delayed neutron. In that case there are only two possibilities which can arise in a fission process:

(a) all ν fission neutrons appear as prompt neutrons (probability $P(\nu)$)

(b) one neutron appears as a delayed neutron and the other neutrons are prompt neutrons (probability $1-P(\nu)$).

If p_ν is the probability that a fission process gives us ν fission neutrons, we must have for the mean number of prompt fission neutrons:

$$(1-\beta)\bar{\nu} = \sum_\nu p_\nu \left[P(\nu)\nu + (1-P(\nu))(\nu-1) \right]$$

or:

$$1-\beta\bar{\nu} = \sum_\nu p_\nu P(\nu)$$

which gives us :

$$P(\nu) = 1-\beta\nu \qquad\qquad \beta = \sum_e \beta_e$$

The p.g.f. can now be written:

$$\Phi_{\nu\mu} = \sum_{\rho} P_{\nu\mu}(\rho)[(1-\beta\rho) \sum_{\eta_1+\eta_2+\ldots=\rho} p_o(\eta_1\eta_2\ldots) \prod_{\lambda} X_{\nu\lambda}^{\eta\lambda}$$

$$+ \sum_{e} \beta_e \rho \sum_{\eta_1+\eta_2+\ldots=\rho-1} p_o(\eta_1\,\eta_2\ldots) \prod_{\lambda} X_{\nu\lambda}^{\eta\lambda} \sum_{k} f_1(k) Z_{\nu ke}]$$

Here

$P_{\nu\mu}(\rho)$: the probability that a fission process in the interval (ν,μ) gives ρ fission neutrons

$$P_{\nu\mu}(\rho) = \left(l - \frac{\Sigma_F}{\Sigma_a}\right) \delta_{\rho o} + \frac{\Sigma_F}{\Sigma_a} p_\rho$$

$p_o(\eta_1\eta_2\ldots)$: the probability that η_i of the prompt fission neutrons lie in the lethargy interval i

$f_1(k)$: the probability that the one delayed neutron has a lethargy in the interval k (fission spectrum for the delayed neutrons).

Let $f_o(k)$ denote the fission spectrum for the prompt neutrons, then:

$$p_o(\eta_1\,\eta_2\ldots) = \frac{\rho!}{\eta_1!\,\eta_2!\,\eta_3!\ldots} f_o(1) f_o(2) f_o(3)\ldots$$

and:

$$\Phi_{\nu\mu} = \sum_{\rho} P_{\nu\mu}(\rho)[(1-\beta\rho) J_{o\nu}^{\rho} + \sum_{e} \beta_e \rho J_{o\nu}^{\rho-1} J_{1\nu e}] \tag{7}$$

with:

$$J_{o\nu} = \sum_{\lambda} f_o(\lambda) X_{\nu\lambda}$$

$$J_{1\nu e} = \sum_{\lambda} f_1(\lambda) Z_{\nu\lambda e}$$

The summation over λ in $J_{o\nu}$ (or $J_{1\nu e}$) covers all lethargy intervals.

For later use we state here the values of some derivatives, when all the variables X, Y, and Z take the value 1:

$$\frac{\partial \Phi_{\nu\mu}}{\partial X_{\nu\lambda}} = \frac{\Sigma_F}{\Sigma_a} \bar{\nu}(1-\beta) f_o(\lambda)\,; \quad \frac{\partial \Phi_{\nu\mu}}{\partial Z_{\nu\lambda e}} = \frac{\Sigma_F(\nu,\mu)}{\Sigma_a(\nu,\mu)} \bar{\nu}\beta_e f_1(\lambda)\,; \quad \frac{\partial^2 \Phi_{\nu\mu}}{\partial Z_{\nu\lambda e}\,\partial Z_{\nu\mu\rho}} = 0\,;$$

$$\frac{\partial^2 \Phi_{\nu\mu}}{\partial X_{\nu\lambda}\,\partial X_{\nu\rho}} = \frac{\Sigma_F}{\Sigma_a} \overline{\nu(\nu-1)} (1-2\beta) f_o(\lambda) f_o(\rho)\,; \quad \frac{\partial^2 \Phi_{\nu\mu}}{\partial X_{\nu\lambda}\,\partial Z_{\nu\rho p}} = \frac{\Sigma_F}{\Sigma_a} \overline{\nu(\nu-1)}\, \beta_p f_o(\lambda) f_1(\rho)$$

For equation (6), Σ_a has to be replaced by Σ_A.

Under these conditions the kinetic reactor equations take the form (analogous to the formulas(1)):

$$\frac{\partial N_{ik}}{\partial t} = \Omega_{i\underline{k}} N_{ik} + \sum_e \lambda_e \Gamma_e(i, k) + S_{ik}$$

$$\frac{\partial \Gamma_e(i,k)}{\partial t} = -\lambda_e \Gamma_e(i,k) + \sum_\mu v_\mu N(i,u) \Sigma_F(i,u) \bar{v} \beta_e f_1(k)$$

where the Operator Ω_{ik} is defined by:

$$\Omega_{ik} = -v_k \sum_{\text{total}}(i,k) + D(k)\Delta_i + \sum_\mu v_\mu \Sigma_s(i,\mu \rightarrow k) + \bar{v}(1-\beta) f_o(k) \sum_\mu v_\mu \Sigma_F(i,\mu)$$

$$\Sigma_t = \Sigma_A + \Sigma_s = \Sigma_c + \Sigma_a + \Sigma_s$$

and the bar under the index k means, that in the summation over the lethargy intervals in the operator Ω this index has to be replaced by μ.

We start now to make use of equation (5) for the computation of the correlation function. Formula (5a) on page 16 shows us that we need for this the Laplace-transform of the function $G(T)$ which was defined by:

$$\overline{Q^2} - \overline{Q}^2 = G(T) + \overline{Q}$$

The mean value

$$\overline{Q} = \sum_{i,k} \overline{Q}(i,k)$$

can now be found out of the p.g.f. by (all variables equal 1)

$$\frac{\partial F}{\partial Y_{ik}} = \overline{Q}(i,k)$$

and the mean value

$$\overline{Q^2} = \sum_{ik,mu} \overline{Q_{ik} Q_{mu}} = \overline{Q}^2 + \sum_{ik,mn} G(ik, mn) + \overline{Q}$$

can be calculated by using:

$$\frac{\partial^2 F}{\partial Y_{ik} \partial Y_{mn}} = \overline{Q_{ik} Q_{mn}} - \overline{Q_{ik}}\, \delta_{\substack{ik\\mn}} = \overline{Q_{ik}}\, \overline{Q_{mn}} + G\ (ik,mn)$$

Our next task is therefore to calculate the two partial derivatives $\partial F/\partial Y_{ik}$ and $\partial^2 F/\partial Y_{ik}\partial Y_{mn}$ of the p.g.f. F. Equation (5) will give us the time derivates of these quantities and the result is:

$$\frac{\partial \overline{Q}_{ik}}{\partial t} = v_k \Sigma_c(ik) N_{ik} \chi'_{ik} = P_{ik}\, \bar{q}_{ik} = I_e(ik) \tag{9}$$

where

$$\chi'_{ik} = \frac{\partial \chi_{ik}}{\partial Y_{ik}} = \overline{q_{ik}}$$

$P_{ik} = v_k \Sigma_c(ik) N_{ik} =$ mean counting rate of the detector D_{ik} for both cases (5) and (6).

We have further:

$$\frac{\partial G(ik,mn)}{\partial t} = Y'_{ik} H(mn,ik) + Y'_{mn} H(ik,mn) + Y''_{ik} N_{ik}\, \delta_{\substack{ik \\ mn}} \tag{10}$$

with:

$$Y'_{ik} = v_k \Sigma_c(ik) \chi'_{ik}$$

$$Y''_{ik} = v_k \Sigma_c(ik) \chi''_{ik}$$

$$\chi''_{ik} = \overline{q^2_{ik}} - \bar{q}_{ik}$$

and where we have put:

$$\overline{Q_{ik} N_{mn}} = \overline{Q}_{ik}\, \overline{N}_{mn} + H(ik,mn)$$

Equation (10) gives us for the function: $G(t) = \sum\limits_{ik,mn} G(ik,mn;\, t)$

$$\frac{dG}{dt} = 2 \sum_{\substack{ik \\ mn}} Y'_{ik} H(mn,ik) + \sum_{ik} Y''_{ik} N_{ik}$$

Now:

$$\Sigma Y''_{ik} N_{ik} = \sum_{ik} v(k) \Sigma_c(i,k) N_{ik} (\overline{q^2_{ik}} - \bar{q}_{ik}) = \sum_{ik} P_{ik}\, \overline{q^2_{ik}} - \sum_{ik} P_{ik}\, \bar{q}_{ik}$$

and the last term is just the total mean current I_e into the collecting device.

Therefore:

$$\frac{dG}{dt} = 2 \sum_{ik,mn} Y'_{ik} H(mn,ik) + \sum_{ik} P_{ik}\, \overline{q^2_{ik}} - I_e$$

and the Laplace-transform is:

$$S^2 G(S) = 2\sum_{\substack{ik\\mn}} Y'_{ik} S\, H(mn,ik) + \sum_{ik} P_{ik}\, \overline{q^2_{ik}} - I_e$$

According to formula (5a) on page 16 we obtain:

$$\Gamma(S) = \frac{1}{2} \sum_{ik} P_{ik}\, \overline{q^2_{ik}} + \sum_{ik,mn} Y'_{ik}\, S\, H(mn,ik;\, S)$$

and with:

$$R_{ik}(S) = \sum_{mn} H(m,n;ik;\, S)$$

$$\Gamma(S) = \frac{1}{2} \sum_{ik} P_{ik}\, \overline{q^2_{ik}} + \sum_{ik} Y'_{ik} R_{ik}(S)\, .\, S \qquad (11)$$

Our aim is then to compute the function R_{ik}.

The next step is to establish the equation for $H(ik,mn)$. In doing this we are again forced to introduce new quantities and we have to find their equations. Instead of proceeding step by step we collect the results with the corresponding definitions:

$$\overline{Q_{ik} N_{mn}} = \overline{Q_{ik}}\; \overline{N_{mn}} + H(ik,mn)$$

$$\overline{Q_{ik} \Gamma_{mne}} = \overline{Q_{ik}}\; \overline{\Gamma_{mne}} + F_e(ik,mn)$$

$$\overline{N_{ik} N_{mn}} = \overline{N_{ik}}\; \overline{N_{mn}} + \Phi(ik,mn) + \overline{N_{ik}}\, \delta_{ik,mn} \qquad (12)$$

$$\overline{N_{ik} \Gamma_{mne}} = \overline{N_{ik}}\; \overline{\Gamma_{mne}} + \Psi_e(ik,mn)$$

$$\overline{\Gamma_{ike} \Gamma_{mnp}} = \overline{\Gamma_{ike}}\; \overline{\Gamma_{mnp}} + F_{ep}(ik,mn) + \overline{\Gamma_{ike}}\, \delta_{ike,mnp}$$

The equations are:

$$\frac{\partial H(ik,mn)}{\partial t} = \underline{\Omega_{mn}}\, H(ik,\underline{mn}) + \sum_e \lambda_e F_e(ik,mn)$$
$$+ Y'_{ik}\, [\Phi(ik,mn) + \bar{\nu}(1-\beta) f_o(n) N_{ik} \delta_{im} \Sigma_F(ik)/\Sigma_A(ik)] \qquad (13)$$

$$\frac{\partial}{\partial t} F_p(ik,mn) = -\lambda_p F_p(ik,mn) + \bar{\nu}\, \beta_p f_1(n) \sum_\mu v(\mu) \sum_F (m,\mu) H(ik,m\mu)$$
$$+ Y'_{ik}\, [\Psi_p(ik,mn) + \bar{\nu}\, \beta_p f_1(n) N_{ik}\, \delta_{im}\, \Sigma_F(ik)/\Sigma_A(ik)] \qquad (14)$$

$$\frac{\partial}{\partial t} \Phi(ik,mn) = \underline{\Omega_{ik}}\, \Phi(\underline{ik},mn) + \underline{\Omega_{mn}}\, \Phi(\underline{mn},ik) + \sum_e \lambda_e \Psi_e(ik,mn) + \sum \lambda_e \Psi_e(mn,ik)$$
$$+ \overline{\nu(\nu-1)}(1-2\beta) f_o(n) f_o(k) \sum_\mu v(\mu) N_{i\mu}\, \Sigma_F(i,\mu) \qquad (15)$$

$$\frac{\partial}{\partial t}\Psi_p(ik,mn) = -\lambda_p\Psi_p(ik,mn) + \sum_c \lambda_c F_{cp}(ik,mn) + \Omega_{\underline{ik}}\Psi_p(\underline{ik},mn)$$

$$+\bar{\nu}\beta_p f_1(n)\sum_\mu v(\mu)\Sigma_F(m,\mu)\Phi(ik,m\mu) + \overline{\nu(\nu-1)}\beta_p f_o(k) f_1(n)\sum_\mu v_\mu N_{m\mu}\sum_F(m\mu) \tag{16}$$

$$\frac{\partial}{\partial t}F_{cp}(ik,mn) = -(\lambda_c + \lambda_p)F_{cp}(ik,mn) + \bar{\nu}\beta_c f_1(k)\sum_\mu v(\mu)\Sigma_F(i\mu)\Psi_p(i\mu,mn)$$

$$+\bar{\nu}\beta_p f_1(n)\sum_\mu v(\mu)\Sigma_F(m,\mu)\Psi_c(m,\mu;ik) \tag{17}$$

Equations (13)-(17) as they stand are valid for the condition of equation (6). In the case of equation (5), only equations (13) and (14) change in that their last term has to be replaced by zero. It is physically plausible that equations (15), (16) and (17) are not influenced by the change of the counter properties because these equations describe only the statistical behaviour of the neutrons in the reactor and can be used to calculate the fluctuations of the number of neutrons in the space and lethargy intervals.

5 — APPLICATION OF THE THEORY OF FLUCTUATIONS IN SOME SPECIAL CASES

For the application of the method developed in the last chapter we shall deal only with the case of equation (5).

We note that if we were to take the case of equation (6) the results would differ only by an additional term from the results for equation (5). This is due to the last terms in equations (13) and (14), which are only different from zero in the case of equation (6).

As the special cases we take:

(a) one-space cell (space-independence)

one-lethargy cell (one-group theory)

no delayed neutrons ($\beta = 0$)

Equation (11) is in this case:

$$\Gamma(s) = \frac{1}{2} P \overline{q^2} + v \Sigma_c \bar{q} R(s)$$

where:

$$R(s) = s H(s)$$

Now (equation (13)):

$$s H(s) = -v \Sigma_A H + \bar{\nu} v \Sigma_F H + \frac{1}{s} v \Sigma_c \bar{q} \Phi$$

and (equation (17) for the stationary state):

$$0 = 2\Phi(-v \Sigma_A + \bar{\nu} v \Sigma_F) + \overline{\nu(\nu-1)} v \Sigma_F N$$

With:

$$k = \bar{\nu} \frac{\Sigma_F}{\Sigma_A}; \qquad l = \frac{1}{v \Sigma_A} \quad \text{and:}$$

$$\alpha = \frac{1-k}{l} \quad \text{we find:}$$

$$\Phi = \overline{\nu(\nu-1)}\, kN / \bar{\nu}\, 2\alpha l \quad \text{and:}$$

$$sH(s) = v \Sigma_c \bar{q} \Phi / (s + \alpha)$$

We take the inverse transformation and obtain:

$$R(\tau) = v \Sigma_c \bar{q} \Phi \exp\{-\alpha\tau\}$$

and with:

$$N\, v\, \Sigma_c = P$$

we finally get:

$$\Gamma(\tau) = P\, \overline{q^2}\, \delta(\tau) + \bar{q}^2 P_F \frac{\overline{\nu(\nu-1)}}{\bar{\nu}^2} \frac{k^2}{2\,\alpha l^2} e^{-\alpha\tau}$$

This result was found in Chapter 3, formula (7):

(b) one-space cell (space-independent)

one-lethargy group (one-group theory)

For this case equations (15)-(19) take the following form:

$$sH(s) = H(s)\,(-1 + k(1-\beta))/l + \sum_l \lambda_l\, F_l(s) + v\, \Sigma_c\, \bar{q}\, \Phi/s$$

$$sF_l(s) = -\lambda_l\, F_l(s) + \bar{\nu}\, \beta_l\, v\, \Sigma_F\, H(s) + v\, \Sigma_c\, \bar{q}\, \Psi_p/s$$

$$0 = 2(\Phi[-1 + k(1-\beta)]/l + \sum_l \lambda_l\, \Psi_l) + \overline{\nu(\nu-1)}\,(1-2\beta)\, N v \Sigma_F$$

$$0 = -\lambda_p\, \Psi_p + \sum_l \lambda_l\, F_{lp}(s) + \Psi_p(s)\,[-1 + k(1-\beta)]/l + \bar{\nu}\, \beta_p\, v\Sigma_F\Phi + \overline{\nu(\nu-1)}\,\beta_p v \Sigma_F N$$

$$0 = -(\lambda_l + \lambda_p) F_{lp} + \bar{\nu}\beta_l\, v\Sigma_F \Psi_p(s) + \bar{\nu}\beta_p\, v\Sigma_F \Psi_l(s)$$

Equation (14) gives:

$$\sum_l \lambda_l\, F_l(s) = \bar{\nu}\, v\, \Sigma_F\, H \sum_l \lambda_l\, \beta_l/(s + \lambda_l) + v\Sigma_c\, \bar{q} \sum_l \lambda_l\, \Psi_l/s\,(s + \lambda_l)$$

We insert this in (13):

$$sH(s)\{s + (1-k(1-\beta))/l - k\sum_l \lambda_l\beta_l/l(s + \lambda_l)\} = v\Sigma_c\bar{q}[\Phi + \sum_l \lambda_l\, \Psi_l/(s + \lambda_l)]$$

and obtain: $(\rho = (1-k)/k)$

$$R(s) = sH(s) = (lv\Sigma_c\bar{q}/k)\left(\Phi + \sum_p \Psi_p - s\sum_p \frac{\Psi p}{s + \lambda_p}\right) \Big/ \left[\rho + s\left(\frac{l}{k} + \sum_l \frac{\beta_l}{s + \lambda_l}\right)\right]$$

Now:

$$\sum_{l,p} \lambda_l\, F_{lp} = \beta \frac{k}{l} \sum \Psi_p \tag{17}$$

$$\sum_p \lambda_p\, \Psi_p = \beta \frac{k}{l} \sum_p \Psi_p + \frac{k(1-\beta)-1}{l} \sum \Psi_p + \frac{k\beta\Phi}{l} + \frac{\overline{\nu(\nu-1)}}{\bar{\nu}\, l}\, \beta\, k\, N \tag{16}$$

$$2\,\Phi\frac{1-k(1-\beta)}{l}=2\sum_l \lambda_l\Psi_l+\frac{\overline{\nu(\nu-1)}}{\bar{\nu}}\;\frac{1-2\beta}{l}\,k\,N \tag{15}$$

We use (16) in (15) and obtain: $\Phi+\sum_l \Psi_l=\dfrac{\overline{\nu(\nu-1)}}{\bar{\nu}}\;\dfrac{kN}{2\alpha l}$

We now write:

$$R(s)=\frac{l v\,\Sigma_c\bar{q}N}{2\alpha l}\;\frac{\overline{\nu(\nu-1)}}{\bar{\nu}}\left[\frac{1-s\sum\dfrac{X_p}{s+\lambda_p}}{s\;\left(\dfrac{l}{k}+\sum\dfrac{\beta_l}{s+\lambda_l}\right)+\rho}\right]$$

with:

$$X_p=\Psi_p\,2\alpha\,l\,\bar{\nu}/k\,N\overline{\nu(\nu-1)}$$

and have:

$$\Gamma(s)=\frac{1}{2}P\,\overline{q^2}+P\,\bar{q}^2\varepsilon\frac{\overline{\nu(\nu-1)}}{\bar{\nu}^2}\;\frac{1}{2\rho}\left[\frac{1-s\sum\dfrac{X_p}{s+\lambda_p}}{s\;\left(\dfrac{l}{k}+\sum\dfrac{\beta_l}{s+\lambda_l}\right)+\rho}\right]$$

The power-density spectrum becomes:

$$\Gamma(\omega)=P\,\overline{q^2}+\bar{q}^{\,2}P\varepsilon\frac{\overline{\nu(\nu-1)}}{\bar{\nu}^2}\;\frac{1}{\rho}\;\text{Real }(T(s))$$

where s has to be replaced by $(-i\omega)$ in:

$$T(s)=\frac{1-s\sum\dfrac{X_p}{s+\lambda_p}}{s\left(\dfrac{l}{k}+\sum_l\dfrac{\beta_l}{s+\lambda_l}\right)+\rho}$$

As:

Real $\{T(-i\omega)\}=\rho\;|T(i\omega)|^2$ (Raievski 4)

we again have Bennet's result (3):

$$\Gamma(\omega)=P\,\overline{q^2}+\bar{q}^2P\varepsilon|T(i\omega)|^2\,\overline{\nu(\nu-1)}/\,\bar{\nu}^2$$

(c) one-lethargy cell (one-group theory).

In the one-group theory it is assumed that all fission neutrons are released with the group energy. This is a point where we can make a slight generalization if we take into account the fact that the fission neutrons are produced as fast neutrons with a high energy and only after moderation reach somewhere in the reactor the slow energy of the group. This means that the fission source in the space cell ν (number of fission neutrons with group energy generated in the space cell ν) is made up of all fission neutrons which are generated somewhere in the reactor and which reach the group energy in the space cell ν. The fission source in space cell ν can therefore be expressed as a sum over the contributions of all the other space cells. If $p_{\mu\nu}$ is the probability that a fast fission neutron born in space cell μ reaches the group energy in space cell ν the prompt fission source is:

$$\bar{\nu}\,(1-\beta)\sum_{\mu} v\,\Sigma_F\,N_\mu\,p_{\mu\nu}$$

where the summation extends over all space cells.

In our general theory the fission source in the lethargy interval ν was expressed as a sum over the contributions of all the other lethargy intervals.

For our problem we therefore need only replace the summation over the lethargy axis in every fission source term by a summation over the reactor cells and to replace the fission spectrum $f_o(\mu)$ resp. $f_1(\mu)$ by the slowing-down kernel $p_{\mu\nu}$:

In this case $J_{o\nu} = X_\nu$ is replaced by $J_\nu = \sum_\lambda p_{\nu\lambda} X_\lambda$ in Φ_ν
(summation over all reactor cells).

If we take for $p_{\nu\mu}$ the slowing-down kernel of the age theory

$$p_{\nu\mu} = \exp\;(-/x_\nu - x_\mu/^2/4\tau)\,d\tau_\mu/(4\pi\tau)^{3/2}$$

our equations (13)-(17) reduce to the fundamental equations on which Raievski's work (4) is based.

It is also very easy to derive with our general method the results obtained by A. Medina (5) and L.I. Pal (6). We mention only that Medina uses directly equation (3) (Chapmann-Kolmogoroff-equation) for some very simple cases (space- and energy-independence; no delayed fission neutrons) and is mainly interested in the solution of our equation (15) in the time-dependent case which gives some information about the time behaviour of the fluctuations in the number of neutrons. L.I. Pal considers the same special case with delayed neutrons and makes some study of the solutions of equations (5) and (15) under the initial condition of one neutron being present at time $t = 0$.

In order to estimate, for instance, the fluctuations in the number of neutrons in the lethargy cells (we consider only the stationary space-independent case and neglect delayed neutrons) we calculate the quantity:

$$\frac{\overline{N(k)^2} - \overline{N(k)}^2}{\overline{N(k)}} = 1 + \Phi(k,k)/\overline{N(k)} = 1 + \Phi(k)/\overline{N(k)}$$
$$= 1 + y(k)$$

where $\Phi(k)$ is the solution of equation (15) for the special case under investigation.

We set:

$$v(k)\Sigma_T(k)\Phi(k) = \chi\,(k)$$

$$v(k)\Sigma_T(k)\overline{N}(k) = \Psi(k) \quad \text{(total collision density in the real system)}$$

and have:

$$y(k) = \chi(k)/\Psi(k)$$

Equation (15) for $\Phi(k)$ takes in this special case the form:

$$\chi(k) = \sum_{\mu} \chi(\mu)\gamma_s(\mu \rightarrow k) + \bar{\nu} f_o(k) \sum_{\mu} \chi(\mu)\gamma_F(\mu) \tag{18}$$

$$+ \overline{\nu(\nu-1)} f_o(k)^2 \sum_{\mu} \Psi(\mu)\gamma_F(\mu)/2$$

where we have put:

$$\gamma_s(\mu \rightarrow k) = \Sigma_s(\mu \rightarrow k)/\Sigma_T(\mu)\,; \; \gamma_F(\mu) = \Sigma_F(\mu)/\Sigma_T(\mu)$$

We write equation (18) in the form:

$$\chi(u) = \int_o^u \chi(u')\gamma_s(u' \rightarrow u)\, du' + S(u) \tag{19}$$

with:

$$S(u) = \bar{\nu} f_o(u)\chi_F + \frac{1}{2}\overline{\nu(\nu-1)} f_o^2(u)\Psi_F$$

$$\chi_F = \int_{-\infty}^{+\infty} \chi(u)\gamma_F(u)du$$

$$\Psi_F = \int_{-o}^{+\infty} \Psi(u)\gamma_F(u)du = \quad \text{(total fission rate in the real system)}$$

Equation (19) is the neutron balance equation for slowing down with $S(u)$ as an external source, and with $\chi(u)$ as the collision density. In our case $\chi(u)$ is a measure of the statistical dependence between the neutrons in the lethargy interval u. We see that this statistical dependence is fed into the system by the source $S(u)$, which in turn owes its existence to the presence of fissions. This is in agreement with the considerations in Chapter 3. If we had no fission the source term $S(u)$ would be zero and the solution of equation (19) would be $\chi \equiv 0$. This is, in other words, the expression for the fact that in this case we have no statistical dependence between the neutrons in the reactor and $y(u) = o$. The neutrons in the system are statistically independent and the number of neutrons in a lethargy interval is therefore distributed according to Poisson's distribution.

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